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A generalized Schrödinger formalism as a Hilbert space representation of a generalized Liouville equation

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Abstract. A generalized Schrödinger formalism has been presented which is obtained as a Hilbert space representation of a Liouville equation generalized to include the action as a dynamical variable, in addition to the positions and the momenta. This formalism applied to a classical mechanical system had been shown to yield a similar set of Schrödinger like equations for the classical dynamical system of charged particles in a magnetic field. The novel quantum-like predictions for this classical mechanical system have been experimentally demonstrated and the results are presented.

Keywords. Classical and quantum mechanics; Hilbert space representation charged particle dynamics; non-Planckian discrete states.

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1. Introduction

Quantum mechanics has, since its advent, revealed itself to be as enigmatic as it is novel. While it is a more general theory, supposed to be containing classical mechanics as a limit, it is not just a generalization of the latter, but represents a fundamentally distinct theory and indeed a paradigm shift from the conceptual framework of classical mechanics.

Attempts have continued since the inception of QM to forge a conceptual unity and continuity between the two theories through a possible unifying covering formalism. The present author has been concerned with the development of such a formalism for a number of years. Clearly, the most distinctive feature of quantum mechanics is the probability amplitude nature of its description and the interference effects resulting therefrom.

While there is obviously a genetic relationship between classical and quantum mechanics, there exist differences in their mathematical form and content which makes them appear so distinct from each other. Classical mechanics, for example, admits the whole continuum of initial values and the resulting dynamical states as 'allowed states', which are propagated as allowed states by the equation of motion. Quantum mechanics, by contrast, being a boundary value theory, permits, for a bounded system, only a discrete set of states of motion determined by the boundary conditions. Quantum mechanical states are thus determined by a set of global constraints (boundary conditions), while there are no such constraints for a classical mechanical system: A classical particle is not affected by a distant boundary, and represents only a local

evolution. One may, therefore, ask the question whether classical and quantum mechanics may be regarded as two different manifestations of the same underlying dynamics, described by the two different mathematical representations: global vs. local.

There are two crucial facts about QM that need to be recalled. One, that it bears a close relationship with the Hamilton–Jacobi formalism of classical mechanics. The second one is that it is a probabilistic theory and ought to have its origin in a probabilistic framework. In classical statistical mechanics, the probability arises, from a lack of knowledge about the initial data. While the orthodox (Copenhagen) point of view ordains that quantum mechanics is intrinsically probabilistic, it has also been argued that the probability may be considered to arise because of a lack of knowledge about certain 'hidden variables' which when specified could make an event deterministic. We shall make two stipulations. One, that the probability in QM has the same origin as in classical statistical mechanics. Second, that the quantum events can be described by a classical Liouville equation, but in a suitably generalized form. To see the manner in which it is generalized we consider here in § 2 the Hamilton–Jacobi formalism via the theory of first order partial differential equation. In § 3, we give the generalized Liouville equation and in § 4 a generalized Schrödinger theory is obtained as a Hilbert space representation of the former. It will be seen that this formalism covers both classical and quantum mechanics suitably.

A fascinating consequence of this formalism, namely, obtaining Schrödinger theory as a Hilbert space representation of the classical (generalized) equation, is that this may be applied to a *suitably defined* classical mechanical system as well. This means that the particular classical mechanical system would be predicted to exhibit properties characteristic of a probability amplitude description. This would appear to be quite heretical. The interesting fact, however, is that such a classical mechanical system had indeed been identified by the present author more than two decades ago and a set of Schrödinger-like equations were obtained then (through a heuristic derivation) [1]. Later they were derived as a Hilbert space representation of the Liouville equation for the system [2]. Its quantum-like observable consequences have indeed been observed over the years. Some of these observed effects will be presented in § 4. In some sense these results are more fascinating than the derivation of the quantum mechanic Schrödinger equation, because they are something entirely new and unexpected and, in fact, cannot be understood in the framework of the standard classical mechanical paradigm. If the latter is indeed the case, as it seems to be, then we have an entirely new physics in the classical mechanical domain of parameters: What it means is that the standard classical mechanical paradigm need to be supplemented by constraints delimiting the allowed values of the initial conditions. It may be emphasized here that the predictions and subsequent observations of a set of entirely new phenomena which cannot be understood even *post-facto* in terms of the standard classical mechanical paradigm in whose domain they fall, do testify to the physical significance and reality of the Schrödinger-like description which led to these predictions. This may also serve as a validation of the procedure for obtaining the quantum mechanic Schrödinger formalism on a Hilbert space representation of a generalized Liouville equation which has been presented here and elsewhere earlier [3, 4].

2. The Hamilton–Jacobi formalism and a generalized Liouville equation in an extended phase space

The standard manner of arriving at the Hamilton–Jacobi formalism given in most standard text books on classical mechanics is via the canonical transformation. A generating function S which makes the new canonically transformed Hamiltonian vanish is essentially the Hamilton principal function and the defining equation for the new Hamiltonian is the Hamilton–Jacobi equation.

