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Black Holes and Rotation

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Abstract. In this article, we first consider briefly the basic properties of the non-rotating Schwarzschild black hole and the rotating Kerr black hole Rotational effects are then described in static and stationary spacetimes with arial symmetry by studying inertial forces, gyroscopic precession and gravi-electromagnetism. The results are applied to the black hole spacetimes.

Key words. Black holes—rotation—inertial forces—gyroscopic precession — gravi-electromagnetism.

1. Introduction

In the last three decades, there has emerged a phenomenal amount of research on black holes. This includes studies on their geometrical structure, their physical aspects and phenomena occurring in their strong gravitational fields. These studies were initially confined to the simpler case of the nonrotating Schwarzschild black hole and later on extended to the more complex case of the rotating Kerr black hole. The effect of rotation inherent to the Kerr spacetime manifests itself in almost all physical phenomena, sometimes in a profound manner. For instance, it is rotation that is responsible for the existence of the ergosphere in the Kerr geometry and the consequent possibility of energy extraction via the Penrose process. The subject of black holes and rotation is quite vast. Here we shall review only a few ideas with emphasis on the work my coworkers and I have done over the years. First we shall very briefly compare and contrast some of the basic attributes of the Schwarzschild and Kerr spacetimes. We shall then discuss the notion of 'rest frames' in the two geometrics which is quite important in studying physical phenomena. In recent years, there has been considerable interest in the general relativistic analogues of inertial forces and their possible reversal in the strong gravitational fields of black holes and ultra compact objects. At the same time there are two other phenomena apparently related to the inertial forces, namely gyroscopic precession and gravito-electro-magnetism. We shall demonstrate and discuss how these three aspects of black hole spacetimes can be related to one another in a covariant and elegant manner utilizing the Killing vector fields. The formalism will be presented at a very general level in the context of arbitrary static and stationary spacetimes with obvious application to the Schwarzschild and Kerr metrics as specific examples.

As has been mentioned already, the above topics are but a small part of an extensive field. They suffice, however, to illustrate the important role played by rotation in black hole physics.

2. Basic properties

In this section we give a very brief account of some of the basic properties of the Schwarzschild and Kerr black holes. These aspects are well known and are given here for the sake of completeness.

2.1 *Line elements*

Schwarzschild:

$$\mathrm{d}s^2 = \left(1 - \frac{2m}{r}\right)\mathrm{d}t^2 - \left(1 - \frac{2m}{r}\right)^{-1}\mathrm{d}r^2 - r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2) \tag{1}$$

where $m = MG/c^2$; M = Mass; $c = G = 1$.

Kerr:

$$\mathrm{d}s^2 = \left(1 - \frac{2m}{\Sigma}\right)\mathrm{d}t^2 + 2\frac{2mar}{\Sigma}\sin^2\theta\mathrm{d}t\,\mathrm{d}\phi - \left(a^2 + r^2\frac{2ma^2r}{\Sigma}\sin^2\theta\right)\sin^2\theta\mathrm{d}\phi^2 - \frac{\Sigma}{\Delta}\mathrm{d}r^2 - \Sigma\,\mathrm{d}\theta^2 \tag{2}$$

where m = Mass, $J = ma$ = *Angular Momentum,* $\Sigma = r^2+a^2\cos^2\theta, \Delta = m^2 - 2mr + a^2$

2.2 *Spacetime symmetries*

Both the Schwarzschild and the Kerr spacetimes admit a globally timelike Killing vector field ξ. The Schwarzschild metric is spherically symmetric with three rotational Killing vector fields.

$$L_x = y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}, \quad L_y = z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}, \quad L_z \equiv \eta = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} \tag{3}$$

satisfying the usual commutation relations $[L_x, L_y] = -L_z$ etc. while ξ and η commute,

$$[\xi, \eta] = 0$$

Here the coordinates (x, y, z) are related to (r, θ, ϕ) by the flat space formulae connecting the Cartesian coordinates to polar coordinates. Furthermore $\xi\ \eta = 0$, i.e $g_{03} = 0$, signifying the absence of rotation. In the case of the stationary, axially symmetric Kerr spacetime only $L_Z \equiv \eta$ exists in addition to the timelike Killing vector ξ^a. They satisfy the commutation relation $[\xi,\eta] = 0$, but $\xi\ .\ \eta = g_{03} \neq 0$, because of the inherent rotation.

2.3 *Source*

The Schwarzschild spacetime can represent the gravitational field exterior to different spherically symmetric sources. These include the static, collapsing or expanding, and oscillating spherical sources. And of course, the Schwarzschild spacetime without a material source of finite extension corresponds to the nonrotating, spherical black hole. In contrast, no realistic source has been matched on to the Kerr spacetime. Because of the unusual multipole structure of the Kerr spacetime involving infinite sequence of multipole moments m, ma^2 , ma^4,... it has been conjectured that no material source exists as the interior for the Kerr metric. This fact has not been proved but seems to be true. Neverthless, the Kerr spacetime corresponds to the rotating, stationary black hole with axial symmetry.

2.4 *Black hole structure*

As is well known, the Schwarzschild black hole is located at $r = 2m$. The global timelike Killing vector becomes null on this surface defining the static limit. Furthermore, the normal to this surace is also null. Consequently, the light cone is tangential to this surface which therefore acts as a one-way membrane. Or equivalently it is the event horizon. In other words, the surface $r = 2m$ is both the static limit and the event-horizon.

This is no longer true in the case of the Kerr spacetime, an important effect of rotation. The stationary limit, namely the surface on which the global timelike Killing vector becomes null, is given by $r = m + (m^2 - a^2 \cos^2 \theta)^{1/2}$ provided $\alpha \leq m$. On the other hand the null event horizon happens to be the surface $r = m + (m^2 - a^2)^{1/2}$. The region between the two surfaces is the ergosphere which makes unusual phenomena like the Penrose process and super-radiance possible at the cost of the rotational energy of the Kerr black hole.

