Proc. Indian Acad. Sci. (Math. Sci.), Vol. 101, No. 2, August 1991, pp. 111–120.

On continuous maps between Grassmann manifolds

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MS received 18 December 1990

Abstract. Let $G_{n,k}$ denote the Grassmann manifold of k-planes in $\mathbb{R}^n$. We show that for any continuous map $f\colon G_{n,k}\to G_{n,l}$ the induced map in $Z/2$-cohomology is either zero in positive dimensions or has image in the subring generated by $w_1(\gamma_{n,k})$, provided $1\leqslant l<k\leqslant[n/2]$ and $n\geqslant k+2l-1$. Our main application is to obtain negative results on the existence of equivariant maps between oriented Grassmann manifolds. We also obtain positive results in many cases on the existence of equivariant maps between oriented Grassmann manifolds.

Keywords. Grassmann manifolds; Steenrod squares; Stiefel–Whitney classes; equivariant maps.

1. Main results

For $1\leqslant k<n$, let $G_{n,k}$ be the Grassmann manifold of all k-dimensional vector subspaces ("k-planes") of $\mathbb{R}^n$, and let $\tilde{G}_{n,k}$ denote the oriented Grassmann manifold of all oriented k-planes in $\mathbb{R}^n$. The double covering map $\pi_{n,k}\colon \tilde{G}_{n,k}\to G_{n,k}$ is a universal covering projection for $n\geqslant 3$. Let $\gamma_{n,k}$ denote the canonical k-plane bundle over $G_{n,k}$ and let $\beta_{n,k}$ denote the "orthogonal complement" bundle over $G_{n,k}$, which is of rank $n-k$. Let $w_i=w_i(\gamma_{n,k})\in H^i(G_{n,k};\,Z/2)$, $1\leqslant i\leqslant k$, denote the i-th Stiefel-Whitney class of $\gamma_{n,k}$, and let $\bar{w}_j=w_j(\beta_{n,k})$, $1\leqslant j\leqslant n-k$. Then, one has the following relation:

$$w\cdot\bar{w}=(1+w_1+\ldots+w_k)(1+\bar{w}_1+\ldots+\bar{w}_{n-k})=1 \tag{1}$$

which can be used to express the $\bar{w}_j$'s in terms of the w_i's.

One has the following description for the $Z/2$-cohomology ring $H^*(G_{n,k})$, (cf. [2], [9]):

$H^*(G_{n,k})$ is generated as an algebra over $Z/2$ by $w_1,\ldots,w_k$ subject only to the relations coming from (1). (2)

Note that $G_{n,k}\cong G_{n,n-k}$ so we assume without loss of generality that $k\leqslant[n/2]$.

Theorem 1. *Let $1\leqslant l<k\leqslant[n/2]$, $n\geqslant k+2l-1$, and let $2^{s-1}<n\leqslant 2^s$. If $f\colon G_{n,k}\to G_{n,l}$ is any continuous map, then $f^*\colon H^*(G_{n,l};Z/2)\to H^*(G_{n,k};Z/2)$ is*

(i) *zero in positive dimensions except possibly when $n=2^s$*, or $(n,k)=(2^s-1,2)$,

(ii) *either zero in positive dimensions or* $f^*(w_i(\gamma_{n,l}))=\begin{bmatrix} l \\ i \end{bmatrix} w_1^i$, $1\leqslant i\leqslant l$, *if* $n=2^s$, *or* $n=2^s-1$, $k=2$.

Note that $G_{\infty,k}=\mathrm{BO}(k)$, a classifying space for real vector bundles of rank k.

We will prove

Theorem 2. *Let* $1\leqslant l<k$. *Then for any* f: $\mathrm{BO}(k)\to\mathrm{BO}(l)$, *the induced map in* $Z/2$-*cohomology is zero in positive dimensions or is given by*

$$f^*(w_j(\gamma_{\infty,l}))=\begin{bmatrix} r \\ j \end{bmatrix} w_1(\gamma_{\infty,k})^j \text{ if } j\leqslant r \text{ and } f^*(w_j(\gamma_{\infty,l}))=0$$

if $j>r$, *for some* $r\leqslant l$. *Conversely, given any* $r\leqslant l$, *there exists a map* f_r: $\mathrm{BO}(k)\to\mathrm{BO}(l)$ *such that*

$$f^*(w_j(\gamma_{\infty,l}))=\begin{bmatrix} r \\ j \end{bmatrix} w_1(\gamma_{\infty,k})^j \text{ for } 1\leqslant j\leqslant r.$$

We will apply Theorem 1 mainly to the question of existence of equivariant maps between oriented Grassmann manifolds. We hope that the above theorems will lead to other applications also.