Following Courant and Hilbert [5] (for example) we introduce the Hamilton–Jacobi equation as a first order partial differential equation, whose characteristics are the Hamilton equation of motion. Thus, consider a partial differential equation of the form

$$\begin{aligned} &F(X_1, X_2, X_N, t, S; p_1, p_2, \ldots, p_N, p_t) \\ &\equiv p_t + H(X_1, X_2, \ldots, X_N, t; p_1, p_2, \ldots, p_N) = 0, \end{aligned} \tag{1}$$

where

$$p_i = \frac{\partial S}{\partial X_i}, \quad p_t = \frac{\partial S}{\partial t}$$

and where to correspond to the H–J equation, F has been taken to be independent of S, and the dependence on p_t to be linear.

With the above first order partial differential equation $F = 0$, one associates the following system of ordinary differential equations for the $(2N + 1)$ functions $(X_i, p_i = \partial S/\partial X_i, S)$ of a parameter τ,

$$\frac{\mathrm{d}X_i}{\mathrm{d}\tau} = \frac{\partial F}{\partial p_i} = \frac{\partial H}{\partial p_i}, \quad \frac{\mathrm{d}t}{\mathrm{d}\tau} = \frac{\partial F}{\partial p_t} = 1, \tag{2a}$$

$$\frac{\mathrm{d}p_i}{\mathrm{d}\tau} = -\frac{\partial F}{\partial X_i} - p_i \frac{\partial F}{\partial S} = -\frac{\partial H}{\partial X_i}, \tag{2b}$$

$$\frac{\mathrm{d}S}{\mathrm{d}\tau} = \sum_i p_i \frac{\partial F}{\partial p_i} + p_t \frac{\partial F}{\partial p_t} = \sum p_i \frac{\partial H}{\partial p_i} - H = L, \tag{2c}$$

where L is the Lagrangian. This system is referred to as the system of 'characteristic differential equations' belonging to the partial differential equation (1). Using the equation $\mathrm{d}t/\mathrm{d}\tau = 1$, the two sets of these equations (2a) and (2b) will be recognized as the Hamilton equations with H as the Hamiltonian function, while (2c) gives the time rate of change of the action S as the Lagrangian on the right. The partial differential equation (the H–J equation) and characteristic equations are thus equivalent to each other. It is this equivalence which was exploited by Jacobi to formulate his theory.

There is an advantage in the above manner of introduction of the H–J equation. It leads naturally to a complete set of characteristic equations: In addition to the standard Hamilton equations (2a) and (2b) one also finds eq. (2c), as one of the characteristic equations. This provides a basis for augmenting the phase space of a dynamical system, and writing down a generalized Liouville equation. As we shall see later, a Hilbert space representation of this generalized Liouville equation yields a generalized Schrödinger theory.

The set of 'characteristic differential equations' (2a,b,c) for the quantities (X_i, p_i, S) define trajectories in the space of the $(2N+1)$ variables. One can then construct a $(2N+1)$ dimensional phase space corresponding to these variables and a probability (phase) function $f(X_i, p_i, S)$ defined over this phase space. (Usually one defines a phase space of the $2N$ variables $(X_i p_i)$. But we include here also the variable S to correspond to the $(2N+1)$ characteristic equations (2a,b,c).) The 'equation of continuity' for $f(X_i, p_i, S)$ can then be written down as

$$\frac{\partial f}{\partial t} + \sum_i \frac{\partial}{\partial X_i}(\dot{X}_i f) + \sum_i \frac{\partial}{\partial p_i}(\dot{p}_i f) + \frac{\partial}{\partial S}(\dot{S} f) = 0. \tag{3}$$

Using the expressions (2a) and (2b) for $\dot{X}_i$ and $\dot{p}_i$ and the fact that $\dot{S} = L$ is independent of S, so that $\partial \dot{S}/\partial S = 0$, eq. (3) becomes

$$\frac{\partial f}{\partial t} + \sum \frac{\partial H}{\partial p_i}\frac{\partial f}{\partial X_i} + \sum \frac{\partial H}{\partial X_i}\frac{\partial f}{\partial p_i} + L\frac{\partial f}{\partial S} = 0. \tag{4}$$