2.5 *Uniqueness and stability*

The Schwarzschild and the Kerr black holes represent uniquely the time independent, asymptotically flat, uncharged black holes without and with rotation respectively. Stability of the Schwarzschild black hole has been established completely. The Kerr black hole has been shown to be stable against all normal modes. However, the completeness of the radial modes has not been proved though it may be reasonable to assume that this is true.

3. The global rest frame

Within the framework of special theory of relativity, or equivalently in the flat space-time, the global rest frame of an inertial observer plays a fundamentally important role. The general relativistic analogue of such a frame of reference in stationary, axisyrnmetric spacetimes is likewise important in studying physical phenomena in a meaningful way. In the case of black holes, the difference between the spacetimes of the nonrotating Schwarzschild black hole and the rotating Kerr black hole shows up clearly in defining these frames.

The rest frame in flat spacetime is adapted to the inertial observer following a worldline along time t. This is the direction of the timelike Killing vector ξ^α. The four velocity of the observers at different spatial points are orthogonal to the hyperspace t = constant. The four velocity is given by

$$u^a = e^{\psi}\xi^a; \quad u^a u_a = 1; \quad e^{\psi} = (\xi^a \xi_a)^{-\frac{1}{2}} \tag{4}$$

and

$$\xi^a = \delta^a_0 \quad \text{and} \quad \xi_a = t_{,a} \tag{5}$$

in Cartesian coordinates. Therefore t is the synchronous time for the rest observers. The worldlines or the four velocities of the rest observers constitute an irrotational congruence. If we define the vorticity of this congruence or that of the vector field ξ^a by

$$\omega^a_\xi \equiv \frac{1}{\sqrt{-g}} \varepsilon^{abcd} \xi_b \xi_{c;d}; \quad \omega^a_u = e^{2\psi} \omega^a_\xi, \tag{6}$$

then

$$\omega^a_\xi = 0. \tag{7}$$

Also u^a is a geodesic.

The concept of the global rest frame is directly extended to a static spacetime like the Schwarzschild. The four velocity u^a defined as in equation (4), is hypersurface orthogonal since,

$$\xi_a = g_{00} t_{,a} = (\xi_b \xi^b) t_{,a}. \tag{8}$$

Once again t is the common synchronous time for the rest observers. Obviously, the vorticity

$$\omega_\xi = 0 = e^{-2\psi} \omega_u \tag{9}$$

and the four velocities form an irrotational congruence. The rest observer's four velocity, however, is no longer geodetic.

Let us now consider the Kerr spacetime. The timelike Killing vector field is no longer irrotational and hence the Killing observers following ξ no longer define the global rest frame. Nevertheless, consider the vector field

$$\chi^a = \xi^a - \frac{(\xi^b \eta_b)}{(\eta^c \eta_c)} \eta^a. \tag{10}$$

We notice,

$$\chi^a \eta_a = 0 \tag{11}$$

so that χ^a is the projection of ξ^a orthogonal to η^a. Furthermore, it is easy to show that the vorticity of the χ^a - congruence

$$\omega^a_\chi = 0. \tag{12}$$

This was first noticed by Bardeen (1970), who called the frames adopted to χ^a as locally nonrotating frames (LNRF). It was recognized that the physical phenomena in the Kerr spacetime could be studied in a significant manner when referred to LNRF. The observers with four velocity

$$u^a = (\chi^b \chi_b)^{-\frac{1}{2}} \chi^a \tag{13}$$

are in fact the 'rest' observers and the frames adapted to them form the global rest frame since χ_a is in fact hypersurface orthogonal:

$$\chi_a = \left[\xi^b \xi_b - \frac{(\xi^b \eta_b)}{(\eta^c \eta_c)}\right] t_{,a}. \tag{14}$$

As before t is the synchronous time for these observers. The apparently paradoxical situation is that in order to be 'rest' observers, those following χ^a will have to be revolving round the black hole! Properties of the global rest frames were studied in detail and generalized to arbitrary stationary, axisymmetric spacetimes by Greene, Schucking & Vishveshwara (1975).

They showed that if the Killing fields ξ^a and η^a satisfied orthogonal transitivity, as in the Kerr spacetime, χ^a became null on the event horizon similar to ξ^a in the Schwarzschild spacetime. Furthermore, t = constant can be shown to be maximal surfaces.

Physical phenomena can be studied meaningfully in the global rest frames, especially since extended systems can be defined only on spatial surfaces of simultaneity like t = constant.

4. Rotational effects

There are three important approaches related to the study of rotational effects in time-independent, axially symmetric spactimes such as those of black holes. These are gravi-electromagnetism, gyroscopic precession and the general relativistic analogues of inertial forces. They manifest themselves especially when considering particle trajectories. They can be studied elegantly when the trajectories follow Killing vector fields as in the case of stationary worldlines or circular orbits. Moreover, the three approaches can be synthesised in a very nice way for such trajectories. We shall present below the formalism in the general case of the stationary, axisymmetric spacetimes and specialize to black holes. These considerations are based on the paper by Nayak & Vishveshwara (1998).

