A MODEL OF THE SUNSPOT UMBRA*

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(Received 22 March, 1967)

Abstract. Equations governing the structure of the umbra of a single spot have been integrated on the spot-axis. It is shown that a consistent umbral model can be obtained only for a narrow range of the electron pressures at the spot surface. The spot center is found to be at a depth of about 400 km below the normal solar surface. The reduced energy flux observed at the surface is assumed to be due to the effect of the spot-magnetic field on subphotospheric convection and an empirical factor β is introduced to take into account the reduction in the convective energy flux. With an inferred expression for β as a function of the internal and magnetic-energy density it is shown that in a consistent model the physical variables in the spot approach their ambient values at about 2330 km below the undisturbed solar surface and at the same depth the energy flux approaches the normal solar value and the magnetic interference with convection vanishes. The present investigation is a refinement of an earlier paper by CHITRE (1963).

1. Introduction

Any theory of the structure of a sunspot must involve the effect of the spot-magnetic field on subphotospheric convection. As emphasized in the work of CHITRE (1963) and DEINZER (1965), the problem is to show that the magnetic field, which supplies the lateral force required to maintain the spot in equilibrium with a reduced internal thermal pressure, is also the cause of this reduced pressure arising from its effect on the transport of convective energy. It is now well established that the darkness of a spot must be due to the magnetic interference with the convective transport of heat. It was BIERMANN (1941) who first suggested that the magnetic field inhibits convection in the hydrogen-convection zone below a sunspot, thus causing the cooling. On the other hand, HOYLE (1949) assumed that convection is not altogether stopped, but is channeled to motions along the lines of forces. As a result of the lateral pressure, $H^2/8\pi$, the spot tube has a larger cross-section at the surface than in the deeper layers, and hence the energy coming from underneath and following the field lines gets spread over a larger area as the surface is approached; and this causes the reduction in the flux.

In Chitre's work both these suggestions were combined, and a consistent umbral model was produced which took into account the inhibiting effect as well as the channeling effect. Anticipating that well below the surface the ratio of the magnetic energy to the internal energy would be a small fraction, it was assumed that the magnetic field is not so efficient in interfering with convection in the deeper layers as near the surface where the magnetic energy exceeds the internal energy by about an order of magnitude. In the absence of an adequate theory for the extent to which

* Supported in part by the Office of Naval Research [Nonr-220(47)] and the National Science Foundation [GP-5391].

Solar Physics **2** (1967) 150–157; © *D. Reidel Publishing Company, Dordrecht-Holland*

the magnetic field interferes with convection, an empirical factor β ($0 \leqslant \beta \leqslant 1$) was introduced in the expression for the convective flux, and a consistent model was produced by increasing the flux from its reduced observed value at the surface to the undisturbed solar value and also by letting β increase from its substantially reduced value at the surface to unity as the base is approached. The base of the spot was defined as the layer where the disturbance to the normal outflow of energy is negligible and where the physical quantities in the spot approach their ambient values. However, a somewhat crude procedure was adopted in obtaining this model: a number of integrations were performed with a series of fluxes with a range of β's associated with each of them, and a consistent model was produced by fitting one solution across with the other by maintaining the continuity of the physical variables such as the temperature, the pressure, etc. In other words, by replacing the continuously varying flux with a series of fluxes and by associating a value of β between 0 and 1 with each of these, it was possible to get a reasonable model which has the flux and β increasing with depth from their reduced values at the surface.

The present investigation is intended to improve this situation. We have assumed for β a reasonable monotonic function involving the magnetic energy and internal energy, and the governing equations are integrated by employing the equation of lateral balance and by varying the energy flux proportional to the magnetic field. We have thus two assumptions built in our model, namely, the flux and β increase continuously from their reduced values at the surface to their undisturbed values (i.e., $F \to F_\odot$ and $\beta \to 1$) as the base is approached, and that at the base

$$\frac{\Delta T}{T} \lesssim 0.1 \quad \text{and} \quad \frac{\Delta P_e}{P_e} \lesssim 0.1\,,$$

where Δ denotes the difference between the value of a physical variable at the same depth in the undisturbed layers and in the spot. Of course, the possibility remains that the ratio of the magnetic and internal energy might increase with depth and the magnetic field will resist any tangling up by turbulence in the deeper layers (MESTEL, 1963; WENTZEL, 1961), in which case β may not necessarily increase with depth. Such a behavior seems rather unlikely and at any rate our model did not give any consistent solution with β decreasing with depth.