This is a generalization of the classical Liouville equation, in the extended phase space of (X_i, p_i, S) resulting in the inclusion of the last term, $L\partial f/\partial S$. This, as usual, describes the conservation of the probability measure along the characteristics described by (2a,b,c). The underlying dynamics is, of course, the classical Hamiltonian flow. Since all the 'flow' components $(\partial H/\partial p_i, -\partial H/\partial X_i, L)$ are independent of S, an integration with respect to S yields the standard classical Liouville equation

$$\frac{\partial \bar{f}}{\partial t} + \frac{\partial H}{\partial p_i}\frac{\partial \bar{f}}{\partial X_i} - \frac{\partial H}{\partial X_i}\frac{\partial \bar{f}}{\partial p_i} = 0 \tag{5}$$

with

$$\bar{f} = \int \mathrm{d}S\, f. \tag{6}$$

2.1 *Classical analogue of the state of motion in quantum mechanics*

Dirac, as the readers would already know, was quite occupied with the question of the nature of relationship between classical and quantum mechanics, and with the question of classical analogues of quantum objects. In his 1933 paper [6], for example, he very beautifully elucidated the role of classical Lagrangian in quantum mechanics, which later culminated into the path integral formulation at the hands of Feynman [7]. Another important remark, made by Dirac [8] was in relation to the significance of the Hamilton principal function as a solution of the Hamilton–Jacobi equation. A principal function, satisfying the Hamilton–Jacobi equation, defines a 'family' [11]. "The family does not have any significance from the point of view of Newtonian mechanics; but it is a family which corresponds to one state of motion in the quantum theory, so presumably the family has some deep significance in nature, not yet properly understood" (quoted from Dirac [8]).

Coming to the generalized Liouville equation (4) the Liouville density function f can be any function of X_i, p_i and S, which solves (4). In my earlier papers, however, f was chosen to be a δ-function in all the initial momenta including the global integrals of

motion. This was done to make the distribution f correspond closely to a quantum state where the momenta (in terms of which the distribution is a δ-function) represent integrals of motion with respect to which the state is prepared. I was not aware of Dirac's remarks then, but it turns out that such a distribution represents what Dirac has called a 'family', defined by a principal function, and which according to him corresponds to one state of motion in the quantum theory.

In accordance with the above considerations, f is regarded as a function of the initial momenta values α_i, rather than of the current momenta p_i. Equation (4) is then transformed accordingly (from the current momenta p_i to the initial momenta α_i). This yields

$$\frac{\partial \hat{f}}{\partial t} + v_i \frac{\partial \hat{f}}{\partial X_i} + \frac{L}{\eta} \frac{\partial \hat{f}}{\partial \Phi} = 0, \tag{7}$$

where now $\hat{f} = \hat{f}(X_i, \Phi, t; \alpha_i)$ and v_i is not to be regarded as independent of the X_i, but as a function of the X_i and the α_i, while the α_i appear as parameters in the argument of $\hat{f}$. Here, we have also introduced a constant η of the dimensions of action to non-dimensionalize the action S, so that

$$S = \eta \Phi \tag{8}$$

or to measure S in units of η. Φ is then a dimensionless quantity which may be termed as 'action-phase'. The meaning of $\hat{f}$ is then also analogous to the meaning of probability in quantum mechanics: $\hat{f}$ gives the probability of finding the system at X_i at time t with a given value of S, if it had initially the momenta α_i.

2.2 *A Hilbert-space representation*

Equation (7) with the definition (8) and the underlying characteristic equations (2a,b,c), it may be recalled, are essentially the equations representing classical dynamics, though the Liouville equation has been generalized to include a term corresponding to the action variable S. It may be recalled that eq. (5) for $\hat{f}$, the action averaged distribution is essentially the classical Liouville equation. One seeks a Hilbert space representation of eq. (7) and writes accordingly:

$$\hat{f} = \psi^2, \tag{9}$$

where ψ is a real quantity (one could let ψ be complex and write $\hat{f} = \psi^* \psi$, but there is no loss of generality if ψ is taken to be real). Next a finite time representation of eq. (7) is written as

$$\hat{f}(X_i, \Phi, t; \alpha_i) = \hat{f}\left[X_i - \int_{t'}^{t} v_i(X_i(t''), \alpha_i) \mathrm{d}t'' \right.$$
$$\left. \Phi - \frac{1}{\eta} \int_{t'}^{t} L(X_i(t''), v_i(t'')), \mathrm{d}t'', t', \alpha_i \right]. \tag{10}$$

Using (9) and taking the square root of (10), choosing the position sign at all space-time points:

$$\psi(X_i, \Phi, t; \alpha_i) = \psi\left(X_i - \int_{t'}^{t} v_i(t'') \mathrm{d}t'', \Phi - \frac{1}{\eta} \int_{t'}^{t} L \, \mathrm{d}t'', t', \alpha_i\right). \tag{11}$$

I shall not repeat the procedure given elsewhere [3,4] which yields, starting from (11), a generalized set of Schrödinger equations

$$\frac{i\hbar}{n}\frac{\partial\Psi(n)}{\partial t} = -\left(\frac{\hbar}{n}\right)^2 \frac{1}{2m}\nabla^2\Psi(n) + V\Psi(n), \quad n = 1,2,3,\ldots \tag{12}$$

with the total probability density

$$G(X,t) = \sum_n \Psi^*(n)\Psi(n). \tag{13}$$

It is, however, pertinent to point out some crucial steps, which represent a point of departure from classical mechanics and which lead to the above set of Schrödinger equations.

