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Vector bundles as direct images of line bundles

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Dedicated to the memory of Professor K G Ramanathan

Abstract. Let X be a smooth irreducible projective variety over an algebraically closed field K and E a vector bundle on X. We prove that, if $\dim X \geqslant 1$, there exist a smooth irreducible projective variety Z over K, a surjective separable morphism $f: Z \to X$ which is finite outside an algebraic subset of codimension $\geqslant 3$ in X and a line bundle L on X such that the direct image of L by f is isomorphic to E. When X is a curve, we show that Z, f, L can be so chosen that f is finite and the canonical map

$$H^1(Z, \mathcal{O}) \to H^1(X, \operatorname{End} E)$$

is surjective.

Keywords. Projective variety; algebraic vector bundle; line bundle; direct image; finite morphism.

1. Introduction

Let X be a smooth irreducible projective variety over an algebraically closed field and E a vector bundle on X. We prove in this paper first that, if $\dim X \geqslant 1$, E is the direct image of a line bundle L on a smooth irreducible projective variety Z by a morphism $f: Z \to X$ which is finite outside an algebraic subset of codimension $\geqslant 3$ in X. Moreover one can choose the morphism f to be separable and to have the property that all higher direct images of L by f are zero [Theorem 4.2].

In particular if $\dim X \leqslant 2$ the morphism f may be chosen to be finite. In the case of surfaces this result has been proved by R.L.E. Schwarzenberger for rank two vector bundles [5, Theorem 3]. We also give an example of a vector bundle on $\mathbf{P}_3$ which cannot be obtained as the direct image of a line bundle on a smooth variety by a finite morphism.

In the second part of the paper we consider the case when X is a curve. We prove in this case that Z, L and f can be so chosen that the canonical homomorphism (see 5.1)

$$H^1(Z, \mathcal{O}_Z) \to H^1(X, \operatorname{End} E)$$

is surjective (Theorem 6.4). This result was proved for a "very stable" vector bundle E by Beauville-Narasimhan-Ramanan in the case of a curve over $\mathbf{C}$ by using the Hitchin map [3]. (For the significance of this result see Remark 6.5).

Let $\pi:\mathbf{P}(E)\to X$ be the projective bundle associated to E. The variety Z is constructed as the subscheme (of $\mathbf{P}(E)$) of zeros of a generic section of the tangent bundle along the fibres of π twisted by a suitable ample line bundle on X pulled up to $\mathbf{P}(E)$; the line bundle L is simply taken to be the restriction of $\mathcal{O}_{\mathbf{P}(E)}(1)$ to Z.

The scheme Z is essentially the scheme of 'eigenstates' of a generic twisted endomorphism of E. However, in general, Z is not the spectral variety of the twisted endomorphism; the canonical map from the spectral variety into X is always a finite morphism.

2. Sections of the tangent bundle of a projective space

Let V be a finite dimensional vector space of dimension $\geqslant 2$ over an algebraically closed field K and $\mathbf{P}=\mathbf{P}(V)$ the projective space of hyperplanes in V. We have the exact sequences of vector bundles on $\mathbf{P}(E)$:

$$0\to\Omega^1(1)\to V_{\mathbf{P}}\to\mathcal{O}(1)\to 0$$

and

$$0\to\mathcal{O}_{\mathbf{P}}\to V^*_{\mathbf{P}}\otimes\mathcal{O}(1)\to\Theta\to 0,$$

where Θ denotes the tangent bundle of $\mathbf{P}$ and V^* the dual of V. We obtain an exact sequence of vector spaces

$$0\to K\to V^*\otimes V\to H^0(\mathbf{P},\Theta)\to 0.$$

If $\mathrm{End}^0(V):=\mathrm{End}(V)/(\text{Scalar endomorphisms})$ we have $\mathrm{End}^0(V)=H^0(\mathbf{P},\Theta)$.