The involution that changes the orientation on each element of $\tilde{G}_{n,k}$ is a smooth $Z/2$ action on $\tilde{G}_{n,k}$. Observe that the "inclusions" i: $\tilde{G}_{n,k}\to\tilde{G}_{n+1,k}$, and j: $\tilde{G}_{n,k}\to\tilde{G}_{n+1,k+1}$ induced by the usual inclusion of $\mathbb{R}^n$ in $\mathbb{R}^{n+1}$ are equivariant. From this it follows that if g: $\tilde{G}_{m,k}\to\tilde{G}_{n,l}$ is equivariant with $m>n$, then there exist equivariant maps g_p: $\tilde{G}_{n,p}\to\tilde{G}_{n,l}$, for $k-m+n\leqslant p\leqslant k$, as can be seen from composing g with i's and j's suitably, $(m-n)$ times. The following theorem is a partial answer to the question: "For what values of n, k and l does there exist an equivariant map $\tilde{G}_{n,k}\to\tilde{G}_{n,l}$, $1\leqslant l, k<n, l\neq k, n-k$"? Since the diffeomorphism $\perp$: $\tilde{G}_{n,k}\to\tilde{G}_{n,n-k}$ is equivariant, we need only consider the case when k, $l\leqslant[n/2]$. When $k=l$, the identity map is equivariant.

Theorem 3. *Let* $2^{s-1}<n\leqslant 2^s$.

(i) *Let* $1\leqslant l<k\leqslant[n/2]$, *and let* $n\geqslant k+2l-1$. *Then there does not exist an equivariant map of* $\tilde{G}_{n,k}$ *into* $\tilde{G}_{n,l}$ *provided that* l *is even when* n *is a power of* 2, *and that* $(n,k)\neq(2^s-1,2)$.

(ii) *Let* $m\leqslant 2^{s-1}$, $1\leqslant p\leqslant[m/2]$, $1\leqslant k\leqslant[n/2]$. *Then there does not exist an equivariant map of* $\tilde{G}_{n,k}$ *into* $\tilde{G}_{m,p}$. *There does not exist an equivariant map of* $\tilde{G}_{n,k}$ *into* $\tilde{G}_{2^s-1,1}$ for $k\geqslant 3$, $(n,k)\neq(2^{s-1}+1,3)$. *No equivariant map exists from* $\tilde{G}_{n,2}$ *or* $\tilde{G}_{2^{s-1}+1,3}$ *into* $\tilde{G}_{2^s-2,1}$.

As for positive results we prove

Theorem 4. (i) *Let* $1\leqslant k\leqslant[n/2]$, *and let* $d=\dim\tilde{G}_{n,k}=k(n-k)$. *For any* r, *there exists an equivariant map* f: $\tilde{G}_{n,k}\to\tilde{G}_{d+r,q}$, $1\leqslant q\leqslant r$. *Further, when* $(n,k)\neq(2^s+1,2)$ *there exists an equivariant map* f': $\tilde{G}_{n,k}\to\tilde{G}_{d-1+r,q}$, $1\leqslant q\leqslant r$. *There does not exist an equivariant map from* $\tilde{G}_{2^s+1,2}\to\tilde{G}_{d,1}$.

(ii) *There exist equivariant maps* $\tilde{G}_{4,2}\to\tilde{G}_{3,1}$, f: $\tilde{G}_{7,2}\to\tilde{G}_{7,1}$, g: $\tilde{G}_{8,3}\to\tilde{G}_{8,1}$, and

$S^{n-1} \cong \tilde{G}_{n,1} \to \tilde{G}_{n,k}$ for k odd and $1 \leqslant k \leqslant 8a + 2^b$, where $n = 2^{4a+b}$. (odd), $0 \leqslant b \leqslant 3$, $a \geqslant 0$. Conversely, if there exists an equivariant map $S^{n-1} \to \tilde{G}_{m,k}$, $2k \leqslant m \leqslant n$, then $m = n$, k is odd and $k \leqslant 2^{4a+b}$.

Recall that the span of a smooth manifold M is the maximum number of linearly independent (tangential) vector fields that M admits. It was known [5] since 1985 that $3 \leqslant \text{Span } G_{6,3} \leqslant 7$. As a further application we prove, as a corollary to Theorem 5 proved in § 4 that

Theorem 6. *Span $G_{6,3} = 7$.*

Theorems 1, 2 and 3 are proved in § 2, Theorem 4 in § 3, and Theorems 5 and 6 in § 4. The following result of Stong [13] will be used in the proofs. Recall that for $w_1 \in H^1(G_{n,k}; Z/2)$, $ht(w_1)$: $=$ height $(w_1) = \sup\{m \mid w_1^m \neq 0\}$.