The first is clearly the generalization of the Liouville equation to include the action S as a variable. However, as the action-averaged Liouville equation (5) (which is the standard classical Liouville equation) shows, the classical mechanics is clearly contained in (4).

The second step is the eq. (9) writing $\hat{f} = \psi^2$ seeking a Hilbert space representation of eq. (7). While it may be an unusual proposition to seek a Hilbert space representation of (7), clearly this by itself could not make it depart in its content from that of classical mechanics, for one could in principle solve for ψ and reconstruct $\hat{f} = \psi^2$ and action-average it to obtain the classical distribution $\bar{f}$.

The third step is the assumption of ψ being periodic in $\Phi = S/\eta$ with the period 2π. It is this assumption (where η is identified with $\hbar$) which enable ψ to be Fourier-decomposed as a series (rather than an integral) and the subsequent procedure to be followed leading up to the derivation of the Schrödinger equations which are second order in space. One should follow the derivation closely to see how the second order operator ∇^2 arises. It is clear that this operator could not have arisen if the action S were not included as a variable.

The important consequence of this structure (second order differential operator) is that the functions $\Psi(n)$ of eq. (12) should be specified through appropriate boundary conditions. This amounts, apart from other consequences, to delimiting the allowed values of certain integrals of motion to a discrete set – that is, to 'quantization'.

It ought to be pointed out that this particular derivation yields a generalized formalism in the form of an infinite set of Schrödinger equations [12] with $(\hbar/n)$, $n = 1,2,3\ldots$ in the place of $\hbar$ and a correspondingly generalized probability connection (13). Some of the direct observable consequences of these additional modes $n = 2,3,\ldots$ have been discussed in ref. [4]. They have not been so far identified nor has there been any attempt made to check their existence experimentally.

3. A classical mechanical system with the Schrödinger-like description

There is nothing special in the procedure given above (where we obtained the Schrödinger-like description as a Hilbert space representation of the Liouville equation) which could not be applied to any suitable classical mechanical system. As pointed out above perhaps the most important step where the quantization really came in was

the assumption of periodicity of ψ with respect to $\Phi(\equiv S/\hbar)$ with the period 2π. What is, of course, required for the classical mechanical system is the justification of periodicity with respect to the action phase and the identification of an appropriate unit of action (analogous to $\hbar$) belonging to the system in terms of which to define the latter.

Such a classical mechanical system was identified by the present author more than two decades back [1]. The dynamical system of charged particles in a magnetic field is such a system. The action in the system which can serve as a unit is the gyroaction $\mu = \frac{1}{2}mv_\perp^2/\Omega$, where $v_\perp$ is the component of velocity perpendicular to the magnetic field at the point of injection and $\Omega = eB/mc$ is the gyro-frequency also at the point of injection, so that μ is essentially an initial value which is an exact constant of motion.

The Liouville equation for the system is given by

$$\frac{\partial f}{\partial t} + \mathrm{v} \cdot \frac{\partial f}{\partial X} - \Omega \frac{\partial f}{\partial \phi} = 0, \tag{14}$$

where ϕ is the gyro-phase.

Transforming the equation to momenta initial values: the total energy E, the canonical angular momentum P_ϕ, and the gyro-action μ, as well as transforming ϕ to the action phase

$$\begin{aligned}\Phi &= \frac{1}{\mu}\int_0^t \frac{1}{2} m v_\parallel^2 \mathrm{d}t' + \phi \\ &= \frac{1}{\mu}\int_0^t \mathrm{d}t' \left(\frac{1}{2} m v_\parallel^2 - \mu\Omega\right) \\ &= S_A/\mu,\end{aligned} \tag{15}$$

where S_A is the time integral of the adiabatic Lagrangian $L_A = (\frac{1}{2}mv_\parallel^2 - \mu\Omega)$, one obtains

$$\frac{\partial f}{\partial t} + v_\parallel \frac{\partial f}{\partial X_\parallel} + \frac{L}{\mu}\frac{\partial f}{\partial \Phi} = 0. \tag{16}$$

This equation has a structure similar to that of eq. (7) except that μ takes the place of η (later identified as $\hbar$). A Hilbert space representation of this equation which can be sought without any prejudice to QM, yields in a similar manner, the following set of Schrödinger-like equations:

$$\frac{i\mu}{n}\frac{\partial \hat{\Psi}(n)}{\partial t} = -\frac{(\mu/n)^2}{2m}\frac{\partial^2 \hat{\Psi}(n)}{\partial X_\parallel^2} + (\mu\Omega)\hat{\Psi}(n), n = 1, 2, 3, \ldots \tag{17}$$

again with the total probability density

$$G(X_\parallel, t) = \sum_n \hat{\Psi}^*(n)\hat{\Psi}(n). \tag{18}$$

The periodicity of $\psi = \sqrt{f}$ with respect to Φ which is crucial for obtaining the Schrödinger-like equation (17) is guaranteed through the periodicity with respect to ϕ (the gyro-phase) which is an additive part of Φ as per eq. (15).