2. Governing Equations

In the usual notation we have for hydrostatic equilibrium

$$\frac{\mathrm{d}P}{\mathrm{d}h} = g\rho$$

where h is measured downwards from the solar surface, and the temperature gradient is given by

$$\frac{\mathrm{d}T}{\mathrm{d}h} = \frac{3\kappa\rho}{4acT^3}(F - C),$$

where F is the total flux and C is the flux carried by convection. For the purpose of computing the flux carried by convection we have made use of the model given by FAULKNER *et al.* (1965) and integrated their equations first for the undisturbed solar surface layers specifying as boundary conditions the chemical composition, the mass, the radius, and the flux; we have always regarded the spot-structure against the background of the undisturbed solar layers.

We have assumed that the convective flux is tied to the magnetic field and follows the lines of force. Considering then cylindrical symmetry with the z-axis along the spot-axis, we have for conservation of the convective and magnetic flux through a ring $(r, r+\mathrm{d}r)$ at a depth z, which emerges through a ring $(r_s, r_s+\mathrm{d}r_s)$ at the surface (CHITRE, 1963)

$$F_z\, 2\pi r\, \mathrm{d}r = (F_z)_s\, 2\pi r_s\, \mathrm{d}r_s\,,$$
$$H_z\, 2\pi r\, \mathrm{d}r = (H_z)_s\, 2\pi r_s\, \mathrm{d}r_s\,,$$

which, when combined, give

$$\frac{F_z}{H_z} = \frac{(F_z)_{\text{surface}}}{(H_z)_{\text{surface}}}$$

To the foregoing set of equations must be added the equation of lateral balance:

$$(P_g)_{\text{spot}} + H^2/8\pi = (P_g)_{\text{sun}}\,,$$

ignoring the curvature of the field lines. The quasi-equilibrium of a spot results from the magnetic interference with convection, which keeps the internal thermal pressure lower and thus allows the condition of lateral balance to be satisfied. We shall use this equation to compute H at every step; the value of H so obtained would naturally be only approximate, but it serves as a good guide to get an estimate of H with depth.

In the absence of any magnetic field there is nothing to impede the transport of energy, and we expect that maximum efficiency of convection will be achieved. However, the presence of a strong magnetic field, such as is present in a spot, will prevent any non-oscillatory motions, and it is reasonable to expect that convection will not be as efficient in the presence of a magnetic field as it would be in its absence. The main difficulty is that we do not know the exact extent to which the convective flux is reduced by the magnetic field. We have therefore adopted an empirical approach by introducing an efficiency factor β $(0 \leqslant \beta \leqslant 1)$ in the expression for C in the absence of the field, and for β we have assumed a monotonic increasing function depending on the magnetic-energy density, E_{mag} and the internal-energy density, E_{int} (which consists of the thermal energy and the ionization-energy density). It turns out that only a certain narrow range of functional dependences of β on E_{mag} and E_{int} yields plausible results. We shall discuss this in detail later.

Our integration proceeds as follows: after fixing the surface values at $\tau = \frac{2}{3}$ the first integration step is performed by assuming the reduced flux and the magnetic field observed at the spot surface. The gas pressure obtained at the end of this step is

compared with the corresponding pressure in the undisturbed sun at the same level and the magnetic field is computed by using the equation

$$H^2/8\pi = (P_g)_{\text{sun}} - (P_g)_{\text{spot}}.$$

The flux is then calculated by employing the equation

$$\frac{F}{H} = \frac{(F)_{\text{surface}}}{(H)_{\text{surface}}},$$

and the next integration step is performed with these new values of H and F, and the procedure is carried till the layer where the flux reaches the undisturbed value, $F_\odot$ and β (which is built in the expression for C) approaches unity.