If an endomorphism T of V leaves a hyperplane ξ invariant, T induces an endomorphism of the one dimensional space V/ξ. The subspace $(V/\xi)^*$ of V^* is an eigenspace of the transpose of T. If s_T is the section of Θ defined by T, we have $s_T(\xi)=0$ if and only if $T(\xi)\subset\xi$. Thus we can view the subscheme $Z=Z_T$ of zeros of s_T as the scheme of "eigenstates" of T. Moreover the "eigenvalue" of T is a section of $\mathcal{O}_Z$; in fact it is the section of $\mathcal{O}_Z$ corresponding to the morphism $\mathcal{O}_Z(1)\to\mathcal{O}_Z(1)$ induced by T from:

$$\begin{array}{ccccccc} 0\to\Omega^1(1)\to & V_{\mathbf{P}} & \to\mathcal{O}(1)\to 0 \\ & T\downarrow & \\ & V_{\mathbf{P}} & \to\mathcal{O}(1)\to 0. \end{array}$$

Observe that the scheme Z_T has dimension $\geqslant 1$ if and only if the transpose of T has an eigenspace of dimension $\geqslant 2$ corresponding to some eigenvalue.

PROPOSITION 2.1

Consider the exact sequence of vector bundles

$$0\to F\to\mathrm{End}^0(V)_{\mathbf{P}}\to\Theta\to 0$$

(F being defined as the kernel of homomorphism $End^0(V)_{\mathbf{P}}\to\Theta$). Let $p:F\to End^0(V)$ be the restriction to F of the projection $End^0(V)\times\mathbf{P}\to End^0(V)$. Then there exists an open subset Ω in $End^0(V)$, whose complement is of codimension $\geqslant 3$, such that the morphism $p:p^{-1}(\Omega)\to\Omega$ is finite.

Proof. Consider the commutative diagram

$$\begin{array}{ccccccccc} & & 0 & & 0 & & & & \\ & & \downarrow & & \downarrow & & & & \\ & & K_{\mathbf{P}} & & K_{\mathbf{P}} & & & & \\ & & \downarrow & & \downarrow & & & & \\ 0 & \to & F^1 & \to & \mathrm{End}(V)_{\mathbf{P}} & \to & \Theta & \to & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \to & F & \to & \mathrm{End}^0(V)_{\mathbf{P}} & \to & \Theta & \to & 0 \\ & & \downarrow & & \downarrow & & & & \\ & & 0 & & 0 & & & & \end{array}$$

Let $q: F^1 \to \mathrm{End}(V)$ be the projection. We shall show that there exists an open set U of $\mathrm{End}(V)$ which is saturated for the map $\mathrm{End}(V) \to \mathrm{End}^0(V)$ and whose complement is of codimension $\geqslant 3$ such that the morphism $q: q^{-1}(U) \to U$ is finite. This will prove the proposition.

For each subspace W of dimension $k \geqslant 2$ of V^*, consider the subspace of $\mathrm{End}(V^*)$ consisting of those endomorphisms whose restriction to W is a scalar endomorphism of W. The dimension of this space is $1 + (r-k)^2 + k(r-k)$. Varying W over the Grassmannian $G(r,k)$ we get a vector bundle $W(r,k)$ over $G(r,k)$ and the dimension of the total space of this bundle is

$$1 + (r-k)^2 + k(r-k) + k(r-k) = 1 + r^2 - k^2.$$

We have a natural morphism π_k: $W(r,k) \to \mathrm{End}(V)$ which maps an endomorphism to its transpose. If $S_k := \overline{\pi_k(W(r,k))}$, we have $\dim S_k \leqslant (1 + r^2 - k^2)$ and $\operatorname{codim} S_k \geqslant k^2 - 1 \geqslant 3$. Let $S = \bigcup_{2 \leqslant k \leqslant r} S_k$ and $U = \mathrm{End}(V) - S$. We have $\operatorname{codim} S \geqslant 3$ and S is saturated for the map $\mathrm{End}(V) \to \mathrm{End}^0(V)$. The fibres of $q: q^{-1}(U) \to U$ are finite and q is proper. Hence q is a finite morphism.