It may be mentioned that $(\mu\Omega)$ which appears in the place of 'potential' is in fact the adiabatic potential which describes the effective parallel motion along the magnetic field through the equation of motion (in the adiabatic approximation)

$$m\frac{\mathrm{d}v_{\parallel}}{\mathrm{d}t} = -\nabla_{\parallel}(\mu\Omega). \tag{19}$$

4. Quantum-like consequences for the classical mechanical system

Even though the above mentioned dynamical system of charged particles in a magnetic field is a classical mechanical one with macroscopic dimension, the above amplitude description governed by the Schrödinger-like equations (18) entails that this system exhibits quantum-like properties for the ensemble of trajectories (the coherent set of trajectories, á la Synge [9] or a 'family' á la Dirac [8]), for which these equations are obtained. Whether this is indeed the case can be ascertained only through appropriate experimentation. Two kinds of predictions were identified for experimental verification.

4.1 *Existence of multiplicity of residence times in an adiabatic trap*

One relates to the residence times of charged particles trapped in an adiabatic potential well as described by the adiabatic equation of motion (19). Charged particles can be confined in a well of the potential $\mu\Omega$, if the total energy E of the particle is $E < (\mu\Omega)_{\max}$. But the trapping is not for ever, because the equation of motion (19) which describes this trapping is only an approximate equation. Particles so trapped have indeed been found experimentally to have leaked out of the trap, with characteristic residence times. No satisfactory theory for the determination of these residence life-times existed when these experimental results were reported during 1968–1969. The Schrödinger-like equations (17) along with (18), which were first given by the author in 1971 as a possible description of the leakage of particles, afforded the possibility of determining the life times through a quantum-tunneling like process. Leaving aside for the moment the question of the interpretation of the amplitude functions $\hat{\Psi}(n)$ vis-a-vis the quantum mechanics Schrödinger wave functions, what is different here from the standard quantum formalism is that we have here not one but a set of equations for the $\hat{\Psi}(n)$ with $n = 1, 2, 3....$ This, as already pointed out, is also true of the generalized Schrödinger formalism for quantum mechanics as contained in the set of equations (12) and (13). Correspondingly, one predicts here the existence of a multiplicity of residence times as determined by the equations corresponding to $n = 1, 2, 3 \ldots$, for the injection of particles with the same energy and the pitch angle. It should, perhaps, be pointed out that no predictions of this kind were available based on the standard approach.

Such multiplicity of residence times were indeed found experimentally in a series of experiments carried out at the Physical Research Laboratory [10a, b, c]. Upto three distinct times were identified, and were found to correspond to the modes $n = 1, 2, 3$ in their dependence on the magnetic field strength for a given energy E and the pitch angle

of injection θ. The experimental identification of these modes $n = 1, 2, 3$ through the existence of the corresponding residence times then constitutes a validation of the Schrödinger-like description. It may also be mentioned that so far no way has been found in terms of the standard approach to the problem to describe these multiplicities of residence times.

4.2 *Existence of discrete allowed and forbidden states in the macro-domain*

The other kind of prediction refers to the wave-like manifestations of the charged particle dynamics in the macro domain, that is the dimensions of tens of centimeters. Such manifestations follow from the wave nature of the Schrödinger-like equations, which include the existence of interference effects and consequent discrete allowed and forbidden energy states in the classical domain of parameters. Astonishing as it may seem such non-Planckian discrete energy states have indeed been observed in recent experiments, as will be described below.

These observations appear to be in manifest violation of the Lorentz equation of motion – initial value paradigm since the latter permits all the continuum of initial values and hence energies. As a matter of fact, the observations demonstrate in conformity with the predictions of the theory, a kind of wave-like nonlocality whereby the relative separation between the discrete states depends on the distance between the electron source – the electron gun, and the detector. Such a distance dependence is quite characteristic of the quantum behaviour, but is completely inexplicable in terms of the standard paradigm, whereby the entire continuum of initial values are allowed values and which takes no cognizance of the distant boundaries.

It seems therefore, that an electron moving in a magnetic field exhibits a wave-like behaviour even in macroscopic dimensions very much like the wave nature of electron in micro-dimensions (de Broglie wave). Such a behaviour follows from the set of Schrödinger-like equations obtained for the system.

What is then the resolution of the apparent paradox? The Lorentz equation is certainly not wrong. But the experiments, on the other hand, cannot be explained in terms of the standard paradigm, but appear to conform to the Schrödinger-like description.

