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Quasi-invariants and generalized Killing vectors-II

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Abstract. We consider here the problem of the existence of a quasi-invariant which is linear in the momenta for Hamiltonians in three degrees of freedom. We show that such quasi-invariants are more constrained in their structure than in the two degrees of freedom case. We also show that some of these quasi-invariants have to be interpreted as 'pseudo-translations', i.e., as translations in a non-orthogonal system of coordinates.

Keywords. Invariants; Killing vectors; Jacobi metric.

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1. Introduction

In a previous paper (Sitaram and Varma 1984) we had shown the connection between quasi-invariants and generalized Killing vectors using the Jacobi metric. We recollect here the general definition of a quasi-invariant:

Definition: Let $L = L(x^i, \dot{x}^i)$ be the Lagrangian of a conservative classical mechanical (CM) system with n degrees of freedom. A function $I(x^i, \dot{x}^i)$ is said to be a quasi-invariant of the system at the energy E if $\mathrm{d}I/\mathrm{d}t = 0$ on the hypersurface $H = E$.

Such a quasi-invariant is a special case of what has been called in the literature as a "configurational invariant" (Sarlet *et al* 1985). The most natural way of dealing with quasi-invariants is through the Jacobi metric.

Definition: Let $L = a_{ij}\dot{x}^i\dot{x}^j - V(x^i)$. Then the associated Jacobi metric at the energy E is defined by

$$\mathrm{d}s^2 = g_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j, \quad g_{ij} = (E - V)a_{ij}. \tag{1}$$

It is well known that the Euler-Lagrange equations given by L are completely equivalent to the geodesic equations on the Jacobi metric. In this language, as we have shown, a quasi-invariant for the CM system is equivalent to a generalized Killing vector field (generalized in the sense that the Killing vector field depends in general on both the position, x^i, and the velocities, $\mathrm{d}x^i/\mathrm{d}s$). For the special case of a quasi-invariant linear in the velocities, the generalized Killing vector field is just an ordinary Killing vector field.

In this paper, we wish to consider the conditions for the existence of quasi-invariants linear in the velocities for the Lagrangian

$$L = \frac{\dot{x}^2}{2} + \frac{\dot{y}^2}{2} + \frac{\dot{z}^2}{2} - V(x, y, z). \tag{2}$$

The corresponding Jacobi metric is $g_{ij} = g\delta_{ij}$, $g = (E - V)$.

2. The Killing equations and their solutions

The Killing equations (Eisenhart 1966) can be readily seen to yield

$$\xi^1 \frac{\partial g}{\partial x} + \xi^2 \frac{\partial g}{\partial y} + \xi^3 \frac{\partial g}{\partial z} + g \frac{\partial \xi^1}{\partial x} = 0, \tag{3a}$$

$$\xi^1 \frac{\partial g}{\partial x} + \xi^2 \frac{\partial g}{\partial y} + \xi^3 \frac{\partial g}{\partial z} + g \frac{\partial \xi^2}{\partial y} = 0, \tag{3b}$$

$$\xi^1 \frac{\partial g}{\partial x} + \xi^2 \frac{\partial g}{\partial y} + \xi^3 \frac{\partial g}{\partial z} + g \frac{\partial \xi^3}{\partial z} = 0, \tag{3c}$$

$$\frac{\partial \xi^1}{\partial y} + \frac{\partial \xi^2}{\partial x} = 0, \quad \frac{\partial \xi^1}{\partial z} + \frac{\partial \xi^3}{\partial x} = 0, \quad \frac{\partial \xi^2}{\partial z} + \frac{\partial \xi^3}{\partial y} = 0, \tag{3d}$$

where the Killing vector field is assumed to have components (ξ^1, ξ^2, ξ^3) and where the quasi-invariant has the form

$$I = \xi_1 \frac{\mathrm{d}x}{\mathrm{d}s} + \xi_2 \frac{\mathrm{d}y}{\mathrm{d}s} + \xi_3 \frac{\mathrm{d}z}{\mathrm{d}s}.$$

It readily follows from the above equations that

$$\frac{\partial \xi^1}{\partial x} = \frac{\partial \xi^2}{\partial y} = \frac{\partial \xi^3}{\partial z}. \tag{3e}$$

From (3d) and (3e), it follows that

$$\frac{\partial^2 \xi^1}{\partial x^2} + \frac{\partial^2 \xi^1}{\partial y^2} = \frac{\partial^2 \xi^1}{\partial x^2} + \frac{\partial^2 \xi^1}{\partial z^2} = 0. \tag{3f}$$

with similar equations for ξ^2 and ξ^3.