3. Surface Boundary Condition

While solving the boundary condition it has been assumed that the mass M, the radius R, the chemical composition and the flux F are specified. The effective temperature then follows from

$$F = \tfrac{1}{4} a c T_e^4.$$

For the spot surface we have assumed $F = 0.33\ F_\odot$ and the same chemical composition as the sun. T_e then comes out to be equal to 4500 °K. To obtain the electron pressure P_e at the surface ($\tau = \frac{2}{3}$) one could solve for the photosphere by assuming thermodynamic equilibrium (cf. CHITRE, 1963). However, as pointed out by MATTIG and SCHRÖTER (1964) the spot photosphere is probably not in thermodynamic equilibrium. Moreover, because of the low temperatures prevailing at the spot photosphere the conditions are rather sensitive to the exact metal abundance. We therefore started the surface calculation by guessing a value of P_e – for example, $P_e = 0.5$ dyne/cm^2. Once the electron pressure, temperature and chemical composition are known at the surface, the rest of the physical variables becomes determinate (cf. FAULKNER *et al.*, 1965). The governing equations were integrated inwards, and it was found that there was an extensive radiative zone extending to about 100 km below the surface where the density increased with temperature quite rapidly, and very shortly the gas pressure in the spot exceeded the corresponding pressure in the undisturbed layers well before the normal flux $F_\odot$ was reached, while because of the equation of lateral balance $(P_g)_{\text{spot}}$ must always be less than $(P_g)_{\text{sun}}$. Thus the surface condition $P_e = 0.5$ dyne/cm^2 or any P_e below this value would not be acceptable. If, on the other hand, we chose too high a P_e at the surface, say $P_e = 10$ dyne/cm^2, we found that the gas pressure at the spot-surface itself exceeded that in the undisturbed layers at the same level, which, in view of the lateral balance, is again not permissible. In this manner, we were able to narrow down the range of acceptable P_e's at the spot-surface to around $1.5^{-1.0}_{+3.0}$ dyne/cm^2. Our calculation does seem to specify the value 1.5 for P_e within a very narrow margin, and this value falls well within the scatter of the observed data for P_e (DE JAGER, 1959).

With $T=4500\,^{\circ}\mathrm{K}$, $P_e=1.5$ dyne/cm^2 and the same chemical composition as the sun, the gas pressure at the spot-surface comes out to be (2.09) 10^4 dyne/cm^2. If we add to this the magnetic pressure corresponding to a field of 3550 gauss at the spot-center, we get

$$(P_g)_{\mathrm{spot}} + H^2/8\pi = (5.21)10^5\,\mathrm{dyne/cm}^2 = (P_g)_{\mathrm{sun}},$$

and this gas pressure is attained at a depth of 400 km below the undisturbed solar surface. This immediately gives a depression of the order of 400 km at the spot-center. For different values of P_e the depression will vary accordingly. However, for sufficiently low P_e's, as in the present situation, the depression would be insensitive to P_e and should not differ appreciably from 400 km. Our result is consistent with the argument by COWLING (1953) that in the cooler region below a spot, the density gradient must fall off with increasing height faster than in the surrounding region, and as a consequence the density that is needed for the escape of radiation from the spot-surface will be reached at a lower level than the undisturbed solar surface and there would be a depression at the spot surface.

4. Discussion

It is important to realise that while comparing the physical quantities at a given level in the spot with the corresponding quantities in the undisturbed layers at the same level, we must adjust the height scales by taking into account the depression at the spot-surface. It is in this connection that the question of adjustment of the height-scale of a spot relative to the photosphere is of considerable importance. Once the surface conditions are obtained the governing equations have been integrated by assuming for β a reasonable monotonic increasing function, and at each step we have computed H in accordance with the equation of lateral balance. Care has to be exercised while computing H from the equation $H^2/8\pi=(P_g)_{\mathrm{sun}}-(P_g)_{\mathrm{spot}}$. In the main body of the programme we had stored the run of the gas pressure with depth in the undisturbed layers, and at each step we compared the gas pressure in the spot with the corresponding pressure in the sun at the same level. In order to do this we had to interpolate for the sun, and when H becomes a small difference between relatively large values of the gas pressures, one has to interpolate with caution to avoid any numerical instabilities from creeping in.