3. Sections of the (twisted) relative tangent bundle of a projective bundle

Let E be a vector bundle of rank $r \geqslant 2$ on a smooth irreducible projective variety X of dimension $\geqslant 1$ over K. Let $\pi: \mathbf{P}(E) \to X$ be the associated projective bundle. We have the exact sequences on $\mathbf{P}(E)$:

$$0 \to \Omega^1_\pi(1) \to \pi^*(E) \to \mathcal{O}_{P(E)}(1) \to 0$$

and

$$0 \to \mathcal{O}_{\mathbf{P}(E)} \to \pi^*(E^*) \otimes \mathcal{O}(1) \to \Theta_\pi \to 0$$

where Θ_π (resp. Ω^1_π) denotes the relative tangent (resp. cotangent) bundle along the fibres of π. Let $\mathrm{End}^0(E)$ denote the vector bundle $\mathrm{End}(E)/\mathcal{O}_X$. We have an exact sequence of vector bundles on $\mathbf{P}(E)$:

$$0 \to F \to \pi^*(\mathrm{End}^0(E)) \to \Theta_\pi \to 0.$$

Let M be a line bundle on X. We obtain the exact sequence:

$$0 \to F \otimes \pi^*(M) \to \pi^*(\mathrm{End}^0(E) \otimes M) \to \Theta_\pi \otimes \pi^*(M) \to 0$$

PROPOSITION 3.1

Let $p: F \otimes \pi^*(M) \to End^0(E) \otimes M$ be the canonical morphism (of total spaces of geometric vector bundles over $\mathbf{P}(E)$ and X respectively). Then there exists an open subset Ω of $End^0(E) \otimes M$ whose complement is of codimension $\geqslant 3$ such that the morphism $p: p^{-1}(\Omega) \to \Omega$ is finite.

Proof. This follows from Proposition 2.1.

PROPOSITION 3.2

There exists an ample line bundle M on X such that a generic section s of $\Theta_\pi \otimes \pi^*(M)$ (i.e. for s in a non-empty open subset of $H^0(\mathbf{P}(E), \Theta_\pi \otimes \pi^*(M))$) satisfies the following conditions:

a) *The scheme Z of zeros of s is smooth and irreducible.*
(b) *The morphism $\pi | Z: Z \to X$ is surjective and separable.*
(c) *There exists a closed subset S of X of codimension $\geqslant 3$ such that the morphism*

$$\pi: Z \backslash \pi^{-1}(S) \to X \backslash S$$

is finite.

Proof. Let ξ be an ample line bundle on X. Then the line bundle $\pi^*(\xi^k) \otimes \mathcal{O}(1)$ is very ample on $\mathbf{P}(E)$ for $k \geqslant k_0$. [4, II, Prop. 7.10, p. 161]. We may also assume that for $k \geqslant k_0$, the bundle $\xi^k \otimes E^*$ is generated by its sections. Let $M = \xi^{2k}$. Since the sections of the very ample line bundle $\pi^*(\xi)^k \otimes \mathcal{O}(1)$ generate its first order jet bundle and $\pi^*(\xi^k \otimes E^*)$ is generated by its sections, we see that the sections of $\pi^*(M \otimes E^*) \otimes \mathcal{O}(1)$ generate its first order jet bundle [7, Lemma 5]. Since $\Theta_\pi \otimes \pi^*(M)$ is a quotient bundle of $\pi^*(M \otimes E^*) \otimes \mathcal{O}(1)$, the sections of $\Theta_\pi \otimes \pi^*(M)$ generate its first order jet bundle. Now by [7, Theorem 1] the zero scheme Z of a generic section s of $\Theta_\pi \otimes \pi^*(M)$ is smooth. We will prove in the next proposition (Proposition 4.1) that Z is irreducible.