Theorem 7. (*Stong* [13]). *Let $2 \leqslant k \leqslant n/2$, $2^{s-1} < n \leqslant 2^s$. Then*

$$ht(w_1) = \begin{cases} 2^s - 2 \text{ if } k = 2; \text{ or } n = 2^{s-1} + 1,\ k = 3 \\ 2^s - 1 \text{ otherwise.} \end{cases} \qquad \square$$

Unless otherwise mentioned, throughout this paper all cohomology groups will have mod 2 coefficients.

2. Proofs of theorems 1, 2 and 3.

Our proofs are quite elementary as they make use of only basic properties of characteristic classes of vector bundles.

Recall, first, that the Steenrod operations on $w_p(\xi)$, the p-th Stiefel-Whitney class of any vector bundle ξ are given by the Wu relations [10]:

$$Sq^j(w_p(\xi)) = \sum_{0 \leqslant i \leqslant j} \begin{bmatrix} p - j + i - 1 \\ i \end{bmatrix} w_{p+i}(\xi) w_{j-i}(\xi), \tag{3}$$

with the usual conventions that $\begin{bmatrix} -1 \\ 0 \end{bmatrix} = 1$ and $w_0(\xi) = 1$. In particular, if $w_q(\xi) = 0$ for all $q > r$, then

$$Sq^j(w_r(\xi)) = w_r(\xi) w_j(\xi),\ 1 \leqslant j \leqslant r. \tag{4}$$

Secondly, notice from (2) that there are no algebraic relations among $w_1, \ldots, w_k$ in $H^*(G_{n,k})$ in dimensions $\leqslant n - k$. Assume $2k \leqslant n$, and order the w_i's by declaring that $1 = w_0 < w_1 < \ldots < w_k$. Extend this to a simple ordering of all monomials of total degree $\leqslant n - k$, as follows: Let $I = i_1, \ldots, i_p$; $J = j_1, \ldots, j_p$ be non-increasing sequences of non-negative numbers with $i_1, j_1 \leqslant k$. We write $w_I = w_{i_1} \ldots w_{i_p}$. Declare that $w_I > w_J$ if for some r, $0 < r \leqslant p$, $i_1 = j_1, \ldots, i_{r-1} = j_{r-1}$ and $i_r > j_r$. For example, when $n = 20$, $k = 7$, we have $w_7 w_6 > w_7 > w_6^2 > w_6 w_1^5 > w_4^2 w_3 > w_2^5 w_1 > w_2 w_1^5 > w_1^8$.

For $0 \neq a \in H^*(G_{n,k})$, deg $(a) \leqslant n - k$, we denote the largest monomial that occurs in a with coefficient $1 \in Z/2$ by $L(a)$, and define $L(0) = 0$. It is easy to check that if $b \in H^*(G_{n,k})$, deg $(a) + \deg(b) \leqslant n - k$, then

$$L(ab) = L(a)\, L(b).$$

In particular, if $b = w_I$, then

$$L(aw_I) = L(a)\cdot w_I.$$

We caution the reader that homomorphisms in $H^*(\ \ ;Z/2)$ induced by a continuous map between Grassmann manifolds need not be order preserving. However, this ordering turns out to be a useful tool in our proofs.

Proof of Theorem 1. Write $v_i = w_i(\gamma_{n,l})$, and let $u_i = w_i(f^*(\gamma_{n,l})) = f^*(w_i(\gamma_{n,l})) = f^*(v_i)$, $1 \leqslant i \leqslant l$. Then $u_i = h_i(w_1, \ldots, w_i)$ for a suitable polynomial h_i, where $w_j = w_j(\gamma_{n,k})$. Let r be the largest integer for which $u_r \neq 0$, and assume $r \geqslant 1$. Write u_r as

$$u_r = w_m^p g_0 + w_m^{p-1} g_1 + \cdots + g_p \tag{5}$$

where $m \geqslant 1$ is the largest integer such that w_m occurs in the expression of u_r as a polynomial in $w_1, \ldots, w_r$, and $g_i = g_i(w_1, \ldots, w_{m-1})$, $0 \leqslant i \leqslant p$, with $g_0 \neq 0$.

At this stage we comment that the proof involves two steps. First we show that Imf^* is contained in the subring generated by w_1 (see Claim below). Then, using Stong's result on the height of w_1 (Theorem 7) we show that f^* is as asserted in the theorem.

Claim: $m = 1$. To get a contradiction, assume that $m > 1$. Then $2m - 1 > m$.