For lack of any theory we had to resort to an empirical approach to take into account the extent to which the magnetic field interferes with convection, and we introduced a function into the expression for the convective flux. Of course, had we known the exact reduction in the convective flux caused by the field, there would have been no need to adopt such an empirical procedure. We could then have deduced from the strength of the spot-magnetic field at the surface all the relevant properties of the spot-structure. As it is the processes of convective transfer in a compressible medium are not very adequately understood, much less their behavior under a constraint such as a magnetic field. We therefore attempted a series of monotonic func-

tions depending on E_{mag} and E_{int} all the time using the equation of lateral balance as a guide, and remembering that in a consistent model the energy flux should reach the undisturbed value $F_{\odot}$ and β should come close to unity as the base is approached. For certain functional β's it turned out that the integration yielded gas pressures in the spot which exceeded the corresponding pressures in the sun very quickly below the surface well before the base was reached, and this was unacceptable. For certain other β's even though the pressure march was acceptable, the flux reached $F_{\odot}$ well before β approached unity, or if β reached unity, the flux was nowhere near $F_{\odot}$. Thus, with the imposition of these two boundary conditions, namely, $F \to F_{\odot}$ and $\beta \to 1$, at the base, we found that the function

$$\beta = \tanh\left(\frac{E_{int}}{2\pi E_{mag}}\right)$$

yielded a consistent model. The question that immediately arises is about the uniqueness of this function. All we can say is that our model seems to indicate this functional dependence within rather a narrow range. Appreciable departures from this behavior of β do not yield a plausible model.

It would be too daring to suggest that the expression $\beta = \tanh\,(E_{int}/2\pi E_{mag})$ represents the exact degree to which the magnetic field reduces the convective flux. The expression should be taken more as a guide of the extent to which convection is affected by the presence of the field rather than an accurate estimate. It probably comes reasonably close to the exact expression because of the underlying assumption on which we have worked, namely β should be very small near the surface where the magnetic energy exceeds the internal energy by an order of magnitude or so, while in the deeper layers, because the ratio of the magnetic energy and internal energy becomes small, β should come close to unity and hence we would expect an hyperbolic tan type of behavior.

For the purpose of present computations, we have adopted the condition that near the base $F \to F_{\odot}$, $\beta \to 1$, and $\Delta T/T \lesssim 0.1$, $\Delta P_e/P_e \lesssim 0.1$. Of course, the gas pressure at the spot-base does not approach the corresponding pressure outside in the undisturbed layers, because there is a finite magnetic field present at the base and hence, as a result of the equation of lateral balance, there will always be a finite-pressure difference between the spot and the undisturbed layers. Rather, as we approach the base, the difference between the gas pressures in the sun and the spot becomes a small fraction of the gas pressure itself; in other words, the magnetic-energy density becomes much smaller than the internal-energy density and this ratio goes on decreasing with depth. The question remains, however: what happens to the magnetic field below the base? Perhaps the lines of force turn around and become horizontal and are no longer able to exert appreciable influence on the convective motions.

The variations of β, H, T, E_{int}/E_{mag}, $\log P_{sun}$, and $\log P_{spot}$ are exhibited in Figure 1, and Table I summarizes the march of the physical quantities with depth on the spot-axis.