The most general solution of (3f) is

$$\xi^1 = f_1(x + iy + iz) + f_2(x + iy - iz) + f_3(x - iy - iz) + f_4(x - iy + iz),$$

where f_1, f_2, f_3 and f_4 are arbitrary functions of their arguments. (Strictly speaking, we should also insist that ξ^1 is real, which yields conditions relating the f's, but this will not be necessary for what follows). It is then easy to see, using (3d) that

$$\xi^2 = -i(f_1 + f_2 - f_3 - f_4) + G_1(z),$$

$$\xi^3 = -i(f_1 - f_2 - f_3 + f_4) + G_2(y),$$

where we have omitted the arguments of functions $f_1 \ldots f_4$ and where we have taken G_1 and G_2 to be functions respectively of z and y alone. This choice then easily satisfies conditions (3e) also. These functions are then determined using the conditions $(\partial \xi^2/\partial z)$

$+(\partial\xi^3/\partial y)=0$ (last of (3d)). This yields

$$\frac{dG_1}{dz}+\frac{dG_2}{dy}+2[\dot{f}_1-\dot{f}_2+\dot{f}_3-\dot{f}_4]=0, \tag{4a}$$

where the dot stands for differentiation w.r.t. the argument of the functions $f_1 \ldots f_4$. We immediately deduce that

$$\frac{d^2G_1}{dz^2}+2i[\ddot{f}_1+\ddot{f}_2-\ddot{f}_3-\ddot{f}_4]=0, \tag{4b}$$

$$\frac{d^2G_2}{dy^2}+2i[\ddot{f}_1-\ddot{f}_2-\ddot{f}_3+\ddot{f}_4]=0. \tag{4c}$$

We now use the following: (a) the second derivative of (4a) w.r.t. x; (b) the derivatives of (4b) w.r.t. x and y; (c) the derivatives of (4c) w.r.t. x and z. Simple algebra yields

$$\dddot{f}_1=\dddot{f}_2=\dddot{f}_3=\dddot{f}_4=0,$$

which implies that $f_1 \ldots f_4$ are at most quadratic in their arguments and hence that ξ^1, ξ^2 and ξ^3 are at the most quadratic in x, y, z. It is then easy to show that the most general solution of (3d) and (3e) is

$$\xi^1=\frac{\alpha}{2}(x^2-y^2-z^2)+\beta xy+\gamma xz+\delta x+\varepsilon y+\mu z+\sigma,$$

$$\xi^2=\frac{\beta}{2}(y^2-x^2-z^2)+\alpha xy+\gamma yz-\varepsilon x+\delta y+\nu z+\tau,$$

$$\xi^3=\frac{\gamma}{2}(z^2-x^2-y^2)+\alpha xy+\beta yz-\mu x-\nu y+\delta z+\rho, \tag{5}$$

where α, β, γ, δ, ε, μ, ν, σ and τ are arbitrary real parameters.

This result already shows the difference between the $n=2$ case and the $n=3$ case. Recall that $n=2$, the components ξ^1 and ξ^2 of the Killing vector field were just constrained to be conjugate solutions of Laplace's equations in two dimensions, or equivalently by the fact that $\xi^1+i\xi^2$ was an analytical function of $x+iy$. On the other hand, here, we have three considerations: (1) $\xi^1+i\xi^2$ analytic in $x+iy$, (2) $\xi^1+i\xi^3$ analytic in $x+iz$ and (3) $\xi^2+i\xi^3$ analytic in $y+iz$. While the first two conditions are similar to the $n=2$ case, the third condition is extremely restrictive and yields the above solutions as the most general solution. We show below that such restrictions would play a similar role in the $n>3$ case also (cf. §4).

Another result of the restrictive nature of the Killing equations is in the simultaneous existence of two independent Killing vectors. Assume that one of the Killing vectors, ξ say, has a quadratic dependence on x, y, z (e.g. $\alpha=1$, all other constants zero in (5)). It can be shown then, that the other Killing vector, say η, cannot have a quadratic dependence on x, y, z. For, if this were so, the commutator of ξ and η, which should also be a Killing vector field, would have a cubic dependence on x, y, z, which is not permitted. This result would also be generalizable to $n>3$.