The resolution of the parodox appears to consist in the realization that the standard paradigm of classical mechanics represents a purely 'local' evolution. That is, the trajectory evolution at a given time t is determined entirely by the value of the field of force at the position of the particle at that time, that is 'locally'. The Schrödinger-like equation on the other hand, seems to provide a global description of the same dynamics. Such a global description takes cognizance of the distant boundaries. As a consequence certain energy values (as initial conditions) are found to be disallowed, as deduced from the theory and verified experimentally. The allowed values on the other hand, could well propagate according to the Lorentz equation of motion. If this point of view is taken there need not be any violation of the Lorentz equation, but only of the paradigm that all the continuum of initial values are the allowed values in the classical mechanical formalism.

It may not be possible for reasons of space to present here the derivation of the formula giving the discrete allowed states in terms of a 'quantum number'. If we consider only the function $\Psi(1)$ for $n = 1$, corresponding to an electron beam from an electron gun

propagating along a magnetic field, then it has been shown (Varma [11]) that the probability density of finding the electrons on a grounded detector after propagating a distance D along the field is given by

$$\Psi^{*}\Psi = \sum_{k} \frac{|\hat{\psi}(k)|^2}{k} + \sum_{k} 2R(K,\bar{k}) \sin\left[\frac{\bar{\Omega}D}{\bar{v}_{\|}} + \phi\right], \tag{20}$$

where $v_{\|}$ is the 'parallel' velocity

$$v_{\|} = [2(E - \mu\Omega)/m]^{1/2}. \tag{21}$$

We see that this expression has an oscillating term (the second term) (over the mean first term) which oscillates with the magnetic field B, with the energy E, and the distance traversed D individually with the other parameters remaining constant. The oscillating term is a consequence of the interference effects. The minima in the expression of $\Psi^{*}\Psi$ are interpreted as 'forbidden states' of the charged particles in a magnetic field. The energies of the 'forbidden states' are given by

$$E_j = \tfrac{1}{2}m(\Omega D/2\pi)^2/(j + \phi_j/2\pi)^2, \tag{22}$$

where j denotes the 'quantum number' labelling the forbidden energy states E_j with ϕ_j being the phase shift.

As is clear from the expression (22) for the 'forbidden energies', they are seen to be determined, very astonishingly, by the distance D over which the electrons travel from the gun to the detector. This is quite astonishing from the point of view of the standard paradigm of classical mechanics. For, the dependence on D signifies a kind of wave-like nonlocality while classical mechanics is known to be a local theory. The predictions contained in this relation were subjected to experimental scrutiny at the Physical Research Laboratoy. We describe below an experiment reported earlier [12(a, b)] to check these predictions.

5. The experiment and its results

An electron beam of very low intensity ($< 0.1\,\mu$A) is injected from an electron gun along a magnetic field ($\sim$200–300 gauss) in an SS vacuum chamber of 27 cm diameter evacuated to $\sim 5.10^{-7}$ torr, and is received at a detector (Faraday cup) at a distance ($L \sim 20$ cm) away from the gun. The injection is almost parallel to the field. The Faraday cup detector consists of a grounded collector plate of $\sim$ 25 cm diameter, 1 cm behind a grid which can be biased to any required potential. The electron current from the beam can be measured by the detector (current received by the plate) for various negative grid potentials from zero to $-\Phi_{\max}$, where $\Phi_{\max} > E/|e|$, E being the electron beam energy. Since according to classical mechanics, the whole continuum of energy states are allowed states, the plate current is expected to exhibit a monotonically increasing response as the negative grid potential is swept from $-\Phi_{\max}$ to zero.

The actual experimentally observed response is quite and astonishingly different from the expected one (à la the standard classical mechanical paradigm) and is shown in

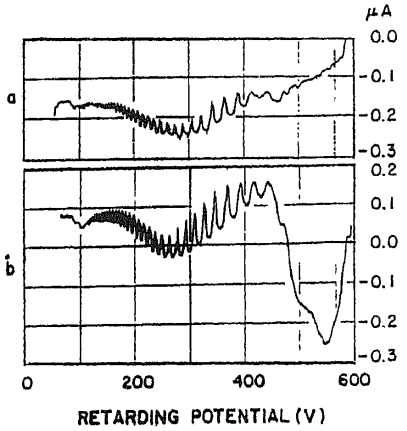

Figure 1. Plate current (**a**) and grid current (**b**) as functions of the retarding potential. $\bar{B} = 170\,\text{G}$, $L = 30\,\text{cm}$ and $E = 600\,\text{eV}$.

