Joint linearization instabilities in general relativity

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When Einstein's equations are supplemented by symmetry conditions, linearization instabilities can occur that are not present in either of the two sets of equations. The general conditions for this joint instability are investigated. This is illustrated with an example where both the Einstein equations and the flatness condition have more linearized solutions than exact solutions. In a minisuperspace model the geometrical reason for these instabilities is shown.

I. INTRODUCTION

The validity of the linearized approximation to nonlinear geometrical equations, such as Einstein's equations, has been largely clarified in the past decade.[1] It is appropriate to think of solutions as points in a suitable function space; linearized objects are then members of the corresponding tangent space. If the tangent space defined by the linear approximation to the nonlinear equations is the same as the tangent space to the manifold of solutions, then the equations are called linearization stable. Thus linearization stability near a solution means that the solution manifold is smooth near that point—for each direction defined by a solution of the linearized equations there is a family of exact solutions (a curve on the solution manifold) whose tangent is that direction.

Einstein's equations have been shown to be linearization stable about most globally defined solutions, both for asymptotically flat and for spatially compact manifolds. The exceptions are solutions on compact manifolds with Killing vector symmetries. Here there are quadratic conditions, in addition to the linearized equations. These conditions must be satisfied to assure that there be exact solutions corresponding, in the above sense, to solutions of the linearized equations. These quadratic conditions are of global type, involving integrals over a spacelike Cauchy surface. In any finite local region (with boundary), Einstein's equations are always linearization stable.[2]

However, in a remarkable paper,[3] Geroch and Lindblom have recently shown that in the context in which exact solutions are typically discussed the linearized approximation is not always reliable; in fact they exhibited linearization instabilities that are characterized by *local* second-order conditions (and therefore have nothing to do with the global conditions mentioned above). The context where this occurs involves existence of fixed Killing vectors in all the metrics under consideration. This restriction to symmetric metrics[4] is by itself linearization stable, and hence cannot be solely responsible for the instability they find. The Geroch–Lindblom example exhibits another surprising feature, namely that to linear order *all* solutions of their class are "gauge," i.e., related by diffeomorphism to flat space-time; yet in higher order genuinely curved, nonflat solutions are obtained.

Although Geroch and Lindblom give a satisfactory and instructive explanation of these circumstances, they do not interpret their results in the standard language of linearization stability theory. It is the aim of the present paper to show that both of the surprising features found by Geroch and Lindblom are a result of linearization instability in the usual sense, and to give a geometrical interpretation of this as lack of smoothness in a function space setting. The key to our interpretation of this instability is the observation that two linearization stable equations may not remain stable when imposed *jointly*. For functions of a finite number of variables this is the easily visualized fact that the intersection of two smooth surfaces is not necessarily smooth (Fig. 1). In Sec. II we explore this phenomenon, which for brevity we call "joint instability," and we discuss the second-order conditions that follow if the joint stability criterion is violated. For simplicity the equations are written for the finite-dimensional case, but they can be generalized easily to function spaces.

In Sec. III we consider the joint stability of the vacuum Einstein equations and certain symmetry conditions, as well as joint stability of space-time flatness and symmetry conditions. We find that both systems are jointly unstable. Thus there are more symmetric solutions of the linearized Einstein equations than symmetric exact solutions, which is one of the Geroch–Lindblom results; and there are more linearly flat symmetric metrics than exactly flat symmetric metrics, which is the other Geroch–Lindblom result.

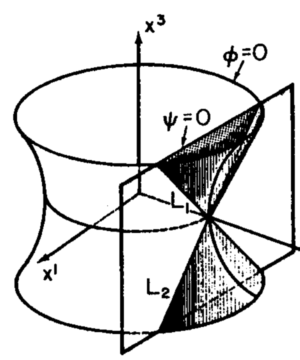

FIG. 1. Simple example of a joint instability. The two surfaces described by Eqs. (4a) and (4b) are everywhere smooth. However, when $a = 1$ their intersection consists of the pair of lines L_1 and L_2. This intersection is not everywhere smooth but has a "conical" singularity at $L_1 \cap L_2$.