3. The conditions on g

Having obtained the general solution for $\xi = (\xi^1, \xi^2, \xi^3)$, it is now necessary to solve the last of the Killing equations,

$$\xi^1 \frac{\partial g}{\partial x} + \xi^2 \frac{\partial g}{\partial y} + \xi^3 \frac{\partial g}{\partial z} + g \frac{\partial \xi^1}{\partial x} = 0. \tag{6}$$

The easiest way to solve this equation is to first make a coordinate transformation $(x, y, z) \to (X, Y, Z)$, where (X, Y, Z), are some functions of x, y, z, such that in the new coordinate system, the vector has coordinates $(1, 0, 0)$. Such a step can always be carried out for any ξ (Eisenhart 1966). In contrast to the $n = 2$ case, it is not possible to write the general solution for (X, Y, Z) in terms of (x, y, z), but the transformation is trivial to carry out in a few special cases where the transformation generated by the quasi-invariant is easy to interpret physically:

(a) σ, τ, ρ non-zero: generates translations along a straight line in the x, y, z space; g has to be independent of the coordinate which measures distances along this line.

(b) ε, μ, ν non-zero: generates rotations along some axis in x, y, z space; g has to be independent of an angle coordinate orthogonal to the rotation axis.

(c) δ non-zero: generates uniform scaling along all three axes; g should depend only on scale independent quantities (e.g. $x^2/y^2 + z^2$, y/z, x^2/yz, etc.).

Combinations of these cases can be discussed in a like manner. For the general case, we proceed as follows: The requirement that in the new coordinates, ξ should have the components $(1, 0, 0)$ leads directly to the differential equations:

$$\boldsymbol{\xi} \cdot \nabla X = 1, \quad \boldsymbol{\xi} \cdot \nabla Y = 0, \quad \boldsymbol{\xi} \cdot \nabla Z = 0. \tag{7}$$

We can distinguish between two cases: (a) $\boldsymbol{\xi} \cdot \nabla \times \boldsymbol{\xi} = 0$ and (b) $\boldsymbol{\xi} \cdot \nabla \times \boldsymbol{\xi} \neq 0$. The condition $\boldsymbol{\xi} \cdot \nabla \times \boldsymbol{\xi} = 0$ can be written in terms of the various parameters appearing in (5) as

$$\begin{aligned} &\nu\alpha + \gamma\varepsilon - \mu\beta = 0; \quad \delta\nu + \rho\beta - \tau\gamma = 0; \quad \mu\delta + \rho\alpha - \sigma\gamma = 0; \\ &\delta\varepsilon - \sigma\beta + \tau\alpha = 0; \quad \rho\varepsilon - \mu\tau + \sigma\nu = 0. \end{aligned} \tag{8}$$

Case (a): When these conditions are satisfied, it is easy to see that the following system of equations is integrable and defines a solution of (7):

$$\nabla X = \frac{\boldsymbol{\xi}}{|\xi|^2}, \quad \nabla Y = \frac{\nabla \times \boldsymbol{\xi}}{|\nabla \times \xi|^2}. \tag{9}$$

Z can then be easily determined to be a solution of the equation

$$\nabla Z = G(x, y, z) \cdot \boldsymbol{\xi} \times (\nabla \times \boldsymbol{\xi}), \tag{10}$$

where G is a solution of the equation

$$(\nabla G) \times [\boldsymbol{\xi} \times (\nabla \times \boldsymbol{\xi})] = -G\{\nabla \times [\boldsymbol{\xi} \times (\nabla \times \boldsymbol{\xi})]\}. \tag{11}$$

An alternative way of proceeding is to use the fact that the surface $Y =$ constant is a solution of the equation

$$dx/\xi^1 = dy/\xi^2 = dz/\xi^3, \tag{12}$$

which is the characteristic equation of the equation $\xi \cdot \nabla Y = 0$ and then use Jacobi's last multiplier theorem (Whittaker 1944) to get Z, the other solution of the characteristic equation. It can be shown that it is possible to choose Z such that $\nabla X \cdot \nabla Z = 0$, $\nabla Y \cdot \nabla Z = 0$. Note also that, by construction, $\nabla X \cdot \nabla Y = 0$.

As an example, assume that $\alpha = 2$, all other parameters $= 0$ in (5). Then one set of coordinates (X, Y, Z) which satisfies (7) is given by

$$X = -\frac{x}{x^2+y^2+z^2}, \quad Y = \frac{x^2+y^2+z^2}{(y^2+z^2)^{1/2}}, \quad Z = y/z. \tag{13}$$

Transforming to X, Y, Z coordinates, we get,

$$H = \frac{1}{2}[(|\nabla X|)(|\nabla Y|)(|\nabla Z|)]\left[\frac{\dot{X}^2}{|\nabla X|^2} + \frac{\dot{Y}^2}{|\nabla Y|^2} + \frac{\dot{Z}^2}{|\nabla Z|^2}\right] + V(X, Y, Z)$$

with the corresponding Jacobi metric given by (14)

$$g_{11} = g\frac{|\nabla Y||\nabla Z|}{|\nabla X|}, \quad g_{22} = g\frac{|\nabla X||\nabla Z|}{|\nabla Y|}, \quad g_{33} = g\frac{|\nabla X||\nabla Y|}{|\nabla Z|}. \tag{15}$$