Let $x_0 \in X$ and $S := \pi^{-1}(x_0)$ the fibre over x_0. Let W be the image of the homomorphism

$$H^0(\mathbf{P}(E), \Theta_\pi \otimes \pi^*(M)) \to H^0(S, \Theta_\pi \otimes \pi^*(M) | S).$$

(We may even assume that $W = H^0(S, \Theta_\pi \otimes \pi^*(M) | S)$, by choosing k large enough). Then the first order jets of elements of W generate the first order jet bundle of $\Theta_\pi \otimes \pi^*(M) | S$; hence, again by [7, Theorem 1], for a generic element σ of W, the zero subscheme (of S) defined by σ is smooth. Thus we see that for a generic section of $\Theta_\pi \otimes \pi^*(M)$ the zero scheme Z is smooth and irreducible and Z intersects S transversally. It follows that there exists a point $z_0 \in Z \cap S$ such that the differential of $\pi | Z$ at z_0 is an isomorphism. This proves that $\pi | Z: Z \to X$ is surjective and (assuming Z to be irreducible) separable. (Observe that Z intersects every fibre $\pi^{-1}(x), x \in X$, for otherwise the tangent bundle of the projective space $\pi^{-1}(x)$ would contain a trivial line bundle.)

$$\begin{aligned}\text{Let } \Sigma := {} & H^0(\mathbf{P}(E), \pi^*(M)\otimes\Theta_\pi)\\ = {} & H^0(X, M\otimes\pi_*(\Theta_\pi))\\ = {} & H^0(X, M\otimes \mathrm{End}^0(E)).\end{aligned}$$

Consider the morphisms

$$\begin{array}{l}\varphi:\Sigma\times X \xrightarrow{\varphi} M\otimes\mathrm{End}^0(E)\\ \quad\downarrow p_X\\ \quad X.\end{array}$$

where the evaluation map φ is a smooth morphism, being a surjection of vector bundles. Let Ω be the open subset of $M\otimes\mathrm{End}^0(E)$ defined in Proposition 3.1 and N its complement. Then

$$\dim\varphi^{-1}(N)\leqslant\dim\Sigma+\dim X-3.$$

By considering $p_X:\varphi^{-1}(N)\to X$ we see that for a generic section s of $M\otimes\mathrm{End}^0(E)$ we have

$$\dim(\varphi^{-1}(N)\cap\{s\times X\})\leqslant(\dim X)-3.$$

Let $S\subset X$ be defined to be $p_X(\varphi^{-1}(N)\cap\{s\times X\})$. Then $\dim S\leqslant(\dim X)-3$ and

$$\pi|Z:Z\backslash\pi^{-1}(S)\to X\backslash S$$

is finite.

Thus for a generic section s of $\Theta_\pi\otimes\pi^*(M)$ all the conditions a), b) and c) are satisfied.

4. Koszul resolution of the zero scheme Z

Proof of Theorem 4.2

PROPOSITION 4.1

Let s be a section of $\Theta_\pi\otimes\pi^*(M)$ *over* $\mathbf{P}(E)$ *such that the zero scheme Z of s is smooth. We then have*

a) $\pi_*(\mathcal{O}_Z\otimes\mathcal{O}_{\mathbf{P}(E)}(1))\simeq E$ *and* $R^i\pi_*(\mathcal{O}_Z(1))=0$ *for* $i\geqslant 1$.

