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Partially conserved axial-vector current in S-matrix theory

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Abstract. Mandelstam's argument that PCAC follows from assigning Lorentz quantum number $M=1$ to the massless pion is examined in the context of multiparticle dual resonance model. We construct a factorisable dual model for pions which is formulated operatorially on the harmonic oscillator Fock space along the lines of Neveu-Schwarz model. The model has both m_π and m_ρ as arbitrary parameters unconstrained by the duality requirement. Adler self-consistency condition is satisfied if and only if the condition $m_\rho^2 - m_\pi^2 = \frac{1}{2}$ is imposed, in which case the model reduces to the chiral dual pion model of Neveu and Thorn, and Schwarz. The Lorentz quantum number of the pion in the dual model is shown to be $M=0$.

Keywords. Partially conserved axial-vector current; Lorentz quantum number; dual pion model; Adler zeros.

1. Introduction

The hypothesis of partially conserved axial-vector current (PCAC) and current algebra lead to strong restrictions on the hadron scattering amplitudes involving pions, usually expressed in the form of low energy theorems. Notable among them is the well-known Adler self-consistency condition which essentially states that hadron scattering amplitudes involving soft pions should vanish. It is of considerable interest to investigate whether these soft-pion theorems can be derived in the S-matrix theory without introducing weak interaction currents. Mandelstam (1968) made the first attempt in this direction and made a significant advance by showing that the Adler self-consistency condition alone is sufficient to obtain most of the consequences of current algebra and PCAC hypothesis for the hadron scattering amplitudes. In other words, if one can construct arguments within the S-matrix framework to derive the Adler self-consistency condition, then most of the restrictions which current algebra placed on hadron scattering amplitudes ought to follow.

Mandelstam (1968) argued that constraints on the couplings at zero four momentum could follow from the conspiracy theory of Regge trajectories and residues. In particular vanishing of the soft pion amplitude follows as a mathematical consequence of the assignment of the Lorentz quantum number (Toller 1965, 1968; Sciarrino and Toller 1967) $M = 1$ to the pion trajectory. However, as noted by himself and discussed in detail by Arbab and Jackson (1968) for the $M = 1$ pion factorisation implies the smallness of both soft pion and hard pion amplitudes. In fact, in the discussion of two-body processes there have been many arguments against assigning $M = 1$ to the pion but favouring the assignment $M = 0$ (see, e.g. Capella 1970; Mueller 1969; Sawyer 1968; Wang and Wang 1970).

In the dual resonance model (DRM), the $\pi\pi$ scattering amplitude of Lovelace (1968) and Shapiro (1969) (LS amplitude) has Adler zeros when the leading ρ-trajectory is constrained to lie half a unit above the π trajectory, that is by imposing the condition*

$$\alpha_\rho - \alpha_\pi = \tfrac{1}{2} \quad \text{or} \quad m_\rho{}^2 - m_\pi{}^2 = 1/2\alpha' \tag{1}$$

where $\alpha' \approx 1(\text{GeV})^{-2}$ is the universal slope of the Regge trajectories. This trajectory splitting condition, eq. (1) occurs naturally as a requirement of duality in the operator formulation of the chiral invariant dual multipion model of Neveu and Thorn (1971) and Schwarz (1972). In this model the Adler condition is satisfied by a mechanism originally discovered by Brower (1971); that is, when the four momentum of one of the pions tends to zero, the N point function can be shown to contain a vanishing beta-function (Neveu and Thorn 1971; Schwarz 1972).

In this paper we investigate whether there is a connection between PCAC and the Lorentz quantum number of the pion in the dual resonance model. (We use the term PCAC to mean existence of Adler zeros in the amplitude). The material of this paper is organised as follows. In section 2 we construct a factorizable dual pion model along the lines of Neveu and Schwarz (1971), Neveu and Thorn (1971) and Halpern and Thorn (1971) which is formulated using boson and fermion oscillators. The model has masses of both ρ and π arbitrary, but still satisfies the requirement of duality unlike the hitherto existing dual pion models. It is shown that Adler self-consistency condition is satisfied if and only if the trajectory splitting condition $m_\rho{}^2 - m_\pi{}^2 = 1/2\alpha'$ is imposed. In section 3 we consider the pion pole in the six point function and using Mandelstam's (1968) argument and Arbab and Jackson's (1968) result on the factorisation of Regge residue we show that the pion pole belongs to the class with Lorentz quantum number $M=0$. In section 4 we state our conclusions.