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In Sec. IV we construct a finite-dimensional "minisuperspace" of symmetric initial data in which we study the conical structure of the subspace of Einstein solutions, and of flat metrics, which is the geometrical feature associated with these instabilities.

II. JOINT INSTABILITY AND SECOND-ORDER CONDITIONS

We consider the nonlinear equations as a map Φ from a domain D to a range R. The space of solutions S is the subset of D that maps to zero, $\Phi(S) = 0$. The linearization of Φ is the differential $d\Phi$ that maps the tangent space at some point of S to the tangent space of the origin of R. Let D be N dimensional, with coordinates x^a, and let S be described by M equations $\phi^i(x^a) = 0$ ($a = 1,...,N$, $i = 1,...,M$). If dx^a are coordinates of the tangent space at a point $P \in S$, then the linearized equations about P are

$$d\phi^i = \frac{\partial \phi^i}{\partial x^a} dx^a = 0.$$

(Here and in the following all partial derivatives are evaluated at P.) A convenient condition for linearization stability[5] is that $d\Phi$ be a surjective map, that is, if dx^a is allowed to range over the whole tangent space of D then $d\phi^i$ will fill the whole of TR, the tangent space of R.

Suppose Φ and Ψ are equations that can be simultaneously imposed on D, so that they map D to two possibly different ranges R_1, R_2. Suppose further that they are separately linearization stable, hence $d\Phi$ and $d\Psi$ are surjective on TR_1 and TR_2, respectively. Let S_1 and S_2 be the subspaces of D corresponding to the solutions, $\Phi(S_1) = 0$, $\Psi(S_2) = 0$. Imposing these equations simultaneously defines the intersection $S_1 \cap S_2$. We can consider the simultaneous set as a single map $\chi = (\Phi, \Psi)$ that maps D to the direct product $R_1 \times R_2$. Then χ is linearization stable—and hence Φ, Ψ are jointly stable—if its differential $d\chi = (d\Phi, d\Psi)$ is surjective on $TR_1 \times TR_2$.

If, on the contrary, $d\chi$ is not surjective, there must be a linear relation between $d\Phi$ and $d\Psi$ (since they are linear and separately surjective). That is, there must be one or more covectors $\underset{A}{w} = \left(\underset{A}{u}, \underset{A}{v}\right)$ in $T^*R_1 \times T^*R_2$ such that, for all $d\chi^a$,

$$\underset{A}{u_i} \frac{\partial \Phi^i}{\partial x^a} dx^a + \underset{A}{v_j} \frac{\partial \Psi^j}{\partial x^a} dx^a = 0. \tag{1}$$

As usual in the theory of linearization stability, existence of such $\underset{A}{w}$ allows us to construct second-order conditions on the dx^a. If these conditions are nonempty, there is joint instability. To construct these conditions we evaluate the simultaneous equations $(\Phi, \Psi) = 0$ to second order (here d^2 denotes the second derivative, not $d \wedge d$):

$$\begin{aligned} 0 &= (d^2\Phi^i, d^2\Psi^j) \\ &= \left(\frac{\partial^2 \Phi^i}{\partial x^a \partial x^b} dx^a dx^b + \frac{\partial \Phi^i}{\partial x^a} d^2 x^a, \right. \\ &\quad \left. \frac{\partial^2 \Psi^j}{\partial x^a \partial x^b} dx^a dx^b + \frac{\partial \Psi^j}{\partial x^a} d^2 x^a \right). \end{aligned} \tag{2}$$