b) $\pi_*(\mathcal{O}_Z)$ *has a filtration*
$0=F_0\subset F_1\subset F_2\subset\ldots\subset F_i\subset\ldots\subset F_r=\pi_*(\mathcal{O}_Z)$
such that $F_i/F_{i-1}\simeq M^{-(i-1)}(:=(M^*)^{\otimes(i-1)})$ *for* $1\leqslant i\leqslant r$ *(In particular* $F_1\simeq\mathcal{O}_Z$*).*

c) *Z is irreducible (if* $\dim X\geqslant 1$ *and M is ample).*

Proof. Using our assumption on Z, we have a Koszul resolution for $\mathcal{O}_z$ on $\mathbf{P}(E)$: [1, Ch I, Lemma 4.2 and Ch. III, Propositions 4.10 and 4.11]

$$(A)\ 0\to\overset{r-1}{\Lambda}(\Omega^1_\pi\otimes\pi^*(M^*))\to\cdots\to\overset{2}{\Lambda}(\Omega^1_\pi\otimes\pi^*(M^*))\to\underset{\pi}{\Omega^1}\otimes\pi^*(M^*)\to\mathcal{O}_{P(E)}\to\mathcal{O}_Z\to 0$$

and also a resolution of $\mathcal{O}_Z(1)$:

(B) $0 \to \Omega_\pi^{r-1} \otimes (\pi^*(M^*))^{r-1} \otimes \mathcal{O}(1) \to \cdots \to \Omega_\pi^1 \otimes \pi^*(M^*) \otimes \mathcal{O}(1) \to \mathcal{O}(1) \to \mathcal{O}(1)|_Z \to 0$

Now we have, for the projective space $\mathbf{P}$, the Bott vanishing theorem:

$$H^i(\mathbf{P}, \Omega^p(1)) = 0 \text{ for } p \geqslant 1 \text{ and all } i \text{ [6, Theórème 1].}$$

Hence we have

$$R^i\pi_*(\Omega_\pi^p \otimes \pi^*((M^*)^{\otimes p}) \otimes \mathcal{O}(1)) = 0$$

for $p \geqslant 1$ and all i. Splitting B into short exact sequences we deduce that

$$\pi_*(\mathcal{O}_Z(1)) = \pi_*(\mathcal{O}_{P(E)}(1)) = E$$

and $R^i\pi_*(\mathcal{O}_Z(1)) = 0$ for $i > 0$.

For proving (b) we observe that $R^q\pi_*(\Omega_\pi^p) = 0$ for $p \neq q$ and

$$R^p\pi_*(\Omega_\pi^p) = \mathcal{O}_X \text{ for } 0 \leqslant p \leqslant (r-1) \text{ [6].}$$

Splitting (A) into short exact sequences we obtain b).

To prove c), since $\dim X \geqslant 1$ and M is ample we have $H^0(X, M^{-k}) = 0$ for $k > 0$. Using the filtration of $\pi_*(\mathcal{O}_Z)$ given in b), we see that

$$H^0(Z, \mathcal{O}_Z) = H^0(X, \pi_*(\mathcal{O}_Z)) = H^0(X, \mathcal{O}_X) = K.$$

Since Z is smooth it follows that Z is irreducible.

Theorem 4.2 *Let X be a smooth irreducible projective variety of dimension $\geqslant 1$ over an algebraically closed field K. Let E be a vector bundle on X. Then there exist a smooth irreducible projective variety Z over K, a line bundle L on Z and a surjective separable morphism $f: Z \to X$ having in addition the following properties:*

1) *there exists a closed subset S in X of codimension $\geqslant 3$ such that the morphism*

$$f: Z \backslash f^{-1}(S) \to X \backslash S$$

is finite.

2) *we have* $f_*(L) \simeq E$ and $R^i f_*(L) = 0$ *for* $i > 0$.