Recently, the two general relativistic phenomena, namely gyroscopic precession and inertial forces have been studied in detail. Iyer & Vishveshwara (1993) have given a comprehensive treatment of gyroscopic precession in axially symmetric stationary spacetimes making use of the elegant Frenet-Serret (FS) formalism. This forms the basis for a covariant description of gyroscopic precession. At the same time, a general formalism defining inertial forces in general relativity has been presented by Abramowicz, Nurowski & Wex (1993). The motivation for this work stemmed from the earlier interest in centrifugal force and its reversal. Such reversal in the Schwarzschild spacetime at the circular photon orbit was first discussed by Abramowicz and Prasanna (1990) and later in the case of the Ernst spacetime by Prasanna (1991). Abramowicz (1990) showed that centrifugal force reversed at the photon orbit in all static spacetimes. He argued, on qualitative grounds, that gyroscopic precession should also reverse at the photon orbit. Taking the Ernst spacetime as a specific example of static spacetimes Nayak & Vishveshwara (1997) have shown that, in fact, both centrifugal force and gyroscopic precession reverse at the photon orbits. A similar study by Nayak & Vishveshwara (1996) in the Kerr-Newman spacetime indicates that the situation in the case of stationary spacetimes is much more complicated than in the case of static spacetimes. Neither centrifugal force nor gyroscopic precession reverses at the photon orbit.

The above studies raise some interesting questions. Is gyroscopic precession directly related to centrifugal force in all static spacetimes? If so, do they both necessarily reverse at the photon orbit? In the case of stationary spacetimes is it possible to make a covariant connection between the gyroscopic precession on the one hand and the inertial forces on the other, not necessarily just the centrifugal force? Does such a connecting formula reveal the individual non-reversal of gyroscopic precession and centrifugal force at the photon orbit? We shall consider these and related questions. We shall then take up gravi-electromagnetism and show how this is related to gyroscopic precession and inertial forces. The case of black holes becomes a specific example of this broad-based formalism.

4.1 *Gyroscopic precession*

4.1.1 *Frenet-Serret description of gyroscopic precession*

The Frenet-Serret (FS) formalism offers a covariant method of treating gyroscopic precession. It turns out to be quite a convenient and elegant description of the phenomenon when the worldlines along which the gyroscopes are transported follow spacetime symmetry directions or Killing vector fields. In fact, in most cases of interest orbits corresponding to such worldlines are considered for simplicity. In the FS formalism the worldlines are characterized in an invariant geometric manner by defining along the curve three parameters κ the curvature and the two torsions τ_1 and τ_2 and an orthonormal tetrad. As we shall see, the torsions τ_1 and τ_2 are directly related to gyroscopic precession. All the above quantities can be expressed in terms of the Killing vectors and their derivatives. These considerations apply to a single trajectory in any specific example. However, additional geometric insight may be gained by identifying the trajectory as a member of one or more congruences generated by combining different Killing vectors. For this purpose the FS formalism is generalized to what may be termed as quasi-Killing trajectories. For the sake of completeness we summarize below relevant formulae taken from Iyer & Vishveshwara (1993).

Let us consider a spacetime that admits a timelike Killing vector ξ^a and a set of spacelike Killing vector $\eta_{(A)}$ ($A = 1,2,...m$).Then a quasi-Killing vector may be defined as

$$\chi^a \equiv \xi^a + \omega_{(A)}\eta^a_{(A)}, \tag{15}$$

where (A) is summed over. The Lie derivative of the functions $\omega_{(A)}$ with respect to χ^a is assumed to vanish,

$$\mathcal{L}_\chi \omega_{(A)} = 0. \tag{16}$$

We adopt the convention that Latin indices $a,b,... = 0 - 3$ and Greek indices α $\beta, \ldots = 1–3$ and the metric signature is $(+, -, -, -)$. Geometrized units with $c = G = 1$ are chosen. A congruence of quasi-Killing trajectories is generated by the integral curves of χ^a. As a special case we obtain a Killing congruence when $\omega_{(A)}$ are constants.

Assuming χ^a to be timelike, we may define the four velocity of a particle following χ^a by

$$e^a_{(0)} \equiv u^a \equiv e^{\psi}\chi^a, \tag{17}$$

so that

$$e^{-2\psi} = \chi^a\chi_a, \quad \psi_{,a}\chi^a = 0 \tag{18}$$

and

$$\dot{e}^a_{(0)} \equiv e^a_{(0);b}e^b_{(0)} = F^a_b e^b_{(0)}, \tag{19}$$

where

$$F_{ab} \equiv e^{\psi}(\xi_{a;b} + \omega_{(A)}\eta_{(A)a;b}). \tag{20}$$

The derivative of $\omega_{(A)}$ drops out of the equation. The Killing equation and the equation $\xi_{a;b;c} \equiv R_{abcd}\,\xi^d$ satisfied by any Killing vector lead to

$$F_{ab} = -F_{ba} \quad \text{and} \quad \dot{F}_{ab} = 0. \tag{21}$$

Now, the FS equations in general are given by

$$\begin{aligned}
\dot{e}^a_{(0)} &= \kappa e^a_{(1)}, \\
\dot{e}^a_{(1)} &= \kappa e^a_{(0)} + \tau_1 e^a_{(2)}, \\
\dot{e}^a_{(2)} &= -\tau_1 e^a_{(1)} + \tau_2 e^a_{(3)}, \\
\dot{e}^a_{(3)} &= -\tau_2 e^a_{(2)}.
\end{aligned} \tag{22}$$

As mentioned earlier k, τ_1 and τ_2 are respectively the curvature, and the first and second torsions while $e^a_{(i)}$ form an orthonormal tetrad. The six quantities describe the worldline completely. In the case of the quasi-Killing trajectories one can show that κ ,τ_1 and τ_2 are constants and that each of $e^a_{(i)}$ satisfies a Lorentz like equation:

$$\dot{\kappa} = \dot{\tau}_1 = \dot{\tau}_2 = 0, \tag{23}$$

$$\dot{e}^a_{(i)} = F^a_b e^b_{(i)}. \tag{24}$$

Further, k, τ_1, τ_2 and $e^a{}_{(\alpha)}$ can be expressed in terms of $e^a_{(0)}$ and $\overset{n}{F}_{ab} \equiv F_a{}^{a_1} F^{a_2}_{a_1} \ldots F_{a_{n-1}b}$.