Now we apply the linear relations (1) to find

$$\left(\underset{A}{u_i} \frac{\partial^2 \Phi^i}{\partial x^a \partial x^b} + \underset{A}{v_j} \frac{\partial^2 \Psi^j}{\partial x^a \partial x^b} \right) dx^a dx^b + \underset{A}{Q_{ab}}\, dx^a dx^b = 0, \tag{3}$$

that is, one quadratic relation for each of the covectors $\underset{A}{w}$. [These are nonempty, that is, independent of the linearized equations, unless there is a matrix $\underset{A}{L_{ij}}$ such that $\underset{A}{Q_{ab}} = (\partial \chi^i \partial x^a) \underset{A}{L_{ij}} (\partial \chi^j / \partial x^b)$. In the latter case, higher-order approximations to the simultaneous equations must be considered to decide about instability.]

A simple example of joint instability is provided by the two surfaces in Euclidean three-space, the hyperboloid

$$\Phi = (x^1)^2 + (x^2)^2 - (x^3)^2 - 1 = 0, \tag{4a}$$

and the plane

$$\Psi = x^1 - a = 0, \tag{4b}$$

as shown in Fig. 1. For a generic value of a their intersection is smooth, a pair of hyperbolas. However, for $a = 1$ it is a pair of intersecting, lines, $x^2 = \pm x^3$, hence the two linearization stable equations (4) must be jointly unstable for this value of a. In fact, any point on the intersections satisfies $x^1 = a$, $(x^2)^2 - (x^3)^2 = 1 - a^2$. For such values of x^2, x^3, the linearization

$$\begin{aligned} dx &= (d\Phi, d\Psi) \\ &= (2a\, dx^1 + 2x^2\, dx^2 - 2x^3\, dx^3, dx^1), \end{aligned} \tag{5}$$

which maps $\mathbf{R}^3$ to $\mathbf{R}^1 \times \mathbf{R}^1$, is surjective whenever x^2 or x^3 differ from zero. However when $a = 1$ and $x^2 = 0 = x^3$, we have the linear relation $d\Phi - 2d\Psi = 0$, which is of the form (1) with $u = 1$, $v = -2$. The corresponding quadratic condition (3),

$$2(dx^1)^2 + 2(dx^2)^2 - 2(dx^3)^2 = 0, \tag{6}$$

is nonempty, hence there is a joint instability: the linearized equations for $a = 1$ are satisfied by $dx^1 = 0$ and dx^2, dx^3 arbitrary; but exact simultaneous solutions of (4a) and (4b) exist only for the directions that also satisfy (6).

III. EINSTEIN'S EQUATIONS AND SYMMETRY CONDITIONS

Since it is sufficient, and more convenient, to discuss the stability of the Einstein constraints, we shall assume that the symmetries are spacelike and that the metric has nontrivial time dependence. (This is not essentially different from the case discussed by Geroch and Lindblom, where the metric is independent of x, y, and t but depends on z.) The space-time is described in terms of the initial data on a Cauchy surface,[6] namely the metric g_{ij} of the surface and its conjugate momentum π^{ij} (related to the second fundamental form). The Geroch–Lindblom symmetry condition demands that there be three commuting Killing vectors that are passive, i.e., the same for all metrics. Without loss of generality we can therefore assume that the three spacelike Killing vectors k are the coordinate directions $\partial/\partial x^i$. The corresponding conditions on the initial data are

$$0=\Psi(g,\pi)=\begin{cases}\mathscr{L}_k g_{ij}, & (7a)\\ \mathscr{L}_k \pi_{ij}. & (7b)\end{cases}$$

The Einstein constraints are

$$0=\Phi_1(g,\pi)=\begin{cases}R(g)+\pi_{ij}\pi^{ij}-\frac{1}{2}(\pi_i{}^i)^2, & (8a)\\ \pi^{ij}{}_{|j}, & (8b)\end{cases}$$

and the flatness conditions are the Gauss–Codazzi equations[7]

$$0=\Phi_2(g,\pi)=\begin{cases}R_{ij}(g)+\pi_{ij}\pi^k{}_j-\frac{1}{2}\pi_{ij}\pi_k{}^k, & (9a)\\ \pi_{ij|k}-\pi_{ik|j}+\frac{1}{2}(g_{ik}\pi_{,j}-g_{ij}\pi_{,k}). & (9b)\end{cases}$$