Proof. We may assume that E is of rank $\geqslant 2$. Choose an ample line bundle M on X satisfying the conditions of Proposition 3.2. Let L be the restriction of $\mathcal{O}_{P(E)}(1)$ to Z and f be the restriction of $\pi: \mathbf{P}(E) \to X$ to Z. Then by Proposition 4.1(a) we have

$$f_*(L) \simeq E \text{ and } R^i f_*(L) = 0 \text{ for } i > 0.$$

5. The map D

Let $f: Z \to X$ be a morphism and L a line bundle on Z such that $f_*(L) = E$ is a vector bundle of rank r on X. The morphism $f^*(f_*(L)) \to L$ gives rise to a morphism

$$f_*(\mathcal{O}_Z) \otimes E \to f_*(L) = E$$

which may be viewed as a morphism

$$D: f_*(\mathcal{O}_Z) \to E^* \otimes E.$$

(D gives the canonical $f_*(\mathcal{O}_Z)$-module structure on $f_*(L)$).

Suppose that $f: Z \to X$ is a finite surjective morphism of smooth varieties. Then f is flat [1, Ch. V, Cor. 3.6]. We have

$$H^1(Z, \mathcal{O}_Z) = H^1(X, f_*(\mathcal{O}_Z)).$$

The homomorphism

$$D_*: H^1(Z, \mathcal{O}_Z) \to H^1(X, \operatorname{End} E) \tag{5.1}$$

induced by D is the infinitesimal deformation map (at L) for the variation of the direct image bundles as L deforms as a line bundle on X [2, Lemma 1.3.1].

Since f is finite, the map $f^*(E) \to L$ is surjective and we have an exact sequence

$$0 \to N \to f^*(E) \to L \to 0$$

of vector bundles on Z. From this we get an exact sequence of vector bundles.

$$0 \to \mathcal{O}_Z \to f^*(E^*) \otimes L \to N^* \otimes L \to 0.$$

Since f is flat and finite, $f_*(\mathcal{O}_Z)$ is a vector bundle on X of rank r and $f_*(N^* \otimes L)$ is a vector bundle. So we have an exact sequence of vector bundles on X:

$$0 \to f_*(\mathcal{O}_Z) \to E^* \otimes E \to f_*(N^* \otimes L) \to 0.$$

Observe that $f_*(\mathcal{O}_Z)/\mathcal{O}_X$ is a vector subbundle of rank $(r-1)$ of the vector bundle $\operatorname{End}(E)/\mathcal{O}_X = \operatorname{End}^0(E)$. Thus we have

Lemma 5.2 *Let $f: Z \to X$ be a finite morphism of smooth varieties and L a line bundle on Z. If $E := f_*(L)$ is a vector bundle of rank r, then the vector bundle $\operatorname{End}^0(E)$ contains a vector subbundle of rank $(r-1)$.*

Let us get back to the situation in §3 and §4.

PROPOSITION 5.3

Let s be a section of $\Theta_\pi \otimes \pi^*(M)$ and Z the zero subscheme of s with the property that the canonical map $E = \pi_*(\mathcal{O}_{\mathbf{P}(E)}(1)) \to f_*(\mathcal{O}_Z(1))$ is an isomorphism, where $f = \pi | Z$. Suppose that T is a section of $\operatorname{End}(E) \otimes M$ such that the image of T in $H^0(\mathbf{P}(E), \Theta_\pi \otimes \pi^*(M))$ is s. Then there is a homomorphism $\mu: M^{-1} \to f_*(\mathcal{O}_Z)$ and a commutative diagram

$$\begin{array}{ccc} f_*(\mathcal{O}_Z) & \xrightarrow{\ D\ } & \operatorname{End} E \\ & {\scriptstyle\mu}\nwarrow \quad \nearrow{\scriptstyle T} & \\ & M^{-1} & \end{array}$$

where D is defined at the beginning of this section (§5).