Case 1: Let $2m - 1 \leqslant k$. Then by (3), $L(Sq(w_j)) = L(Sq^{j-1}(w_j)) = w_{2j-1}$, for $j \leqslant m$. Note that $(m-1)p + mp \leqslant 2r - 1 \leqslant n - k$. Now consider

$$\begin{aligned} Sq^{(m-1)p}(w_m^p) &= (Sq^{(m-1)}(w_m))^p + \text{ terms involving } w_m^2 \\ &= (w_{2m-1} + \cdots + w_m w_{m-1})^p + \text{ terms involving } w_m^2. \\ &= w_{2m-1}^p + \text{ lower terms.} \end{aligned}$$

Again $(m-1)p + r \leqslant 2r - 1 \leqslant n - k$. Now

$$\begin{aligned} Sq^{(m-1)p}(u_r) &= Sq^{(m-1)p}(w_m^p g_0) + \text{ terms smaller than } w_{2m-1}^p, \\ &= Sq^{(m-1)p}(w_m^p) g_0 + \text{ terms smaller than } w_{2m-1}^p, \\ &= w_{2m-1}^p g_0 + \text{ terms smaller than } w_{2m-1}^p. \end{aligned}$$

Therefore if $L(g_0) = w_I$ and $L(u_{(m-1)p}) = w_J$, then by comparing both sides of the equality $L(Sq^{(m-1)p}(u_r)) = L(u_r u_{(m-1)p})$, we get

$$w_{2m-1}^p w_I = L(Sq^{(m-1)p}(u_r)) = L(u_r u_{(m-1)p}) = w_m^p w_I w_J.$$

This is a contradiction as $2m - 1 > m$ and $(m-1)p + r \leqslant n - k$, and there are no algebraic relations among the w_i's in dimensions up to $n - k$. This shows that $m = 1$ in case $2m - 1 \leqslant k$.

Case 2: $2m - 1 > k$. Let j be the smallest integer such that $2j - 1 \geqslant k$. Then u_r can be written as

$$u_r = w_m g_0 + w_{m-1} f_1 + \cdots + w_j f_{m-j} + f_0$$

where $g_0 \neq 0$, f_i, $0 \leqslant i \leqslant m-j$, are polynomials in $w_1, \ldots, w_{j-1}$ only by dimension considerations. (Some of the f_i's can be zero.)
For $j \leqslant \alpha < m$,

$$\begin{aligned} Sq^{r-1}(w_\alpha f_{m-\alpha}) &= Sq^{\alpha-1}(w_\alpha) f_{m-\alpha}^2 + w_\alpha^2 Sq^{r-\alpha-1}(f_{m-\alpha}) \\ &= (w_k w_{2\alpha-1-k} + \cdots + w_\alpha w_{\alpha-1}) f_{m-\alpha}^2 + w_\alpha^2 Sq^{r-\alpha-1}(f_{m-\alpha}) \\ &= w_k w_{2\alpha-1-k} f_{m-\alpha}^2 + \text{ terms not involving } w_k. \end{aligned}$$

Therefore,

$$Sq^{r-1}(u_r) = w_k(w_{2m-1-k} g_0^2 + \Sigma w_{2\alpha-1-k} f_{m-\alpha}^2) + \text{ terms not involving } w_k.$$

The coefficient of w_k in the above is non-zero because in $w_{2m-1-k} g_0^2$, w_{2m-1-k} occurs with odd exponent whereas in other terms it occurs, if at all, with even exponent. Therefore w_k divides $L(Sq^{r-1}(u_r))$. As in the previous case this leads to a contradiction.
This establishes our claim that $m=1$.

Now if $u_r = w_1^r$, then for $j<r$, $u_j u_r = Sq^j(u_r) = \begin{bmatrix} r \\ j \end{bmatrix} w_1^{j+r} = \begin{bmatrix} r \\ j \end{bmatrix} w_1^j u_r$. As $j<r$, $j+r \leqslant 2r-1 \leqslant n-k$, we deduce that $u_j = \begin{bmatrix} r \\ j \end{bmatrix} w_1^j$.