The linearizations about flat space-time (described by $g_{ij}=\delta_{ij}$, $\pi_{ij}=0$), written in the usual notation $dg_{ij}=h_{ij}$, $d\pi^{ij}=\omega^{ij}$ take the form

$$d\Psi=\begin{cases}d\Gamma^k_{ij}(h)=\frac{1}{2}(h_{ki,j}+h_{kj,i}-h_{ij,k}), & (10a)\\ \omega^{ij}{}_{,k}, & (10b)\end{cases}$$

$$d\Phi_1=\begin{cases}dR(h)=d\Gamma^k_{ii,k}-d\Gamma^k_{ik,i}, & (11a)\\ \omega^{ij}{}_{,j}, & (11b)\end{cases}$$

$$d\Phi_2=\begin{cases}dR_{ij}(h)=d\Gamma^k_{ij,k}-d\Gamma^k_{ik,j}, & (12a)\\ (\omega_{ij,k}-\omega_{ik,j})+\frac{1}{2}(\delta_{ik}\omega_{,j}-\delta_{ij}\omega_{,k}). & (12b)\end{cases}$$

Note that the equations for h and for ω decouple. Since the exact equations (7b), (8b), (9b) are linear in π, we do not get an instability from the linear relations between (10b), (11b), and (12b). However, the corresponding linear relations between (10a) and (11a) and between (10a) and (12a) do result in second-order equations of the type (3). For example, (11a) is a kind of divergence of (10a) and we have

$$\underset{x}{u}=\delta(x-x'),$$

$$\underset{x}{v}{}^{ij}{}_k=-\delta^{ij}\delta_{,k}(x-x')+\delta_k{}^j\delta_{,i}(x-x')$$

[where the index A of Eq. (1) corresponds to the continuous index x, the index i to the continuous index x', and the index j to i, j, k, and x'], so that the equation corresponding to (3) obtained from Eqs. (7a) and (8a) becomes

$$\begin{aligned}d^2R(h,h)&+\omega_{ij}\omega^{ij}-(\omega_i{}^i)^2-d^2\Gamma^k_{ij}(h,h)_{,k}\\&+d^2\Gamma^k_{ik}(h,h)_{,j}=0\\&=\omega_{ij}\omega^{ij}-(\omega_i{}^i)^2.\end{aligned}\qquad(13)$$

Similarly from Eqs. (10a) and (12a) we get

$$\omega^{ik}\omega_k{}^j-\omega^{ij}\omega_k{}^k=0.\qquad(14)$$

Since the exact equation $\Psi=0$ actually implies $R_{ij}=0$ (flat three-space), the exact $\Phi_1=0$ and $\Phi_2=0$ equations reduce to Eqs. (13) and (14), with ω^{ij} replaced by π^{ij}. It is then easily seen that there are no further conditions on the linearized solutions beyond (13) and (14).

What are the consequences of this joint instability? For Ψ and Φ_2 it means that there are too many linearized symmetric and flat space-time metrics. In fact, once $D\Psi=0$ is imposed, $d\Phi_2$ always vanishes; that is, any constant h_{ij},ω^{ij} satisfy the first-order space-time flatness condition. All the corresponding space-time metrics are therefore "gauge," i.e., diffeomorphic (to linear order) to Minkowski space. This is the first surprising feature noted by Geroch and Lindblom.

Similarly, the joint instability of Ψ and Φ_1 means that there are too many linearized symmetric vacuum Einstein metrics; again, all solutions of $d\Psi=0$ also solve Einstein's equations to linear order. Those corresponding to actual solutions have to satisfy the local second-order condition, Eq. (13). This is the second surprising feature noted by Geroch and Lindblom. [However, their distinction between true and apparent gauge is not represented precisely by our flatness condition (9). For example, all nonvanishing solutions of Eq. (13) would be apparent gauge, but these still include some flat space-times, namely those of Eq. (17) below.]