Proof. Consider the diagram

$$\begin{array}{ccccccccc} 0 & \to & \Omega^1_\pi \otimes \pi^*(M)^{-1} & \xrightarrow{i} & \pi^*(E \otimes M^{-1}) & \to & \mathcal{O}(1) \otimes \pi^*(M^{-1}) & \to & 0 \\ & & & & \downarrow \tilde{T} & & & & \\ 0 & \longrightarrow & \Omega^1_\pi & \longrightarrow & \pi^*(E) & \xrightarrow{g} & \mathcal{O}(1) & \longrightarrow & 0 \end{array}$$

where $\tilde{T}$ is induced by T. The homomorphism $g \circ \tilde{T} \circ i : \Omega^1_\pi \otimes M^{-1} \to \mathcal{O}(1)$ gives the section s of $\Theta_\pi \otimes \pi^*(M)$. So $\tilde{T}$ induces on Z a homomorphism $\lambda : \mathcal{O}_Z(1) \otimes f^*(M^{-1}) \to \mathcal{O}_Z(1)$ and we have a commutative diagram

$$\begin{array}{ccc} \pi^*(E) \otimes \pi^*(M^{-1}) & \xrightarrow{q \otimes 1} & \mathcal{O}_Z(1) \otimes \pi^*(M^{-1}) \\ \downarrow \tilde{T} & & \downarrow \lambda \\ \pi^*(E) & \xrightarrow{q} & \mathcal{O}_Z(1) \end{array}$$

Considering λ as a section of $\mathcal{O}_Z \otimes \pi^*(M)$ we obtain the section $\pi_*(\lambda)$ of $M \otimes \pi_*(\mathcal{O}_Z)$ which we view as a homomorphism $\mu : M^{-1} \to f_*(\mathcal{O}_Z)$. Since by assumption $E \to \pi_*(\mathcal{O}_Z(1))$ is an isomorphism, it follows, from the above diagram, that T corresponds to $\pi_*(\lambda) : M^{-1} \otimes E \to E$. But by the definitions of D and μ we see that $\pi_*(\lambda)$ corresponds to $D \circ \mu$. This means that $T = D \circ \mu$.

6. Vector bundles on curves

Lemma 6.1. *Let X be a smooth, projective irreducible curve over K and F a vector bundle of rank $\geqslant 2$ on X. Let ξ be an ample line bundle on X. Then there exists an integer l_0 such that for $l \geqslant l_0$, ξ^{-l} is a subbundle of F and the induced homomorphism $H^1(X, \xi^{-l}) \to H^1(X, F)$ is surjective.*

Proof. Let first F be of rank 2. Since for all large l, $\xi^l \otimes F^*$ contains a trivial line subbundle, we get an exact sequence

$$0 \to \xi^{-l} \to F \to \xi^l \otimes \det F \to 0.$$

Choose l large enough so that $H^1(X, \xi^l \otimes \det F) = 0$.

Now suppose that F is a vector bundle of rank $\geqslant 3$. Then we can find a filtration of F by subbundles

$$0 \subset F_1 \subset F_2 \ldots \subset F_i \subset \ldots F_{r-1} = F$$

such that rank $(F_i) = i + 1$ (in particular rank $F_1 = 2$) and such that $H^1(X, F_i/F_{i-1}) = 0$, for $i \geqslant 2$. Now choose a line subbundle ξ^{-l} of F_1 with $H^1(F_1/\xi^{-l}) = 0$. We see easily, by induction on i, that $H^1(X, F/\xi^{-l}) = 0$.

Remark 6.2 Note that $H^1(X, \xi^{-l}) \to H^1(X, F)$ is surjective if and only if $H^1(F/\xi^{-l}) = 0$, as $H^2(X, \xi^{-l}) = 0$.

COROLLARY 6.3

Let F be a vector bundle on X. Then there exists an integer l_0 such that, for $l \geqslant l_0$, for a generic section σ of $\xi^l \otimes F$ the map $H^1(X, \xi^{-l}) \to H^1(X, F)$ induced by σ is surjective.

Theorem 6.4 *Let X be a smooth projective irreducible curve over an algebraically closed field K and let E be a vector bundle on X. Then there exist a smooth projective irreducible curve Z over K, a line bundle L on Z and a finite surjective separable morphism $f: Z \to X$ such that*

1) $f_*(L) \simeq E$
2) *and the homomorphism (defined in 5.1)*

$$H^1(Z, \mathcal{O}_Z) \to H^1(X, \operatorname{End} E)$$

is surjective.