It follows that for the dual Stiefel-Whitney class, $\bar{v}_j$, which is a certain polynomial in $v_1, \ldots, v_l$, we must have

$$f^*(\bar{v}_j) = a_j w_1^j, \text{ for some } a_j \in Z/2.$$

Applying f^* to the relation (1) for the bundle $\gamma_{n,l}$, we see that

$$\left(1 + \begin{bmatrix} r \\ 1 \end{bmatrix} w_1 + \cdots + \begin{bmatrix} r \\ r \end{bmatrix} w_1^r\right)(1 + a_1 w_1 + \cdots + a_{n-l} w_1^{n-l}) = 1.$$

If p is the largest integer for which $a_p = 1$, then we get $w_1^{r+p} = 0$. But $r \leqslant l, p \leqslant n-l \Rightarrow r+p \leqslant n$. Thus $w_1^n = 0$. This contradicts Theorem 7, the result of Stong [13] on the height of w_1 unless $n = 2^s$ and $r = l$; or $n = 2^s - 1$ and $k = 2$. This implies that in case $n \neq 2^s$, and $(n,k) \neq (2^s - 1, 2)$, we must have $r = 0$. It follows that $u_i = 0$ for all $i \geqslant 1$. Since $H^*(G_{n,l})$ is generated by $v_1, \ldots, v_l$, we see that f^* is zero in positive dimensions. If $n = 2^s$, and $r \geqslant 1$, then $r = l$. Thus $u_l = w_1^l$, and $u_i = \begin{bmatrix} l \\ i \end{bmatrix} w_1^i$. If $n = 2^s - 1$, $k = 2$, then $l = 1$, and, $f^*(v_1) = w_1$, when f^* is non-zero in positive dimensions.

Proof of Theorem 2. The proof of our claim that $m=1$ in the above holds even if $n = \infty$. Hence, writing $v_i = w_i(\gamma_{\infty,l})$ and $w_i = w_i(\gamma_{\infty,k})$, for any f: BO(k)$\rightarrow$BO(l), $f^*(v_i) = \begin{bmatrix} r \\ i \end{bmatrix} w_1^i$, $1 \leqslant i \leqslant r$ and $f^*(v_j) = 0$, $j > r$ for some $r \leqslant l$ as before, when f^* is non-zero in positive dimensions.

As for the converse, let $1 \leqslant r \leqslant l$. Let ζ be the line bundle with $w_1(\zeta) = w_1$, the generator of $H^1(G_{\infty,k})$. Let $\eta = r\zeta \oplus (l-r)\varepsilon$, the Whitney sum of r copies of ζ and a trivial vector bundle of rank $(l-r)$. Let f_r: BO(k)$\rightarrow$BO(l) be a classifying map for η, that is $f^*(\gamma_{\infty,l}) = \eta$. Then $f^*(w(\gamma_{\infty,l})) = w(\eta) = (w(\zeta))^r = (1 + w_1)^r$. Hence, $f^*(v_i) = \binom{r}{i} w_1^i$, $1 \leqslant i \leqslant r$, and $f^*(v_j) = 0$ for $j > r$. □

NOTE: R R Patterson [11] has characterized all algebra homomorphisms from $H^*(\mathrm{BO}; Z/2)$ to $H^*(\mathrm{BO}(k); Z/2)$ which respect the Steenrod operations.

Proof of Theorem 3. (i) It is easy to see that if g: $\tilde{G}_{n,k} \to \tilde{G}_{n,l}$ is equivariant then for the induced map f: $G_{n,k} \to G_{n,l}$, $f^*(w_1(\gamma_{n,l})) = w_1(\gamma_{n,k})$. Therefore it follows from Theorem 1, that either $n = 2^s$ and l is odd, or $n = 2^s - 1$, and $k = 2$, $l = 1$.
(ii) Assume the contrary and consider the induced map between the Grassmann manifolds. A contradiction is obtained on comparing the heights of w_1 using Theorem 7 stated in the introduction. □

3. Construction of equivariant maps

Let $\zeta_{n,k}$ (or simply ζ) denote the unique (up to bundle isomorphism) non-trivial line bundle over $G_{n,k}$, so that $w_1(\zeta_{n,k}) = w_1 \in H^1(G_{n,k}) \cong Z/2$. Notice that a map f: $G_{n,k} \to G_{m,p}$ is covered by an equivariant map $\tilde{f}$: $\tilde{G}_{n,k} \to \tilde{G}_{m,p}$ (i.e. $\pi_{m,p} \circ \tilde{f} = f \circ \pi_{n,k}$) if and only if $f^*(\zeta_{m,p}) \approx \zeta_{n,k}$, or equivalently $f^*(w_1(\gamma_{m,p})) = w_1(\gamma_{n,k})$. Another useful characterization of the existence of equivariant maps between oriented Grassmannians in terms of vector bundles over Grassmann manifolds is the following:

PROPOSITION 3.1.

There exists an equivariant map $\tilde{f}$: $\tilde{G}_{n,k} \to \tilde{G}_{m,p}$ if and only if there exist vector bundles ξ and η over $G_{n,k}$ of ranks p and $(m-p)$ respectively such that ξ is non-orientable and $\xi \oplus \eta \approx m\varepsilon$.