$$\begin{aligned}
\kappa^2 &= F^2_{ab}e^a_{(0)}e^b_{(0)}, \\
\tau_1^2 &= \kappa^2 - \frac{F^4_{ab}e^a_{(0)}e^b_{(0)}}{\kappa^2}, \\
\tau_2^2 &= \frac{F^6_{ab}e^a_{(0)}e^b_{(0)}}{\kappa^2\tau_1^2} - \frac{\left(\kappa^2 - \tau_1^2\right)^2}{\tau_1^2},
\end{aligned} \tag{25}$$

$$\begin{aligned}
e^a_{(1)} &= \frac{1}{\kappa}F^a_b e^b_{(0)}, \\
e^a_{(2)} &= \frac{1}{\kappa\tau_1}[F^{2a}_b - \kappa^2\delta^a_b]e^b_{(0)}, \\
e^a_{(3)} &= \frac{1}{\kappa\tau_1\tau_2}[F^{3a}_b + (\tau_1^2 - \kappa^2)F^a_b]e^b_{(0)}.
\end{aligned} \tag{26}$$

The above equations were first derived by Honig, Schiicking & Vishveshwara (1974) to describe charged particle motion in a homogeneous electromagnetic field. Interestingly, they are identical to those that arise in the case of quasi-Killing trajectories.

Next let us consider an inertial frame of tetrad ($e^a_{(0)}, f^{\ a}_{(\alpha)}$) which undergoes Fermi-Walker (FW) transport along the worldline. The triad $f_{(\alpha)}$ may be physically realized by a set of three mutually orthogonal gyroscopes. Then, the angular velocity of the FS triad $e^a_{(\alpha)}$ with respect to the FW triad $f^a_{(\alpha)}$ is given by Iyer & Vishveshwara (1993)

$$\omega^a_{\mathrm{FS}} = \tau_2 e^a_{(1)} + \tau_1 e^a_{(3)}. \tag{27}$$

Or the gyroscopes precess with respect to the FS frame at a rate given by $\Omega(g) = -\omega_{\mathrm{FS}}$. In case of the Killing congruence ω_{FS} is identical to the vorticity of the congruence.

4.1.2 *Axially symmetric stationary spacetimes*

An axially symmetric stationary metric admits a timelike Killing vector ξ^α and a spacelike Killing vector η^a with closed circular orbits around the axis of symmetry. Assuming orthogonal transitivity, in coordinates ($x^0 \equiv t$, $x^3 \equiv \phi$) adapted to ξ^α and η^a respectively the metric takes on its canonical form

$$\mathrm{d}s^2 = g_{00}\,\mathrm{d}t^2 + 2g_{03}\,\mathrm{d}t\,\mathrm{d}\phi + g_{33}\,\mathrm{d}\phi^2 + g_{11}\,\mathrm{d}r^2 + g_{22}\,\mathrm{d}\theta^2 \tag{28}$$

with g_{ab} functions of $x^1 \equiv r$ and $x^2 \equiv \theta$ only. The quasi-Killing vector field

$$\chi^a = \xi^a + \omega\eta^a \tag{29}$$

generates closed circular orbits around the symmetry axis with constant angular speed w along each orbit. The FS parameters and the tetrad can be determined either by the direct substitution of χ^a or by transforming to a rotating coordinate frame as discussed by Iyer & Vishveshwara (1993). They can be written in terms of the Killing vectors and their derivatives as follows.

$$\kappa^2 = -g^{ab} a_a a_b, \tag{30}$$

$$\tau_1^2 = [g^{ab} a_a d_b]^2, \tag{31}$$

$$\tau_2^2 = \left[\frac{\varepsilon^{abcd}}{\sqrt{-g}}\, n_a \tau_b a_c d_d\right]^2, \tag{32}$$

$$\begin{aligned} e^a_{(0)} &= \frac{1}{\sqrt{\mathcal{A}}}(1,0,0,\omega), \\ e^a_{(1)} &= -\frac{1}{\kappa}\,(0, g^{11}a_1, g^{22}a_2, 0), \\ e^a_{(2)} &= \frac{1}{\sqrt{\mathcal{A}}\sqrt{-\Delta_3}}(\mathcal{B},0,0,-\mathcal{C}), \\ e^a_{(3)} &= \frac{\sqrt{g^{11}g^{22}}}{\kappa}(0,-a_2,a_1,0). \end{aligned} \tag{33}$$

In the above,

$$d_a = \left(\frac{\mathcal{B}}{2\sqrt{-\Delta_3}\kappa}\right)\left[\frac{\mathcal{B}_a}{\mathcal{B}} - \frac{\mathcal{A}_a}{\mathcal{A}}\right] = \left(\frac{\mathcal{B}}{\sqrt{-\Delta_3}\kappa}\right)[b_a - a_a],$$

$$a_a = \frac{\mathcal{A}_a}{2\mathcal{A}},$$
$$b_a = \frac{\mathcal{B}_a}{2\mathcal{B}},$$
$$\mathcal{A} = (\xi^a\xi_a) + 2\omega(\eta^a\xi_a) + \omega^2(\eta^a\eta_a),$$
$$\mathcal{B} = (\eta^a\xi_a) + \omega(\eta^a\eta_a),$$
$$\mathcal{C} = (\xi^a\xi_a) + \omega(\eta^a\xi_a),$$
$$\mathcal{A}_a = (\xi^b\xi_b)_{,a} + 2\omega(\eta^b\xi_b)_{,a} + \omega^2(\eta^b\eta_b)_{,a}; \quad a = 1, 2,$$
$$\mathcal{B}_b = (\eta^a\xi_a)_{,b} + \omega(\eta^a\eta_a)_{,b}; \quad b = 1, 2,$$
$$\Delta_3 = (\xi^a\xi_a)(\eta^b\eta_b) - (\eta^a\xi_a)^2 \tag{34}$$

where n^a is the unit vector along $\zeta_a = \xi_a - (\xi^b\eta_b)/(\eta^c\eta_c)\,\eta_a$ and τ^i is the unit vector along the rotational killing vector η^a. We may note that all the above equations can be specialized to a static spacetime by setting $\xi^a\eta_a=0$ and $\zeta^a \equiv \xi^a$