Proof. Choose an ample line bundle M as in the proof of Theorem 4.2. We may also choose M to have the further properties:

a) $H^1(X, M) = 0$

and

b) a generic section σ of $H^0(X, \operatorname{End} E \otimes M)$ verifies the condition that the homomorphism

$$H^1(X, M^{-1}) \to H^1(X, \operatorname{End} E)$$

is surjective (use Corollary 6.3).

By condition a) the map

$$H^0(X, \operatorname{End} E \otimes M) \to H^0(X, \operatorname{End}^0 E \otimes M)$$

is surjective.

Now a generic section s of $H^0(\mathbf{P}(E), \Theta_\pi \otimes \pi^*(M)) = H^0(X, \operatorname{End}^0(E) \otimes M)$ is the image of a section T of $\operatorname{End}(E) \otimes M$ with the property that the homomorphism

$$H^1(X, M^{-1}) \to H^1(X, \operatorname{End} E)$$

induced by T is surjective and satisfies conditions a), b) and c) of Proposition 3.2. Choose Z, L, f as in the proof of Theorem 4.2. Then $f_*(L) = E$.

To prove 2), observe that the factorisation, given in Proposition 5.3,

$$\begin{array}{ccc} f_*(\mathcal{O}_Z) & \xrightarrow{D} & \operatorname{End} E \\ {\scriptstyle \mu}\nwarrow & & \nearrow{\scriptstyle T} \\ & M^{-1} & \end{array}$$

induces a commutative diagram:

$$\begin{array}{ccc} H^1(Z, \mathcal{O}_Z) = H^1(X, f_*(\mathcal{O}_Z)) & \xrightarrow{D_*} & H^1(X, \operatorname{End} E) \\ {\scriptstyle \mu_*}\nwarrow & & \nearrow{\scriptstyle T_*} \\ & H^1(X, M^{-1}) & \end{array}$$

Since, by choice, T_* is surjective, the homomorphism D_* is forced to be surjective.

Remark 6.5 If E is a stable bundle on X and if Z, f and L are chosen as in Theorem 6.4, we see that f_* gives a *dominant* separable rational morphism from an appropriate component of $Pic(Z)$ into the moduli space of vector bundles on X of rank $r = rkE$ and degree $d = degree\, E$ (compare [3]). We thus obtain 'most' stable bundles of a given rank and degree as direct images of line bundles on a *fixed* covering Z of X.

7. The example

We now give an example of a rank 2 vector bundle on the projective space $\mathbf{P}_3(\mathbf{C})$ which cannot be obtained as the direct image of a line bundle by a *finite* morphism $f: Z \to \mathbf{P}_3(\mathbf{C})$, with Z smooth.

Let E be a stable vector bundle of rank 2 on $\mathbf{P}_3(\mathbf{C})$ with $c_1(E) = 0$ and $c_2(E) > 0$. If E were the direct image of a line bundle by $f: Z \to X$, with Z smooth and f finite, the bundle $\mathrm{End}^0(E)$ would contain a line *subbundle* L by Lemma 5.2. If $\xi = L^{-1}$, we would have

$$c_3(\xi \otimes \mathrm{End}^0 E) = 0. \text{ We have}$$

$$c_3(\xi \otimes \mathrm{End}^0 E) = 4c_1(\xi)c_2(E) + c_1(\xi)^3.$$

But the bundle L and hence ξ, is non-trivial, since $h^0(\mathbf{P}_3, \mathrm{End}^0 E) = 0$, E being stable. So $c_1(\xi) \neq 0$ and we would have

$$c_1(\xi)(4c_2(E) + c_1(\xi)^2) = 0,$$

a contradiction.

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