Proof. Suppose that $\tilde{f}$:$\tilde{G}_{n,k} \to \tilde{G}_{m,p}$ is equivariant and f: $G_{n,k} \to G_{m,p}$ is the map induced by $\tilde{f}$. Let $\xi = f^*(\gamma_{m,p})$, and $\eta = f^*(\beta_{m,p})$. Then $\xi \oplus \eta \approx f^*(\gamma_{m,p} \oplus \beta_{m,p}) \approx f^*(m\varepsilon) \approx m\varepsilon$. Also since $\tilde{f}$ is equivariant, $w_1(\xi) = f^*(w_1(\gamma_{m,p})) = w_1(\gamma_{n,k}) \neq 0$. Hence ξ is non-orientable.

Conversely, assume that ξ^p and η^{m-p} are vector bundles over $G_{n,k}$ with ξ non-orientable and $\xi \oplus \eta \approx m\varepsilon$. Choose a trivialization φ: $E(\xi \oplus \eta) \to G_{n,k} \times \mathbb{R}^m$. Define f: $G_{n,k} \to G_{m,p}$ by $f(V) = pr_2 \circ \varphi(F_\xi(V))$, where $F_\xi(V)$ denotes the fibre of ξ over $V \in G_{n,k}$, and pr_2 is the projection $G_{n,k} \times \mathbb{R}^m \to \mathbb{R}^m$. Then f is continuous, and $f^*(\gamma_{m,p}) = \xi$. Since ξ is non-orientable, it follows that $w_1(\gamma_{n,k}) = w_1(\xi) = w_1(f^*(\gamma_{m,p})) = f^*(w_1(\gamma_{m,p}))$. Hence f is covered by an equivariant map $\tilde{f}$: $\tilde{G}_{n,k} \to \tilde{G}_{m,p}$.

Proof of Theorem 4(i). Recall from Theorem 1.2 Ch. 8 [4], that any real vector bundle of rank $(d+1)$ over a d-dimensional CW-complex admits a nowhere vanishing section. Therefore, for a suitable d-plane bundle η,

$$(d+1)\zeta \approx \eta \oplus \varepsilon,$$

where ζ is the non-trivial line bundle over $G_{n,k}$.

Tensoring both sides by ζ and observing that $\zeta\otimes\zeta\approx\varepsilon$ as ζ is a line bundle, we obtain

$$(d+1)\varepsilon\approx(\eta\otimes\zeta)\oplus\zeta.$$

Applying Proposition 3.1 we see that there exists an equivariant map

$$h\colon \tilde{G}_{n,k}\to\tilde{G}_{d+1,1}.$$

For any r, and q, $1\leqslant q\leqslant r$, one obtains an equivariant map $\tilde{G}_{n,k}\to\tilde{G}_{d+r,q}$ by suitably composing with h the equivariant inclusions i's and j's mentioned in the introduction.

When $(n,k)=(2^s+1,2)$, from Stong's Theorem, $ht(w_1)=2^{s+1}-2=d$. Therefore there is no map $g\colon G_{n,k}\to G_{d,1}=\mathbb{R}P^{d-1}$ with the property that $g^*(w_1(\gamma_{d,1}))=w_1(\gamma_{n,k})$. Consequently there exists no equivariant map $\tilde{G}_{n,k}\to\tilde{G}_{d,1}$ in this case.

When $(n,k)\neq(2^s+1,2)$, $2\leqslant k\leqslant[n/2]$, we see that $w_d(d\zeta)=w_1(\zeta))^d=w_1^d=0$. The manifold $G_{n,k}$ is orientable if and only if n is even, whereas $d\zeta$ is orientable if and only if d is even. It follows that $w_1(G_{n,k})\neq w_1(d\zeta)$ if and only if n or k is odd, as $d=k(n-k)$. Applying Proposition 3.10(i) of [6], we see that $d\zeta$ admits a nowhere vanishing section, providing $(n,k)\neq(2^s+1,2)$ and n or k is odd.

If n and k are both even, then $d=k(n-k)$ is even, and $G_{n,k}$ is orientable. Write $d=2m$. Then $d\zeta=2m\zeta=m$-fold Whitney sum of the oriented 2-plane bundle 2ζ. The Euler class $e(2\zeta)\in H^2(G_{n,k};Z)$ can be shown to be a torsion element. In fact $2e(2\zeta)=0$. (See Prob. 9A, [9].) It follows that $2e(2m\zeta)=2\cdot(e(2\zeta))^m=0$ in $H^{2m}(G_{n,k};Z)=H^d(G_{n,k};Z)\cong Z$. Hence $e(2m\zeta)=0$. Therefore $2m\zeta$ must admit a nowhere vanishing section over the d-skeleton of $G_{n,k}$, which is the whole of $G_{n,k}$. As before we conclude that there exists an equivariant map $\tilde{G}_{n,k}\to\tilde{G}_{d-1+r,q}$, $1\leqslant q\leqslant r$ in this case, completing the proof of 4(i).