2. PCAC in dual pion model and Lorentz quantum number of the pion

2.1. *General remarks*

The dual pion model of Neveu and Thorn (1971) and Schwarz (1972) (NTS model) is known to possess Adler zeros and reduce (Schwarz and Wallace 1972) to the non-linear σ-model in the zero-slope limit, that is in the limit $\alpha' \to 0$. Let us consider the six-point function in this model, $A_6\,(p_1, p_2, p_3, p_4, p_5, p_6)$ and look at the residue of the pion-pole in a three pion channel, say at $(p_1+p_2+p_3)^2 = m_\pi{}^2$. We shall assume $m_\pi=0$. In the limit of the four-momentum $Q=p_1+p_2+p_3 \to 0$, following Mandelstam's argument (which uses only the properties of the groups O (3, 1) and O (2, 1)) the residue of the pion pole will vanish if the pion belongs to the class with Lorentz quantum number $M = 1$. On the other hand by the property of factorisation, in dual resonance model, the residue is a product of two four-pion amplitudes and will

*As pointed out by Brower (1971) the Adler condition may also be satisfied by imposing a condition different from eq. (1). However there has been no operator for mulation of Brower's model so far; so it lacks fundamental significance.

again vanish in the above limit ($Q \rightarrow 0$) if the amplitude possesses Adler zeros due to condition like eq.(1). This situation tends to mask the difference, if any, between the two hypotheses. Any possible difference between the two is however likely to emerge more distinctly in a situation where one of them is made to hold only approximately. Since the PCAC hypothesis is of an approximate nature it is instructive to investigate the Lorentz quantum number of the pion in a model in which there is a small departure from the Adler self-consistency condition but the departure can be made arbitrarily small. To this end we construct a dual pion model which has a continuously variable parameter, in a certain limit of which, the model satisfies the Adler self-consistency condition. In the model which we describe below such a parameter turns out to be $d^2 = (1/2\alpha') + m_\pi^2 - m_\rho^2$. The Adler zeros appear only in the limit $d^2 \rightarrow 0$.

2.2. *Dual pion model with two mass variables*

Fubini and Veneziano (1971) (FV) pointed out that in the conventional dual resonance model, the trajectory intercepts in different channels can be varied by introducing an extra space-like component, the so-called ' fifth component ', of the momenta and a corresponding set of oscillators without destroying duality properties and still preserving the full gauge group. Adopting the same procedure for the dual pion model of Neveu and Schwarz (1971) (NS model) Halpern and Thorn (1971) (HT) shifted the pion mass by an arbitrary amount (to make $m_\pi^2 \geqslant 0$) from its value $m_\pi^2 = -\frac{1}{2}$ in the original NS model. However, to preserve duality ρ was still massless as in the original NS model. Neveu and Thorn (1971) (NT) adopted this technique of introducing extra components to the momenta and oscillators, to obtain another variation of the original NS model. In their model all masses are shifted equally from their values in the NS model. This also allows us to vary only one mass, say m_π^2, and masses of all other particles are constrained in relation to this variable. For example the mass of the ρ is still constrained by $m_\rho^2 - m_\pi^2 = \frac{1}{2}$, a trajectory splitting condition which endows Adler zeros to the model. The $\pi\pi \rightarrow \pi\pi$ amplitude in this model is the well-known Lovelace-Shapiro amplitude. Schwartz (1972) used a slightly different operator formulation to obtain the same model as Neveu and Thorn.

A useful distinction exists between the FV or HT way of introducing extra components and the NT way. The fifth component of FV and HT is introduced independent of the number of external particles N and hence the preservation of the factorisation property is obvious. In the NT way the number of extra dimensions depends on N, yet the extra components of momenta are assigned in such a way that there is only nearest neighbour coupling of the extra components, so that only one set of oscillators contribute at any pole (Neveu and Thorn 1971). This mechanism of nearest neighbour coupling preserves the factorisation property and is closely related to the implementation of Adler's self consistency condition. The introduction of a fifth component along the lines of HT results in shifting only the masses in the odd-G channels, that too by an equal amount from their values in the original NS model, whereas Neveu-Thorn's method of introducing extra components results in shifting all masses equally.

It is easy to combine the NTS model and HT model to construct yet another modification of the NS model in which both m_ρ^2 and m_π^2 can be varied independently.