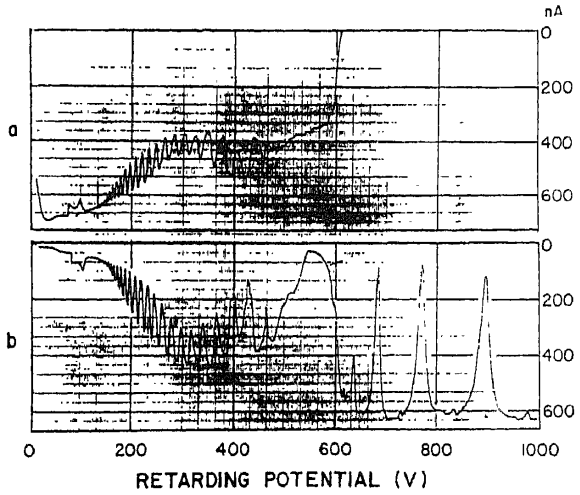

Figure 2. Plate current (**a**) and anode current (**b**) as functions of the retarding potential $\bar{B} = 177\,\text{G}$, $L = 19\,\text{cm}$ and $E = 600\,\text{eV}$.

figure 1. The upper curve (a) is the plate current and quite clearly exhibits a series of sharply defined dips. The lower curve (b) is the grid current for the same set of parameters and also exhibits a series of dips which are found to be exactly correlated with the plate current dips. From the point of view of classical mechanics, these dips are quite unexpected and astonishing as they signify the existence of 'forbidden states' which have no place in the formalism of classical mechanics.

To further check that these are indeed the forbidden states, both the plate current and the anode current were measured simultaneously as the grid potential is swept. (It should be explained that the anode which is grounded, is a part of the electron gun and which accelerates the electrons emanating from a negatively biased hot cathode placed about one cm away from it.) Thus the electrons encounter the anode much before ($\sim 20\,\text{cm}$) on the way to grid-plate assembly. Figure 2 shows both the simultaneously recorded plate

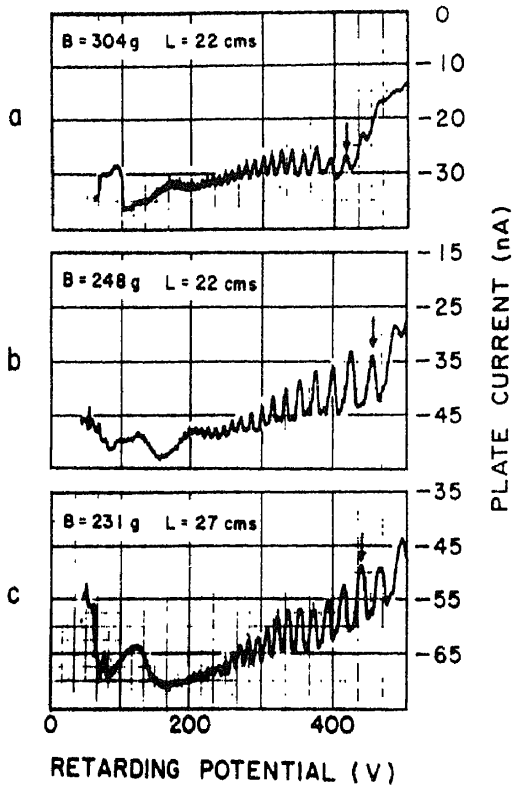

Figure 3. Plate current as a function of the retarding potential for different values of $\bar{B}$ and L (as shown in the plots) and $E = 650$ eV.

Table 1. The energies E_j and the j values for plots in figure 3(a, b, c).

Peaks	Energy E_j (eV)			$j + \phi/2\pi$		
	Plot 3a	Plot 3b	Plot 3c	Plot 3a	Plot 3b	Plot 3c
$N(\downarrow)$	417	453	438	$41 + 0.15$	$31 + 0.32$	$36 + 0.56$
$N+3$	357	377	373	$44 + 0.48$	$34 + 0.36$	$39 + 0.62$
$N+6$	313	317	323	$47 + 0.45$	$37 + 0.47$	$42 + 0.57$
$N+9$	277	272	283	$50 + 0.50$	$40 + 0.46$	$45 + 0.47$
$N+12$	247	237	250	$53 + 0.48$	$43 + 0.35$	$48 + 0.41$

and anode currents, the upper curve being the plate and the lower one being the anode current. The two curves are clearly anti-correlated, showing that electrons that did not (or could not) pass through the system in a certain energy state (forbidden state) have found their way to the anode, confirming in a sense the interpretation of the 'dips' in the plate current as forbidden states.

The experiment was repeated with different distances between the gun and the detector and magnetic fields. The plots so obtained are shown in figure 3. We see that the positions of the dips change both with the distance L and the magnetic field B. A little reflection

shows that the dependence on the distance L is most enigmatic because it signifies a kind of wave-like nonlocality.