The above expressions when specialized to the equatiorial planes of black hole spacetimes are as follows. We have $\tau_2 = 0$ so that gyroscopic precession is given by τ_1 alone.

Kerr:

$$\tau_1^2 = \frac{1}{r^2}\,\frac{\left[\frac{Ma}{r^2} - \left(\frac{(r^2+2a^2)M}{r^2} - r\left(1 - \frac{2M}{r}\right)\right)\omega + \frac{Ma(3r^2+a^2)\omega^2}{r^2}\right]^2}{\left[1 - (r^2+a^2)\omega^2 - \frac{2M(a\omega)^2}{r}\right]^2}. \tag{35}$$

Schwarzschild:

$$\tau_1^2 = \omega^2 \frac{\left(1 - \frac{3M}{r}\right)^2}{\left(1 - \frac{2M}{r} - r^2\omega^2\right)^2}. \tag{36}$$

4.2 *Inertial forces*

4.2.1 *General formalism*

As has been mentioned earlier, in a recent paper Abramowicz *et al.* (1993) have formulated the general relativistic analogues of inertial forces in an arbitrary space-time. The particle four velocity u^a is decomposed as

$$u^a = \gamma(n^a + v\tau^a). \tag{37}$$

In the above, n^a is a globally hypersurface orthogonal timelike unit vector, τ^a is the unit vector orthogonal to it along which the spatial three velocity v of the particle is aligned and γ is the normalization factor that makes $u^a u_a = 1$.
Then the forces acting on the particle are written down as:

Gravitational force $G_k = \phi_{;k}$,
Centrifugal force $Z_k = -(\gamma v)^2 \tilde{\tau}^i \tilde{\nabla}_i \tilde{\tau}_k$,

$$\text{Euler force} \quad E_k = -\dot{V}\tilde{\tau}_k,$$
$$\text{Coriolis-Lense-Thirring force} \quad C_k = \gamma^2 v X_k, \tag{38}$$

where

$$\dot{V} = (ve^{\phi}\gamma)_{,i}u^i,$$
$$X_k = n^i(\tau_{k;i} - \tau_{i;k}),$$
$$\phi_{,k} = -n^i n_{k,i}. \tag{39}$$

Here $\tilde{\tau}^i$ is the unit vector along τ^i in the conformal space orthogonal to n^i with the metric

$$\tilde{h}_{ik} = e^{-2\phi}(g_{ik} - n_i n_k). \tag{40}$$

One can show that the covariant derivatives in the two spaces are related by

$$\tilde{\tau}^i\tilde{\nabla}_i\tilde{\tau}_k = \tau^i\nabla_i\tau_k - \tau^i\tau_k\nabla_i\phi - \nabla_k\phi. \tag{41}$$

We shall now apply this formalism to axially symmetric stationary Spacetimes.

4.2.2 *Inertial forces in axially symmetric stationary spacetimes*

As has been shown by Greene, Schiicking & Vishveshwara (1975), axially symmetric stationary spacetimes with orthogonal transitivity admit a globally hypersurface orthogonal timelike vector field

$$\zeta^a = \xi^a + \omega_0\eta^a, \tag{42}$$

where the fundamental angular speed of the irrotational congruence is

$$\omega_0 = -(\xi^a\eta_a)/(\eta^b\eta_b). \tag{43}$$

The unit vector along ζ^a is identified with n. Further, if u^a follows a quasi Killing circular trajectory, then τ^i is along the rotational Killing vector η^a. In this case it is easy to show that $\dot{V} = 0$ and hence the Euler force does not exist.

More specifically,

$$u^a = e^{\psi}(\xi^a + \omega\eta^a) = e^{\psi}\chi^a = \gamma(n^a + v\tau^a). \tag{44}$$

Then we have

$$n^a = e^{-\phi}\zeta^a,$$
$$\tau^a = e^{-\alpha}\eta^a,$$
$$\gamma = e^{\psi+\phi},$$
$$v = e^{-\phi+\alpha}(\omega - \omega_0), \tag{45}$$

where

$$\phi = \tfrac{1}{2}\ln(\zeta^a\zeta_a), \alpha = \tfrac{1}{2}\ln(-\eta^a\eta_a), \psi = \tfrac{1}{2}\ln(\chi^a\chi_a). \tag{46}$$

From the above relations, we can write down the inertial forces from their definitions as follows.