Introducing the fifth component of momenta as in the Halpern-Thorn (1971) model as well as the N extra space-like components as in the Neveu-Thorn (1971) model we write

$$\hat{k}_1 = (k_1, -d, c/\sqrt{2}, -c/\sqrt{2}, 0, 0 \ldots, 0, 0)$$

$$\hat{k}_2 = (k_2, +d, 0, c/\sqrt{2}, -c/\sqrt{2}, 0 \ldots, 0, 0)$$

$$\vdots$$

$$\hat{k}_i = (k_i, (-1)^i d, 0, \ldots c/\sqrt{2}, -c/\sqrt{2}, 0 \ldots, 0, 0)$$

$$\vdots$$

$$\hat{k}_{N-1} = (k_{N-1}, -d, 0, 0 \ldots 0, c/\sqrt{2}, -c/\sqrt{2})$$

$$\hat{k}_N = (k_N, +d, -c/\sqrt{2}, 0 \ldots, 0, c/\sqrt{2}) \tag{2}$$

where $k_1, k_2, \ldots, k_N$ are the four-momenta of the particles $1, 2, 3, \ldots, N$ respectively. The model can now be easily written down by replacing the momenta and oscillators in the original Neveu-Schwarz model by the above $(N+5)$-dimensional momenta and the corresponding $(N+5)$-dimensional set of operators. We briefly describe below the construction of the N-point function. The $(N+5)$-analogues of the usual harmonic oscillator operators satisfy the algebra

$$\left[\hat{a}^\mu_n, \hat{a}^\nu_m\right] = -ng^{\mu\nu}\,\delta_{n,\,-m} \tag{3}$$

$$\left\{\hat{b}^\mu_m, \hat{b}^\nu_n\right\} = -g^{\mu\nu}\,\delta_{m,\,-n} \tag{4}$$

$$\left[\hat{a}^\mu_m, \hat{b}^\nu_n\right] = 0 \tag{5}$$

where the indices μ and ν run over the $(N+5)$ components. In addition we also have the momentum operator $a^\mu_0 = \sqrt{2}\,p^\mu$ and its canonical conjugate (position) operator $\hat{x}^\nu$ satisfying

$$\left[\hat{a}^\mu_0, \hat{x}^\nu\right] = -\sqrt{2}ig^{\mu\nu}. \tag{6}$$

In analogy with the standard procedure (Neveu and Schwarz 1971; Schwarz 1973) the vertex operator is given by

$$\hat{V}(\hat{k}) = \hat{k}\cdot\hat{H}\hat{V}_0(\hat{k}) \tag{7}$$

where

$$\hat{V}_0(\hat{k}) = \exp\,(i\hat{k}\cdot\hat{x})\,\exp\,(\sqrt{2}\hat{k}\cdot\textstyle\sum_{n=1}^{\infty}\hat{a}_{-n}/n)$$

$$\times\,\exp\,(-\sqrt{2}\hat{k}\cdot\textstyle\sum_{n=1}^{\infty}\hat{a}_n/n) \tag{8}$$

and

$$\hat{H}^{\mu} = \sum_{n=-\infty}^{\infty} \hat{b}_{n}^{\mu}, \quad n=\pm\tfrac{1}{2}, \pm\tfrac{3}{2}, \ldots \tag{9}$$

In addition we define

$$\hat{L}_0 = -\tfrac{1}{2}\hat{a}_0 \cdot \hat{a}_0 - \sum_{n=1}^{\infty} -\hat{a}_{-n} \cdot \hat{a}_n - \sum_{n=\frac{1}{2}}^{\infty} n\hat{b}_{-n} \cdot \hat{b}_n. \tag{10}$$

In terms of these operators, the N-point function (in the original F_1-space formulation of NS model (Neveu and Schwarz 1971)) is given by

$$A_N = \langle 0, -\hat{k}_1 | \hat{k}_1 \cdot \hat{b}_{1/2} \hat{V}(\hat{k}_2) (\hat{L}_0 - 1)^{-1} \hat{V}(\hat{k}_3) \ldots$$
$$\ldots (\hat{L}_0 - 1)^{-1} \hat{V}(\hat{k}_N) \hat{k}_N \cdot \hat{b}_{-1/2} | 0, \hat{k}_N \rangle. \tag{11}$$

The extra components of momenta are constrained by the requirement of conformal symmetry $\hat{k}_i^2 = -\frac{1}{2}$, that is

$$m_{\pi}^2 - c^2 - d^2 = -\tfrac{1}{2}. \tag{12}$$

The trajectory functions in the even and odd G-parity channels are

$$\alpha_{\rho}(s) = 1 + (s - c^2) \quad \text{for even-}G\text{ channels} \tag{13a}$$

and

$$\alpha_{\pi}(s) = \tfrac{1}{2} + (s - c^2 - d^2) \quad \text{for odd-}G\text{ channels} \tag{13b}$$

from which it follows that

$$m_{\rho}^2 = c^2; \; m_{\rho}^2 - m_{\pi}^2 = \tfrac{1}{2} - d^2. \tag{14}$$

The amplitude given by eq. (11) has the properties of duality, factorisations and incorporates the tachyon killing mechanism of the Neveu-Schwarz model. The latter is apparent when we rewrite eq. (11) in the F_2-formulation (Neveu *et al* 1971) as

$$A_N = \langle 0, -\hat{k}_1 | \hat{V}(\hat{k}_2) (\hat{L}_0 - \tfrac{1}{2})^{-1} \hat{V}(\hat{k}_3) \ldots (\hat{L}_0 - \tfrac{1}{2})^{-1} \hat{V}(\hat{k}_{N-1}) | 0, \hat{k}_N \rangle. \tag{15}$$