One may now finally examine whether the dips so observed can be described by the relation (23). To do so, we read the positions of the dips (in energy) from the plots and using these values on the left of eq. (23) with the magnetic field and distance D inserted on the right, we calculate the quantity $(j + \phi/2\pi)$ for every third dip counted from an (arbitrary) arrowed peak in three respective plots of figure 3. The results are shown in table 1. (It should be pointed out that $D = 3L$ was used in the calculation for reasons to be explained elsewhere.) The whole number in the value of $(j + \phi/2\pi)$ so obtained is identified with the 'quantum number' characterizing the dip and the fraction with $(\phi/2\pi)$. It is clearly seen that the j values do differ by 3 corresponding to the fact that every third dip was chosen. Since the different curves correspond to different B and L values, the dependence on B and, in particular, on the distance L, is well borne out by the experimental results.

In summary, it is important to highlight the following facts that we have demonstrated:

a) The discrete forbidden states of motion do exist in the domain of parameters where one would use classical equation of motion to determine the motion.
b) The energies of these 'forbidden states' are well represented by the relation (5) which is obviously nonquantal as there is no Planck quantum h appearing in it.
c) The forbidden states E_j form a hydrogen-like sequence for which 'quantum numbers' j and the phases can be identified as in table 1.
d) The forbidden states E_j and the associated quantum numbers j depend on the distance between the gun and the detector. This is a manifestation of wave-like behaviour which is not known to be a characteristic of the standard initial value paradigm of classical mechanics.

Based on the above described experimental results, one may now conclude that the elèctrons moving in a magnetic field do appear to exhibit a wave-like behaviour in macroscopic dimension which is known to be a domain of operation of classical dynamics and which, therefore, admits a continuum of energy states. The existence of discrete forbidden states as described above appears to be in contradiction with the latter.

6. Implication of these results

To summarize, it has been predicted theoretically and demonstrated experimentally, that electrons moving along a magnetic field within the classical mechanical domain of parameters, exhibit the existence of discrete forbidden states, a feature which is not admitted by the standard initial value paradigm of classical dynamics. The latter admits the entire continuum of initial conditions as allowed states. This is an entirely new feature of classical dynamics, not hitherto pointed out to the best of the author's knowledge. What is most astonishing and violative of the classical mechanical intuition is that the forbidden (as also the allowed) states are determined nonlocally by the distant boundaries. The allowed states may well evolve à la the Lorentz equation of motion. The new description à la the Schrödinger-like equation merely constrains the allowed initial data.

The Schrödinger-like formalism may thus be regarded as providing a global description of the classical mechanical system as it takes cognizance of the boundaries which the standard paradigm does not. It is of course, interesting and significant that a classical mechanical system, such as the one under consideration, admits a Schrödinger wave-like description, so far considered as the sole preserve of quantum mechanics. Of course, not every classical mechanical system may be so describable. We have not been able to identify general criteria for such systems.

We have thus shown how the generalized Liouville equation for the coherent system gives the Schrödinger formalism (generalized) as its Hilbert space representation provided the Liouville density and the square-root thereof is periodic in the action phase $\Phi = S/\hbar$. The classical Liouville equation is obtained as an action-phase average of the generalized Liouville equation. It is in this manner that classical mechanics is contained in this generalized formalism. It will be seen that formally the limit $\hbar \to 0$ (which is usually identified as the classical limit) also amounts, in the lowest order, to an averaging over Φ.

The question that is most likely to be raised is that of the validity and the validation of the formalism obtained and the essential steps and arguments leading to it. Such a validation of the formalism has been provided in a rather spectacular fashion.

As discussed earlier the main thrust of our investigation was to provide a formalism which could cover both classical and quantum mechanics in a more natural and logically connected manner. The same formalism applied to an appropriate classical mechanical system (charged particle dynamics in a magnetic field) has yielded a Schrödinger-like description whose quantum-like predictions have been experimentally verified in a rather spectacular fashion. These are astonishing observations in as much as they cannot be understood in terms of the standard classical mechanical paradigm. Our line of investigation has thus helped unearth entirely new physical effects in the classical mechanical domain of parameters. More than just that, they seem to point to a new paradigm in classical mechanics whereby not the whole of the continuum of initial values are allowed values, just like in the quantum domain.

The question that presents itself is that what is it that leads a classical mechanical system, to the discreteness of the allowed states as against the continuum as per the standard paradigm. It has been shown by the author [13] recently that the topological considerations of the system configuration space leads to conditions which constrain the allowed initial values to a discrete set. These conditions turn out to be nothing but the eigenvalue equations in the EBK form of the Schrödinger-like equations (17). It is conjectured that the same that is, the topology, may also be the origin of discreteness of allowed energy values in quantum mechanics.

These investigations thus provide another new point of view in terms of which to consider the question of classical-quantum relationship: Quantum mechanics as a Hilbert space representation of the (generalized) classical Liouville equation may thus be considered as a global description of the same dynamics, with the boundary conditions simply helping to determine the wave functions and corresponding discrete energies as eigenvalues permitted by the overall topology of the system. This in turn translates into an appropriate Liouville density function being determined by the boundary conditions.

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