Gravitational force

$$G_k = \phi_{,k}, \tag{47}$$

Centrifugal force

$$Z_k = \frac{1}{2} e^{2(\psi+\phi)} \tilde{\omega}^2 \left(\frac{\eta^a \eta_a}{\zeta^b \zeta_b} \right)_{,k}, \tag{48}$$

Coriolis-Lense-Thirring force

$$C_k = e^{2(\psi+\alpha)} \tilde{\omega} \left(\frac{\xi^a \eta_a}{\eta^b \eta_b} \right)_{,k}, \tag{49}$$

Where $\tilde{\omega} = (\omega - \omega_0)$.
On the equatorial plane of the Kerr spacetime they reduce to

$$G_k = \frac{(r-M)\{(r^2+a^2)r + 2Ma^2\} - \frac{\Delta}{r}\{r^3 - Ma^2\}}{\Delta\{(r^2+a^2)r + 2Ma^2\}} (0,1,0,0), \tag{50}$$

$$C_k = \frac{2a\mathcal{W}}{\mathcal{A}\mathcal{G}_3} \left\{ \frac{M}{r^2} a^2 + 3M \right\} (0,1,0,0), \tag{51}$$

$$Z_k = \frac{\mathcal{W}^2}{\mathcal{A}} (0, z_1, 0, 0), \tag{52}$$

where

$$\begin{aligned}
\Delta &\equiv r^2 + a^2 - 2Mr, \\
\mathcal{A} &= 1 - \omega^2(r^2 + a^2) - \frac{2M}{r}(1 - \omega a)^2, \\
\mathcal{G}_3 &= (r^2 + a^2) + \frac{2M}{r} a^2, \\
\mathcal{W} &= \omega - \frac{2Ma}{(r^2 + a^2)r + 2Ma^2}, \\
z_1 &= \frac{1}{\Delta r^2} [(r-M)\{(r^2+a^2)r^2 + 2Mra^2\} - 2\Delta\{r^3 - Ma^2\}].
\end{aligned} \tag{53}$$

4.2.3 *Specialization to static spacetimes*

In a static spacetime the global timelike Killing vector ξ^a itself is hypersurface orthogonal. The unit vector n^a is now aligned along ξ^a,

$$n^a = e^{-\phi} \xi^a. \tag{54}$$

Then we have the inertial forces as follows:

Gravitational force

$$G_k = \phi_{,k}, \tag{55}$$

where $\phi = \frac{1}{2} \text{In} \, (\xi^a \xi^a)$, Centrifugal force

$$Z_k = -\frac{\omega^2}{2} e^{2(\psi+\alpha)} \left[\ln \left(\frac{\eta^i \eta_i}{\xi^j \xi_j} \right) \right]_{,k}, \tag{56}$$

Coriolis-Lense-Thirring force is identically zero,

$$C_k = 0. \tag{57}$$

In the specific example of the Schwarschild spacetime we have:

$$G_k = \left(1 - \frac{2M}{r}\right)^{-1} \frac{M}{r^2} (0, 1, 0, 0), \tag{58}$$

$$Z_k = \frac{(r - 3M)}{\left(1 - \frac{2M}{r} - \omega^2 r^2\right)\left(1 - \frac{2M}{r}\right)} (0, 1, 0, 0). \tag{59}$$

4.3 *Covariant connections*

In the preceding section we have derived expressions for τ_1 and τ_2 which give gyroscopic precession rate in terms of the Killing vectors. Similarly, inertial forces in an arbitrary axisymmetric stationary spacetime have also been written down in terms of the Killing vectors. All these quantities have been defined in a completely covariant manner. We shall now proceed to establish covariant connections between gyroscopic precession, i.e. the FS torsions τ_1 and τ_2, on the one hand and the inertial forces on the other. First, we shall consider the simpler case of static Spacetimes.

4.3.1 *Static spacetimes*

We have derived in equation (31) and (32), the FS torsions τ_1 and τ_2 for a stationary spacetime. As has been mentioned earlier, for a static spacetime $\xi^a \eta_a = 0$ and $\zeta^\alpha = \xi^\alpha$ in the above equations as well as in the expressions for inertial forces. With this specialization; centrifugal force can be written from equation (56) as

$$Z_b = e^{-(\phi-\alpha)} \omega \kappa d_b. \tag{60}$$

Substituting equation (60) in equations (31) and (32) we arrive at the relations

$$\tau_1^2 = \frac{\beta^2}{\omega^2} [a^b Z_b]^2, \tag{61}$$

and

$$\tau_2^2 = \frac{\beta^2}{\omega^2} \left[\frac{\varepsilon^{abcd}}{\sqrt{-g}} n_a \tau_b a_c Z_d \right]^2, \tag{62}$$

where

$$\beta = \frac{e^{(\phi-\alpha)}}{\kappa}. \tag{63}$$

The equations above relate gyroscopic precession directly to the centrifugal force The two torsions τ_1 and τ_2 , equivalent to the two components of precession, are respectively proportional to the scalar and cross products of acceleration and the centrifugal force. We shall discuss the consequences of these relations later on.

4.3.2 *Stationary spacetimes*

From equation (34) we have

$$\mathcal{A} = (\xi^a \xi_a) + 2\omega(\eta^a \xi_a) + \omega^2 \ (\eta^a \eta_a),$$
$$\mathcal{B} = (\eta^a \xi_a) + \omega(\eta^a \eta_a).$$

We decompose the angular speed ω with reference to the fundamental angular speed of the irrortational congruence $\omega_0 = -(\xi^a \eta_a)/(\eta^a \eta_a)$,

$$\omega = \tilde{\omega} + \omega_0. \tag{64}$$

Then we have

$$\mathcal{A} = \zeta^a \zeta_a + \tilde{\omega}^2 \eta^a \eta_a,$$
$$\mathcal{B} = \tilde{\omega} \eta^a \eta_a. \tag{65}$$