Using the fact that in these coordinates, ξ has components $(1, 0, 0)$ leads to the conditions

$$\frac{\partial g_{11}}{\partial X} = \frac{\partial g_{22}}{\partial X} = \frac{\partial g_{33}}{\partial X} = 0. \tag{16}$$

If we define $\tilde{g} = g\,(|\nabla X| \cdot |\nabla Y| \cdot |\nabla Z|)$, then these equations require

$$\frac{\partial}{\partial X}\left(\frac{\tilde{g}}{|\nabla X|^2}\right) = \frac{\partial}{\partial X}\left(\frac{\tilde{g}}{|\nabla Y|^2}\right) = \frac{\partial}{\partial X}\left(\frac{\tilde{g}}{|\nabla Z|^2}\right) \tag{17}$$

and hence that

$$\frac{\partial}{\partial X}\ln|\nabla X|^2 = \frac{\partial}{\partial X}\ln|\nabla Y|^2 = \frac{\partial}{\partial X}\ln|\nabla Z|^2, \tag{18}$$

which can be easily verified to be true. The solution of (17) can then be written as $\tilde{g} = |\nabla X|^2 \cdot F(Y, Z)$, where F is an arbitrary function of its arguments and hence,

$$g = E - V = F(Y, Z) \cdot \frac{|\nabla X|}{|\nabla Y||\nabla Z|}. \tag{19}$$

It is worthwhile remarking here that using, e.g.,

$$g = E - V = F(X, Z) \cdot \frac{|\nabla Y|}{|\nabla X||\nabla Z|}$$

does not lead to a Killing vector field, as the conditions

$$\frac{\partial}{\partial Y}\ln|\nabla X|^2=\frac{\partial}{\partial Y}\ln|\nabla Y|^2=\frac{\partial}{\partial Y}\ln|\nabla Z|^2$$

are not satisfied. This is similar to the case of polar coordinates in $n=2$, where a potential independent of r does not lead to an invariant.
case (b): When $\xi\cdot\nabla\times\xi\neq 0$, we come across a feature which is not to be found in the $n=2$ case. In general, for case (b), it is not possible to find an orthogonal system of coordinates X, Y, Z satisfying (7). Take for example the Killing vector defined by

$$\xi=(x, y+z, z-y).$$

One choice of coordinates satisfying (7) can be easily seen to be

$$X=\ln x,\quad Y=\frac{\exp(2\tan^{-1}y/z)}{y^2+z^2},\quad Z=x^2/(y^2+z^2).$$

It is quite straightforward to verify that while $\nabla Y\cdot\nabla Z=0$, $\nabla X\cdot\nabla Y\neq 0$ and $\nabla X\cdot\nabla Z\neq 0$. It is worth noting here that X is uniquely defined by (7), as is the plane containing Y and Z. We have the freedom to choose Y, Z to be mutually orthogonal, i.e. $\nabla Y\cdot\nabla Z=0$, as has been done above, but there is no way of choosing X to be orthogonal to the $Y-Z$ plane and still have the property of satisfying (7). In this non-orthogonal coordinate system defined by (X, Y, Z), the Killing vector generates a "pseudo-translation" along the X-axis. The fact that this transformation is a pseudo-translation can be seen by considering any transformation acting in a plane orthogonal to the X-axis, e.g., a rotation around the X-axis. In contrast to the usual case, it is quite clear that the pseudo-translation fails to commute with the rotation.

4. Generalization for $n>2$

The major results of the previous sections are, firstly, the severe constraints on the form of the quasi-invariants linear in the momenta, in contrast to the $n=2$ case, and, secondly, the existence of a new class of translations, i.e., the pseudo-translations considered in the previous section. Both these results generalize to the $n=3$ case as follows:

Let $$L=\sum_{i=1}^{n}\frac{(\dot{x}^i)^2}{2}-V(x^1,\ldots,x^n).$$

It is straightforward to show that the Killing equations imply the following set of equations:

$$\frac{\partial\xi^i}{\partial x^j}+\frac{\partial\xi^j}{\partial x^i}=0,\quad i\neq j, \tag{20a}$$

$$\frac{\partial\xi^i}{\partial x^i}=\frac{\partial\xi^j}{\partial x^j}\quad\text{(no summation)}, \tag{20b}$$

$$\xi\cdot\nabla g+\frac{g}{n}(\nabla\cdot\xi)=0, \tag{20c}$$

where $g = (E - V)$. Equations (20a, b), for any triple i, j, k ($i \neq j \neq k$) coincide with (3d)–(3e) and hence the analysis following equations (3) applies, *mutatis mutandis*, to the above equations, leading to the conclusion that each component of ξ is at the most quadratic in each of the x^i. It is also possible to write down the most general solutions of equations (20a, b) as follows, where the summation convention has been temporarily dropped:

$$\xi^i = \frac{\alpha_i}{2}[(x^i)^2 - \Sigma'(x^j)^2] + x^i\Sigma'\alpha_j x^j + \beta x^i + \varepsilon_{ij}x^j + \delta^i,$$

where α_i, β, $\varepsilon_{ij} = -\varepsilon_{ji}$, δ are arbitrary constants, and where Σ' stands for a summation over j, with the $j = i$ term dropped.

Given such a ξ, we now investigate the conditions under which there exists an orthogonal coordinate system (X^i) such that the components of ξ become $(1, 0, \ldots, 0)$. This condition is easily seen to be $\nabla X^1 = \xi/|\xi|^2$. If we define a 1-form η on the space defined by the (X^i) by $\eta = \xi^i \mathrm{d}x^i$, then this condition can be written as $\mathrm{d}X' = \eta/|\xi|^2$ which implies that $\mathrm{d}\eta = \eta \wedge \mathrm{d}(\ln|\xi|^2)$ and hence that $\eta \wedge \mathrm{d}\eta = 0$. This last condition is not always satisfied by ξ, as for example for $\eta = x^i\mathrm{d}x^i + x^3\mathrm{d}x^2 - x^2\mathrm{d}x^3$, for which $\eta \wedge \mathrm{d}\eta = 2x^i\mathrm{d}x^i \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 \neq 0$, showing the existence of pseudo-translations for all $n > 2$. Note that this analysis is valid for all n; for $n = 2$, $\eta \wedge \mathrm{d}\eta$ is always 0, while for $n = 3$, the condition $\eta \wedge \mathrm{d}\eta = 0$ is equivalent to the condition $\xi \cdot (\nabla \times \xi) = 0$.

5. Conclusions

As noted in our earlier work (Sitaram and Varma 1984), CM systems, while being non-integrable in the sense of absence of sufficient number of integrals of motion, could still be quasi-integrable in the sense that for a given set of initial conditions (e.g., those defined by a given value of the energy), the equations of motion are integrable (i.e. reducible to quadrature). A change in the set of initial conditions, (e.g. a change in the energy) however, could result in the destruction of the quasi-invariants, leading to the possibility of chaos. This leads us to a picture of transition to chaos in CM systems as being a series of energies where the system is quasi-integrable, followed by regions of chaos due to the destruction of the quasi-invariants. As shown in our earlier work, it is easy to construct examples of the CM systems which are quasi-integrable for some energy in the case $n = 2$ degrees of freedom. For $n > 2$, it is rather difficult to show quasi-integrability except in special cases, as in general, it is necessary to construct $n - 1$ quasi-invariants. However, the existence of even a single quasi-invariant for $n > 2$ is likely to make a qualitative change in the dynamics, as the allowed phase space is smaller than in the general case. Note that in the general case (i.e. where there do not exist any quasi-invariants), the allowed phase space is $2n - 1$ dimensional, while the existence of a m quasi-invariants brings down the allowed phase space dimension to $2n - m - 1$. For $m = n$, a theorem due to Sarlet *et al* (1985) shows that the allowed phase space dimension is in fact one-dimensional.

It is possible, in principle, to extend our analysis to the case of quasi-invariants of order > 1 in the momenta for $n > 1$. (The case of second order invariants for $n = 2$ has already been discussed in our earlier paper). We see two problems with such an extension:

(a) The equations become rapidly intractable. (Note, however, in this connection the work of Holt (1982) and Kaushal *et al* (1985), where some invariants of order 3 are discussed. Both the works confine themselves to the case of invariants and do not discuss quasi-invariants).

(b) For studying quasi-invariants of order $\geqslant 2$, it is advantageous to use a method which allows the full use of canonical transformations to simplify the equations. In fact, we know that while the first order quasi-invariants generate coordinate transformations, higher order invariants generate non-trivial canonical transformations (non-trivial in the sense that the transformations mix up coordinates and momenta). Our approach, being essentially Lagrangian in nature, does not permit the natural use of such canonical transformations and hence it is not possible to simplify the quasi-invariants sufficiently so as to solve the generalized Killing equations.

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