Using the standard technique of writing the propagator as

$$(\hat{L}_0 - \tfrac{1}{2})^{-1} = \int_0^1 dx \, x^{\hat{L}_0 - \frac{3}{2}}$$

we have the following integral representation of the N-point function

$$A_N = \int_0^1 \Pi_{i=2}^{N-2} dx_i \, x_i^{-\hat{P}_i^2 - \frac{3}{2}} \times \prod_{2 \le i < j \le N-1} (1 - x_i x_{i+1} \ldots x_j)^{-2\hat{k}_i \cdot \hat{k}_j}$$
$$\times \langle 0 | \hat{k}_2 \cdot \hat{H}(1) \hat{k}_3 \cdot \hat{H}(x_2) \hat{k}_4 \cdot \hat{H}(x_2 x_3) \ldots \hat{k}_{N-1} \cdot \hat{H}(x_2 x_3 x_4 \ldots x_{N-2}) | 0 \rangle \tag{16}$$

where

$$\hat{P}_i = (\hat{k}_1+\hat{k}_2+\ldots+\hat{k}_i);\ \hat{H}(z) = \Sigma_n\, \hat{b}_n\, z^{-n}$$

Adler zeros: If we now set $d=0$ the above model reduces to the NTS model, which is known to possess Adler zeros. In $d=0$ case it can be easily shown (Neveu and Thorn 1971) from eq. (16) that for example when $k_2 \to 0$

$$A_N \to \left[\int_0^1 dx_2\, x_2^{-1+c^2}\,(1-x_2)^{-1-c^2}\right] \times \left[\int_0^1 dx_3\ldots dx_{N-2}\, F(x_3, x_4\ldots x_{N-2})\right]. \tag{17}$$

The first factor in eq. (17) is

$$\int_0^1 dx_2\, x_2^{-1+c^2}\,(1-x_2)^{-1-c^2} = B(c^2, -c^2) = 0. \tag{18}$$

This is the Brower mechanism (Brower 1971) for implementing the Adler self-consistency condition. This depends crucially on the fact that when $d=0$ in eq. (16) the product $\hat{k}_i\cdot\hat{k}_j = k_i\cdot k_j$ except for the nearest neighbours, so that as the four vector $k_2 \to 0$, $\hat{k}_2\cdot\hat{k}_j \to 0$ for $j \neq 1,3$. For $d \neq 0$ the latter circumstance does not hold so that the amplitude given by eq. (16) does not reduce to the form given by eq. (17) when $k_2 \to 0$. Therefore the factorisable dual model constructed by us [eq. (11) or eq. (15) or eq. (16)] does not possess Adler zeros in general unless $d=0$.

3. The pion pole and its Lorentz quantum number

Let us now look at the pion pole in the six-point function of the present model. Following standard procedure we obtain from eq. (11) the six-point function with external momenta ordered cyclically,

$$A_6(k_1, k_2, k_3, k_4, k_5, k_6) = \int_0^1 \frac{du_{12}\, du_{13}\, du_{14}}{(1-u_{12}\, u_{13})\,(1-u_{13}\, u_{14})}$$

$$u_{12}^{-\alpha_\rho(s_{12})}\; u_{13}^{-\alpha_\pi(s_{13})-1}\; u_{14}^{-\alpha_\rho(s_{14})}\; u_{23}^{-\alpha_\rho(s_{23})}\; u_{25}^{-\alpha_\rho(s_{25})}$$

$$u_{24}^{-\alpha_\pi(s_{24})-1}\; u_{34}^{-\alpha_\rho(s_{34})}\; u_{35}^{-\alpha_\pi(s_{35})-1}\; Z(\hat{k}, u) \tag{19}$$

where $s_{ij} = (k_i + k_{i+1} + \ldots + k_j)^2$ and u_{ij}'s are the usual Chan variables, and $Z(\hat{k}, u)$ is given by