Proof of 4(ii). It can be shown that $\tilde{G}_{4,2}\approx S^2\times S^2$ (cf. p. 104, [3]) under a $Z/2$-equivariant diffeomorphism. Here the $Z/2$-action on $S^2\times S^2$ is given by the map $(a,b)\mapsto(-a,-b)$ for $(a,b)\in S^2\times S^2$. Then the composite

$$\tilde{G}_{4,2}\xrightarrow{\cong}S^2\times S^2\xrightarrow{\text{proj}}S^2$$

is $Z/2$-equivariant in our sense.

Using the properties of the 2-fold vector product $v\cdot\mathbb{R}^7\times\mathbb{R}^7\to\mathbb{R}^7$ and the 3-fold vector product $\mu\colon \mathbb{R}^8\times\mathbb{R}^8\times\mathbb{R}^8\to\mathbb{R}^8$, as given by Zvengrowski [14], we construct equivariant maps $f\colon \tilde{G}_{7,2}\to\tilde{G}_{7,1}$ and $g\colon \tilde{G}_{8,3}\to\tilde{G}_{8,1}$ as follows: if (a,b) is an ordered basis in the orientation on $V\in\tilde{G}_{7,2}$, then $v(a,b)\in S^6=\tilde{G}_{7,1}$ depends only on the oriented vector space V and not on the specific choice of the basis in the orientation of V. We let $f(V)=v(a,b)$. Then f is continuous. If $-V$ is the same vector space as V but with opposite orientation, then $f(-V)=v(-a,b)=-v(a,b)=-f(V)$ (cf. [14]). Hence f is equivariant. The map $g\colon \tilde{G}_{8,3}\to\tilde{G}_{8,1}$ is defined similarly. See [14].

We now construct equivariant maps $S^{n-1}=\tilde{G}_{n,1}\to\tilde{G}_{n,k}$ for k odd and $k\leqslant\rho(n)$, where $\rho(n)-1$ is the span of S^{n-1}. From Adams [1], $\rho(n)=8a+2^b$ where $n=2^{4a+b}\cdot$(odd), $0\leqslant b\leqslant 3$, $a\geqslant 0$. If $V_{n,k}$ denotes the Stiefel manifold of orthonormal k-frames in $\mathbb{R}^n$, the bundle projection $p\colon V_{n,k}\to S^{n-1}$, $p(v_1,\ldots,v_k)=v_1$ admits a $Z/2$-equivariant section $s\colon S^{n-1}\to V_{n,k}$ if $k\leqslant\rho(n)$. Here the $Z/2$ action on $V_{n,k}$ is given by $(a_1,\ldots,a_k)\mapsto(-a_1,\ldots,-a_k)$ for each $(a_1,\ldots,a_k)\in V_{n,k}$. The projection map $q\colon V_{n,k}\to\tilde{G}_{n,k}$ is equivariant if k is odd. Therefore the composite $q\circ s\colon \tilde{G}_{n,1}=S^{n-1}\to\tilde{G}_{n,k}$ is equivariant if k is odd, $k\leqslant\rho(n)$.

Now suppose that h: $G_{n,1} = \mathbb{R}P^{n-1} \to G_{m,k}$ is induced by an equivariant map $\tilde{h}$: $S^{n-1} \to \tilde{G}_{m,k}$ with $2k \leqslant m \leqslant n$. Then the map h^*: $H^*(G_{m,k}) \to H^*(\mathbb{R}P^{n-1}) = (Z/2)[x]/\langle x^n \rangle$ has the property that $h^*(w_1) = x$, where $w_i = w_i(\gamma_{m,k})$. Let $r \leqslant k$, and $s \leqslant m-k$ be the largest integers such that $h^*(w_r) \neq 0$ and $h^*(\bar{w}_s) \neq 0$, respectively. Applying h^* to the relation $w \cdot \bar{w} = 1$, and comparing $(r+s)$-th degree terms on both sides we obtain $0 = h^*(w_r) \cdot h^*(\bar{w}_s) = x^r \cdot x^s = x^{r+s}$. Hence $r + s \geqslant n$. Since $r + s \leqslant m \leqslant n$, it follows that $r = k$, $s = m - k$ and $m = n$. Now $h^*(w_k) = x^k$ implies, by Wu's formula, that $h^*(w) = (1+x)^k$. Since $x = h^*(w_1) = \binom{k}{1} x$ we see that k must be odd. Also, $h^*(w) = (1+x)^k$ implies $h^*(\bar{w}) = (1+x)^{-k} = (1+x)^{2^N - k}$ for large enough N, as $(1+x)^{2^N} = 1$ for N large. This implies that

$$h^*(\bar{w}_j) = \begin{bmatrix} 2^N - k \\ j \end{bmatrix} x^j, \quad 1 \leqslant j \leqslant n-k.$$