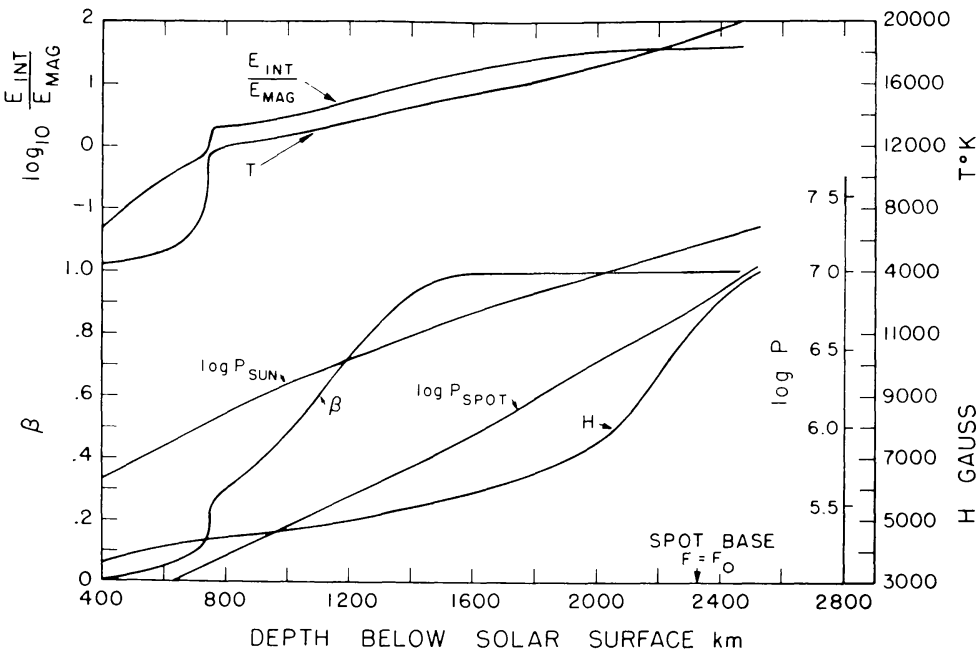

Fig. 1. The run of β, H, $E_{\text{int}}/E_{\text{mag}}$, T, $\log P_{\text{sun}}$ and $\log P_{\text{spot}}$ with depth on the spot axis. Observe the steep gradients at about 8000°–12000 °K due to hydrogen ionization.

TABLE I

Run of Physical Quantities in the Umbra

h (km)	P_g (dyne/cm^2)	T (°K)	H (gauss)	$F/F_\odot$	β
400	2.09×10^4	4500	3550	0.330	0.0099
444	3.22×10^4	4534	3696	0.343	0.014
570	1.05×10^5	4955	4077	0.378	0.038
638	1.86×10^5	5577	4194	0.389	0.063
732	3.60×10^5	8950	4272	0.396	0.134
826	5.36×10^5	12060	4433	0.411	0.325
1077	1.18×10^6	13179	4934	0.457	0.568
1260	1.97×10^6	13981	5030	0.466	0.792
1765	6.75×10^6	16208	6783	0.627	0.906
2216	1.68×10^7	18242	10111	0.934	0.985
2335	2.00×10^7	18790	11027	1.000	1.000

5. Conclusions

The base of the spot in our model is reached at about 2330 km, where the field strength comes out to be 11027 gauss. A spot is thus a shallow phenomenon not extending very deep in the subsurface layers. As argued by COWLING (1953), its sharp boundary and the mechanical unbalance resulting from a long column of cool material do tend to support this conclusion. The fact that the magnetic field is bounded between the two thermal pressures and is a small fraction of them suggests that pressure fluctuations that exist in convective regions are liable to induce fluctuations in the magnetic field and this will give rise to hydromagnetic waves.

One direct result of the spot coolness is the depression at its surface, of the order of 400 km. Such a depression has been observed by a number of workers, and is probably the cause of the foreshortening of the penumbra (the Wilson effect). On the model of convection adopted for the present investigation, the velocity field in the spot is found to be not very different from that in the sun.

It is possible to produce a consistent umbral model where the flux increases from its reduced value at the surface to the undisturbed value $F_{\odot}$ and β rises from 0.01 to unity as the base is approached. The expression $\beta = \tanh (E_{\text{int}}/2\pi\, E_{\text{mag}})$ seems to suggest the way in which the magnetic field affects the convective flux; the expression should, however, be taken more as an indication of this dependence on E_{int} and E_{mag} rather than the exact manner in which the convective processes are affected by the magnetic field.

In this investigation we have taken into account the variation of flux and β with depth, but we have ignored the magnetic stresses arising from the curvature of the field lines. In DEINZER's (1965) model the curvature of the lines of force was considered, while no account was taken of the variation of flux over the depth of the spot. Deinzer's argument to rule out the Hoyle mechanism is based on the assumption that there is a relationship between the observed flux and the magnetic-field intensity at the spot-surface. The recent observational evidence, however, does not support this contention. For relatively small spots, ZWAAN (1965) and MAKITA and MORIMOTO (1964) find that the stray light leaking from the sides rapidly increases with decreasing area, and consequently the decrease of the field strength with area may be in part spurious. On the other hand, for large spots the flux and the magnetic field are found to be practically constant, and hence Deinzer's argument relating the flux with the magnetic-field intensity is hardly sufficient to invalidate the Hoyle mechanism. In a more realistic model, one should take into account both the variation of flux and β with depth and also the curvature of the field lines through the equation of lateral balance.

Acknowledgement

It is a pleasure to express our grateful thanks to professors Zirin, Leighton, and Hoyle for helpful discussions.

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