$$Z(\hat{k}, u) = \{ [(\hat{k}_1\cdot\hat{k}_2\; \hat{k}_3\cdot\hat{k}_4\; \hat{k}_5\cdot\hat{k}_6)\,(u_{12}\, u_{34}\, u_{14})^{-1} + \mathbf{P}_c\,(1)]$$

$$+ [(\hat{k}_1\cdot\hat{k}_2\; \hat{k}_3\cdot\hat{k}_6\; \hat{k}_4\cdot\hat{k}_5)\, u_{24}\,(u_{12}\, u_{45})^{-1} + \mathbf{P}_c\,(2)]$$

$$-[(\hat{k}_1\cdot\hat{k}_2\,\hat{k}_3{}'\,\hat{k}_5\,\hat{k}_4\cdot\hat{k}_6)\,u_{24}\,u_{12}^{-1}+\mathbf{P}_c\,(5)]$$

$$+[(\hat{k}_1\cdot\hat{k}_4\,\hat{k}_2\cdot\hat{k}_6\,\hat{k}_3\cdot\hat{k}_5\,u_{13}\,u_{24}+\mathbf{P}_c\,(2)]$$

$$-(\hat{k}_1\cdot\hat{k}_4\,\hat{k}_2\cdot\hat{k}_5\,\hat{k}_3\cdot\hat{k}_6\,u_{13}\,u_{24}\,u_{35})\,\}\tag{20}$$

where the symbol $\mathbf{P}_c(n)$ means one must add the n independent terms obtained by cyclic permutation of the preceding expression. Computing the residue of A_6 $(k_1,k_2,k_3,k_4,k_5,k_6)$ at the pion pole corresponding to $\alpha_\pi(s_{13})=0$ we get the pion pole residue in the factorised form

$$[(2k_2\cdot Q+2d^2)\,B\,(1-\alpha_{12},\,1-\alpha_{23})]\,[(-2k_5\cdot Q+2d^2)\,B\,(1-\alpha_{14},\,1-\alpha_{45})]$$

$$=A_4\,(k_1,\,k_2,\,k_3,\,-Q)\,A_4\,(Q,\,k_4,\,k_5,\,k_6)\tag{21}$$

where $Q=k_1+k_2+k_3$ and $B(x,y)$ is the Euler beta-function and A_4 is the four-point function,

$$A_4\,(k_1,k_2,\,k_3,\,-Q)=(2\,k_2\cdot Q+2d^2)\,B\,(1-\alpha_{12},\,1-\alpha_{23})$$

$$=(1-\alpha_{12}-\alpha_{23})\,B\,(1-\alpha_{12},\,1-\alpha_{23})\tag{22}$$

which is the canonical Lovelace-Shapiro four-point function. It is evident from eq. (22) that our dual model has Adler zeros if and only if $d^2=0$.

The M quantum number: Returning to the six point function given by eq. (19) one can perform an O(3,1) analysis (Toller 1965, 1968) corresponding to $Q_\nu=(k_1+k_2+k_3) = -(k_4+k_5+k_6)_\nu=0$ to find the M quantum number of pion. However without going through the mathematics we can show that pion belongs to $M=0$ representation as follows. We can vary the parameters c and d without destroying duality and factorisation so that we can choose

$$d\neq 0\ \text{ but }\ m_\pi{}^2=-\tfrac{1}{2}+c^2+d^2=0.$$

We have seen that when $d\neq 0$ our model does not possess an Adler zero and must belong to $M=0$ representation as otherwise there will be a contradiction with Mandelstam's argument that if $|M|\geqslant 1$, Adler zeros are present. Although by continuity, we expect that as we vary the parameter d in our model the pion will continue to have $M=0$, it remains to be checked that the M value does not abruptly change, for example to $M=1$ as d becomes zero. This latter fact can be established, using the results of Arbab and Jackson (1968) who showed that if pion has $M=1$, then by factorization even the hard pion amplitudes should vanish linearly as $\sqrt{Q^2}$, Q_μ being the pion four momentum. Returning to eq. (21) and setting $d=0$, we see that the pion residue does not vanish even if $Q^2=0$ unless the four momentum Q_μ also vanishes.

4. Discussion

We are thus led to conclude that for all values of d^2 in our model, the pion has the Lorentz quantum number $M=0$. The Adler self consistency conditon is satisfied if and only if the trajectory splitting condition eq. (1) is satisfied. Our model does not suffer from the defect of requiring that hard pion amplitudes also vanish if $Q^2=0$ (but $Q_\mu \neq 0$), which would be the case if pion had $M=1$. Since our model satisfies the requirement of duality and factorisation for all values of m_π^2 and m_ρ^2, many other interesting questions like the spectrum of physical states, the field theory corresponding to the zero slope limit of the model, should be investigated.

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