IV. A MINISUPERSPACE MODEL

The Geroch and Lindblom example that we discussed in Sec. III can be used to construct a minisuperspace[8] that illustrates the conical nature of the solution manifold at points of instability. As initial data we consider only the Euclidean spatial metric δ_{ij} and spatially constant momenta π^{ij}. (We do not consider the corresponding mini-phase space of all spatially constant g_{ij} and π^{ij}, with positive definite g_{ij}, because it has a conical singularity itself at $g_{ij}\to 0$.) These form a six-dimensional space on which the six independent components of π_{ij} are smooth coordinates. We investigate the subspaces $\Phi_1=0$ and $\Phi_2=0$. Since the symmetry is presupposed, the instability will be exhibited by singularities of these subspaces.

The subspace Σ_1 of solutions of Einstein's equations is described by

$$0=\Phi_1=\pi_{ij}\pi^{ij}-(\pi_i{}^i)^2=\pi^{ij}G_{ijkl}\pi^{kl},\qquad(15)$$

where the DeWitt metric

$$G_{ijkl}=\delta_{ik}\delta_{jl}-\delta_{ij}\delta_{kl}\qquad(16)$$

has signature $+\,+\,+\,+\,+\,-$. Therefore Σ_1 is a five-dimensional "light" cone over a four-sphere S^4. [The four-sphere can be obtained by intersecting (15) with the five-plane $\pi_i{}^i=1$.] The singular point of this cone occurs at $\pi_{ij}=0$, i.e., at the Minkowski metric for the space-time generated by these initial data. Tangents at that point satisfy the second-order condition (13), but they span the entire six-dimensional space. All directions at the origin that are not on the cone Σ_1 represent unstable solutions of the linearized Einstein constraints.

The subspace Σ_2 representing flat space-time metrics is described by

$$0=\Phi_2=\pi^{ik}\pi_k{}^j-\pi_k{}^k\pi^{ij}.\qquad(17)$$

The general solution of (17) is

$$\pi^{ij}=v^iv^j,\qquad(18)$$

where v^i are the components of an arbitrary spatially constant vector on the initial surface. Therefore Σ_2 is a three-dimensional cone over the two-dimensional surface P described by points π^{ij} of type (18) with v^i a unit vector. Since any such v^i corresponds to a point on the two-sphere S^2, and since v^i and $-v^i$ map via (18) to the same point on P, P is

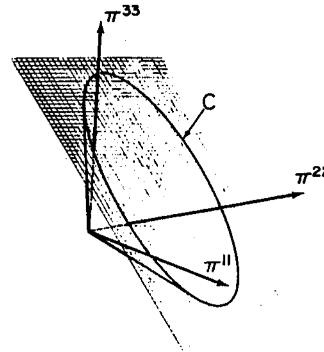

FIG. 2. Minisuperspace of Kasner space-times. The three axes (labeled π^{11}, π^{22}, π^{33}, respectively) represent flat space-times. All other points on the cone represent nonflat Kasner space-times. The plane represents the time coordinate condition $\pi^i_i = \text{const}$. All Kasner solutions satisfying this coordinate condition lie on the circle C.

topologically the projective plane. Since (17) implies (15), Σ_2 is contained in Σ_1 (and $P \subset S^4$). The singular point of Σ_2 is again $\pi^{ij} = 0$, and its tangents again span the entire six-dimensional space. All directions at the origin that are not on Σ_2 represent unstable ("apparently flat") solutions of the linearized flatness conditions.