Similarly, we get

$$\mathcal{A}_a = (\zeta^b \zeta_b)_{,a} + 2\tilde{\omega}\mathcal{C}_a + \tilde{\omega}^2 (\eta^b \eta_b)_{,a},$$
$$\mathcal{B}_a = \mathcal{C}_a + \tilde{\omega}(\eta^b \eta_b)_{,a}, \tag{66}$$

where

$$\mathcal{C}_a \equiv (\xi^b \eta_b)_{,a} + \omega_0 (\eta^b \eta_b)_{,a} \tag{67}$$

or equivalently

$$\mathcal{C}_a = -(\xi^b \eta_b)\omega_{0,a}. \tag{68}$$

From equations (34), (65) and (66) we can show

$$\mathrm{d}_a = -e^{2\psi} \frac{e^{-(\phi+\alpha)}\tilde{\omega}}{2\kappa} \{(\zeta^p \zeta_p)\mathcal{C}_a + \tilde{\omega}[(\zeta^p \zeta_p)(\eta^q \eta_q)_{,a} - (\eta^p \eta_p)(\zeta^q \zeta_q)_{,a}] - \tilde{\omega}^2 (\eta^p \eta_p)\mathcal{C}_a\}. \tag{69}$$

Further, it is easy to see that C_a is directly proportional to C_a,

$$\mathcal{C}_a = -e^{-2\psi}\tilde{\omega}^{-1} C_a \tag{70}$$

where C_a is the Coriolis-Lense-Thirring force. Then equation (69) takes on the form where Z_a is the centrifugal force.

$$\mathrm{d}_a = \frac{e^{(\phi-\alpha)}}{\tilde{\omega}\kappa} \left\{ Z_a - \frac{1}{2}[1 + \tilde{\omega}^2 e^{2(\alpha-\phi)}] C_a \right\} \tag{71}$$

where Z_a is the centrifugal force.

Substituting this in equation (31) for τ_1^2 we get the relation,

$$\tau_1^2 = \frac{\beta^2}{\tilde{\omega}^2} [g^{ab} a_a (Z_b + \beta_1 C_a)]^2, \tag{72}$$

where

$$\beta = \frac{e^{(\phi-\alpha)}}{\kappa},$$
$$\beta_1 = -\tfrac{1}{2}[1 + \tilde{\omega}^2 e^{2(\alpha-\phi)}]. \tag{73}$$

Again, from equation (32), we obtain the expression

$$\tau_2^2 = \frac{\beta^2}{\tilde{\omega}^2}\left[\frac{\varepsilon^{abcd}}{\sqrt{-g}} n_a \tau_b a_c (Z_d + \beta_1 C_d)\right]^2. \tag{74}$$

These relations are more complicated than those we have derived in the static case Nevertheless, they closely resemble the latter with the centrifugal force replaced by the combination of centrifugal and Coriolis forces ($Z_a + \beta_1 C_a$). The static case formulae are obtained from those of stationary case by setting the Coriolis force to zero.

A formula for gyroscopic precession in the case of circular orbits in axially symmetric stationary spacetimes was derived by Abramowicz, Nurowski & Wex (1995) within a different framework. We note that gyroscopics precession does not involve the gravitational force. In case of geodetic orbits, total force is zero but not the centrifugal and Coriolis force individually. Therefore gyroscopic precession is also nonzero even for geodetic orbits.

4.4 *Reversal of gyroscopic precession and inertial forces*

The condition for the reversal of gyroscopic precession is given by

$$\omega^a_{\rm FS} = \tau_1 e^a_{(3)} + \tau_2 e^a_{(1)} = 0. \tag{75}$$

Since $e^a_{(1)}$ and $e^a_{(3)}$ are linearly independent vector fields at each point, this condition is the same as requiring

$$\tau_1 = \tau_2 = 0. \tag{76}$$

In the case of static spacetimes, τ_1 ans τ_2 are directly related to the centrifugal force Z_k. Therefore gyroscopic precession and centrifugal force reverse simultaneously. It can be shown that this happens at a photon orbit as borne out by the Schwarschild spacetime. In the case of stationary spacetimes there is no such correlations. This is true in the case of the Kerr spacetime.

5. Gravi-electric and Gravi-magnetic fields

Gravi-electric and gravi-magnetic fields are closely related to the idea of inertial forces. These fields with respect to observers following the integral curves of n^a can be defined as follows.

Gravi-electric field:

$$E^a = F^{ab} n_b, \tag{77}$$

Gravi-magnetic field:

$$H^a = \tilde{F}^{ab} n_b \tag{78}$$

where $\tilde{F}^{ab}$ is the dual of F^{ab},

$$\tilde{F}^{ab} = \tfrac{1}{2}\left(\sqrt{-g}\right)^{-1} \varepsilon^{abcd} F_{cd}. \tag{79}$$

In the above, as before, $F^{ab} = e^{\psi}(\xi_{a;b}+\omega\eta_{a;b})$The equation of motion is

$$\dot{u}^a = F^{ab} u_b. \tag{80}$$

Projecting onto the space orthogonal to n^a with $h_{ab} = g_{ab} - \eta_a\eta_b$ and decomposing u_a as given in (44), we get

$$\dot{u}_{\perp a} = \gamma[F_{ac}n^c + (v(F_{ac}\tau^c - n_a F_{bc} n^b \tau^c)] \tag{81}$$

where 7 is the normalization factor. This equation can be written in the form

$$\dot{u}_{\perp a} = \gamma[F_{ac}n^c + v\sqrt{-g}\varepsilon_{abcd} n^b \tau^c H^d] \tag{82}$$

or

$$\dot{u}_{\perp a} = \gamma[E + v \times H]. \tag{83}$$

We can therefore define

Gravi-electric force:

$$f_{GEa} = \gamma F_{ac} n^c, \tag{84}$$

Gravi-magnetic force:

$$f_{GHa} = \gamma v \sqrt{-g}\varepsilon_{abcd} n^b \tau^c H^d = \gamma v (F_{bc}\tau^c - n_a F_{bc} n^b \tau^c). \tag{85}$$