To recover the Kasner solutions in their usual, diagonal form we reduce the number of minisuperspace dimensions to three by setting

$$\pi^{12} = \pi^{23} = \pi^{13} = 0.$$

The intersection with Σ_1 is a two-dimensional cone; that with Σ_2 consists of three lines (see Fig. 2). Both of these surfaces have a singularity at the origin, illustrating the geometrical reason for the instability in this restricted case. To regain the usual description of the Kasner solutions, we impose the time coordinates condition, $\pi_i{}^i = \text{const}$ (see Ref. 9). The resulting family of solutions form a circle, as shown in the figure. Since these metrics cannot be continuously connected to the Minkowski metric, the geometrical reason for the instability is not apparent with this coordinate condition.

V. CONCLUSIONS

We have seen that two individually linearization stable equations can be jointly unstable when imposed simultaneously. This situation may of course arise in cases other than the example we discussed above. Thus, joint instability may become important in Kaluza–Klein-type theories, where Einstein's equations are supplemented by further conditions.

Our results also have implications for the perturbative approach to quantum gravity. For example, since the Kasner-type perturbation we discussed in first order correspond to gauge transformations, only the phase of the quantum wave functional will vary in first order. Another quantum implication of linearization instability has been pointed out by Moncrief[10]: the operator form of the second-order conditions should be imposed on the wave function of linearized quantum gravity. An analogous procedure has to be followed where there is a joint instability, provided of course that a consistent quantum analog exists of the jointly unstable equations. Similarly, Moncrief conditions arise if there is a classical joint instability, and if the supplementary conditions are solved *before* quantizing (as in the minisuperspace approach).

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[1] For instance, see the following papers and the references cited therein: J. M. Arms, J. E. Marsden, and V. Moncrief, Commun. Math. Phys. **78**, 455 (1981); D. R. Brill and S. Deser, Commun. Math. Phys. **32**, 291 (1973); D. R. Brill, "Linearization stability," in *Spacetime and Geometry*, edited by R. Matzner and L. Shepley (U. Texas Press, Austin, 1982); A. E. Fischer, J. Marsden, and V. Moncrief, Ann. Inst. H. Poincaré **33**, 147 (1980).

[2] D. R. Brill, O. Reula, and B. Schmidt, "Local linearization stability" (to be published); GR11 abstracts, RESO, Stockholm, 1986.

[3] R. Geroch and L. Lindblom, J. Math. Phys. **26**, 2581 (1985).

[4] In the present discussion metric shall mean metric tensors; that is a description of the geometry in a particular coordinate system. Metrics with a given symmetry form a smooth submanifold of all metrics. In contrast, in the space of geometries, where diffeomorphic metrics are identified, the symmetric geometries lie on strata.

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[6] See, for example, Chap. 21 in C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (Freeman, San Francisco, 1973).

[7] See, for example, N. J. Hicks, *Notes on Differential Geometry* (Van Nostrand, Princeton, NJ, 1965).

[8] This name was originally given to finite-dimensional spaces whose points are symmetric three-geometries [C. W. Misner, in *Magic Without Magic: John Archibald Wheeler*, edited by J. Klauder (Freeman, San Francisco, 1972), pp. 441–473]. Here we consider a finite-dimensional space of symmetric initial data that have not all been identified by diffeomorphisms. It can be regarded as a "minimomentum space" or as a minisuperspace of four-geometries (generated by the Einstein time development equations for *all* initial data, whether solutions of the constraints of not).

[9] The family of metrics used by Geroch and Lindblom,

$$ds^2 = -(\lambda z + 1)^{2p_1} dt^2 + (\lambda z + 1)^{2p_1} dx^2 + (\lambda z + 1)^{2p_1} dy^2 + dz^2,$$

parametrized by λ, p_1, p_2, p_3, also corresponds to points in this minisuperspace if we identify those metrics that are equivalent under a z-translation. Three independent parameters that label these metrics smoothly are $dg_{ii}/dz = 2\lambda p_i$ (related to the diagonal components of π_{ij}).

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