5.1 *Relations among gravi-electric, gravi-magnetic and inertial forces*

5.1.1 *Static case*

We have defined the gravi-electric field E_a by

$$yE_a = yF_{ac}n^c$$

If we substitute for $F_{ab} = e\ (^{\psi}\xi_{a;b} + \omega\eta_{a;b})$, we get

$$f_{GEa} = \gamma E_a = \gamma F_{ac} n^c = -e^{2(\psi+\phi)} G_a. \tag{86}$$

So,

$$E_a = -e^{(\psi+\phi)} G_a. \tag{87}$$

Here G_a is the gravitational force. Similarly we have for the gravi-magnetic field

$$f_{GHa} = \gamma v (F_{ac}\tau^c - n_a n^b F_{bc} \tau^c).$$

The second term in this equation is identically zero because the Killing vector field ξ^α and η^a commute and we get

$$
\begin{aligned}
f_{GHa} &= \gamma v\sqrt{-g}\varepsilon_{abcd}n^b\tau^c H^d,\\
&= \gamma v F_{ac}\tau^c,\\
&= [e^{2(\psi+\alpha)}\omega^2 G_a - Z_a].
\end{aligned} \tag{88}
$$

The above relation clearly shows the connection between the gravi-magnetic force on the one hand and the gravitational and centrifugal forces on the other.

5.1.2 *Stationary case*

In the stationary case, n^a is given by equation (45). As before we decompose $\omega = \varpi + \omega_0$ where ω_0 is given by (43). Then a straightforward computation gives the expression for the gravi-electric field.

$$
E_a = -e^{(\psi+\phi)}G_a + e^{-(\psi+\phi)}C_a \tag{89}
$$

and the gravi-electric force,

$$
f_{GEa} = \gamma E_a = -e^{2(\psi+\phi)}G_a + C_a. \tag{90}
$$

This shows the relation of gravi-electric field or force to both gravitational and centrifugal forces. In the stationary case also we have

$$
n_a n^b F_{bc}\tau^c \equiv 0. \tag{91}
$$

Then it follows

$$
\begin{aligned}
f_{GHa} &\equiv \gamma v\sqrt{-g}\varepsilon_{abcd}n^d\tau^c H^d,\\
&= \gamma v F_{ac}\tau^c,\\
&= \left[\frac{C_a}{2} + e^{2(\psi+\alpha)}\tilde{\omega}^2 G_a - Z_a\right].
\end{aligned} \tag{92}
$$

Hence gravi-magnetic force is related to all the three inertial forces—gravitational, centrifugal and Coriolis.

5.2 *Gravi-electric and Gravi-magnetic fields with respect to comoving frame*

In the previous section we have defined gravi-electric and gravi-magnetic fields with respect to the irrotational congruence. Similarly these fields can be defined with respect to the four velocity u^a of the particle as follows.

Gravi-electric field:

$$
\tilde{E}^a = F^{ab}u_b, \tag{93}
$$

Gravi-magnetic field:

$$
\tilde{H}^a = \tilde{F}^{ab}u_b. \tag{94}
$$

Where $\tilde{F}^{ab}$ is dual to F^{ab} as before. The equation of motion takes the form

$$
a^a = \tilde{E}^a. \tag{95}
$$

Precession frequency can be written simply as

$$\omega^a = \tilde{H}^a. \tag{96}$$

Following Honig, Schücking & Vishveshwara (1974), Frenet-Serret parameters κ, τ_1 and τ_2 can be expressed in terms of gravi-electric and gravi-magnetic fields.

$$\kappa = |\tilde{E}| \tag{97}$$

where

$$|\tilde{E}| = \sqrt{-\tilde{E}^a \tilde{E}_a}, \tag{98}$$

$$\tau_1 = \frac{|\tilde{P}|}{|\tilde{E}|}, \tag{99}$$

where

$$\tilde{P}^a = \varepsilon^{abcd} \tilde{E}_b \tilde{H}_a u_d = \tilde{E} \times \tilde{H}, \tag{100}$$

$$|\tilde{P}| = \sqrt{-\tilde{P}^a \tilde{P}_a} \tag{101}$$

and

$$\tau_2 = -\frac{\tilde{H}^a \tilde{E}_a}{|\tilde{E}|}. \tag{102}$$

Frenet-Serret tetrad components can also be expressed in terms of

$$e^a_{(1)} = \frac{\tilde{E}^a}{|\tilde{E}|}, \quad e^a_{(2)} = \frac{\tilde{P}^a}{|\tilde{P}|}, \quad e^a_{(3)} = \frac{\varepsilon^{abcd} \tilde{E}_b \tilde{P}_c u_d}{\tilde{P}^r \tilde{E}_r}. \tag{103}$$

In reference (Honig, Schücking & Vishveshwara 1974), these expressions had been derived for charged particle motion in a constant electromagnetic field. We have now demonstrated the exact analogues in the case of gravi-electric and gravi-magnetic fields. The one-to-one correspondence is indeed remarkable.

All this can be translated easily to the specific example of black holes since the required expressions have been given already.

6. Conclusion

The geometric structure and the physical phenomena associated with black holes offer a striking example of the general relativistic effects engendered by strong gravitational fields. Furthermore, rotation plays a pivotal role in distinguishing the properties of the Kerr black hole from those of the Schwarzschild black hole. In comparing and contrasting their properties and the consequent effects, the Killing fields admitted by the two spacetimes provide an elegant, simple and yet a powerful basis for detailed analysis. This is utilized in defining fundamental concepts and formalisms as in the definition of the global rest frame. Again, the Killing symmetries provide a covariant method for treating gravi-electromagnetism, gyroscopic precession and inertial forces. They are interrelated and can be synthesized in an appealing manner. There are many other topics in black hole physics that carry the stamp of rotation: radiation, thermodynamics, Mach's principle, astrophysical applications such as accretion and so on. All this is way beyond the scope of the present article.

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