On the other hand,

$$\begin{aligned} h^*(\bar{w}_j)x^{n-k} &= h^*(\bar{w}_j \bar{w}_{n-k}) = h^*(Sq^j(\bar{w}_{n-k})) \\ &= Sq^j(h^*(\bar{w}_{n-k})) = Sq^j(x^{n-k}) = \begin{bmatrix} n-k \\ j \end{bmatrix} x^{n-k+j}. \end{aligned}$$

Therefore we must have

$$\begin{bmatrix} 2^N - k \\ j \end{bmatrix} \equiv \begin{bmatrix} n-k \\ j \end{bmatrix} \bmod 2, \quad 1 \leqslant j < k.$$

Using Lucas' Theorem [12], p. 5, it can now be seen that if $2^{p-1} < k \leqslant 2^p$, then $n \equiv 0 \bmod 2^p$. This completes the proof. □

Remarks. i) It is possible that Theorems 1 and 3 are true even without the restriction that $n \geqslant k + 2l - 1$. For $l = 2$, (and $l < k \leqslant [n/2]$) this condition is automatically satisfied. For $l = 3$, the only exception is the case $k = 4$, $n = 8$ in Theorem 1. In this case one directly shows that Theorem 1 holds.
ii) The question of existence of equivariant maps of $\tilde{G}_{n,k}$ into $\tilde{G}_{n,l}$ in the case $1 \leqslant k < l \leqslant [n/2]$ seems to be much more difficult to handle, and perhaps requires a quite different approach in order to obtain better results than Theorem 4(ii).
iii) Theorem 2 is perhaps well-known to experts in the field, but has been included here for the sake of completeness.

4. Span of $m\zeta_{n,k}$

For a vector bundle ξ and let Span (ξ) denote the maximum number of everywhere linearly independent cross sections that ξ admits. Using our results on the existence of equivariant maps, in this section we obtain estimates for Span $(m\zeta_{n,k})$, where $\zeta_{n,k}$ denotes the non-trivial line bundle over $G_{n,k}$. Note that when $k = 1$, this is the generalized vector field problem. We will assume that $2 \leqslant k \leqslant [n/2]$, and that $m \geqslant 1$.

Theorem 5. *Let $2^{s-1} < n \leqslant 2^s$, $2 \leqslant k \leqslant [n/2]$, and let $d = \dim G_{n,k}$.*

i) *Span* $((2^s-2)\zeta_{n,2}) =$ *Span* $((2^s-2)\zeta_{2^{s-1}+1,3}) = 0$, *and for* $k \geqslant 3$, $(n,k) \neq (2^{s-1}+1, 3)$, *Span* $((2^s-1)\zeta_{n,k}) = 0$.
ii) *Span* $(m\zeta_{n,k}) \geqslant$ *Span* $(m\zeta_{d,1})$ *for all* $m \geqslant 1$ *provided* $(n,k) \neq ((2^{s-1}+1), 2)$. *Also, Span* $(m\zeta_{2^{s-1}+1,2}) \geqslant$ *Span* $(m\zeta_{d+1,1})$, and $(d\zeta_{2^{s-1}+1,2}) = 0$.

$$\text{iii) } \mathit{Span}\,(m\zeta_{4,2}) = \begin{cases} 4[m/4] & \text{if } m \equiv 0,1,2 \bmod 4 \\ 4[m/4]+1 & \text{if } m \equiv 3 \bmod 4 \end{cases}.$$

$$\begin{aligned} \mathit{Span}\,(m\zeta_{5,2}) &= \mathit{Span}\,(m\zeta_{6,2}) = \mathit{Span}\,(m\zeta_{7,2}) \\ &= \mathit{Span}\,(m\zeta_{7,1}) = \begin{cases} 8[m/8]+1 & \text{if } m \equiv 7 \bmod 8 \\ 8[m/8] & \text{otherwise} \end{cases}. \end{aligned}$$

$$\begin{aligned} \mathit{Span}\,(m\zeta_{6,3}) &= \mathit{Span}\,(m\zeta_{7,3}) = \mathit{Span}\,(m\zeta_{8,3}) \\ &= \mathit{Span}\,(m\zeta_{8,1}) = 8[m/8]. \end{aligned}$$

Proof: i) $w_{2^s-2}((2^s-2)\zeta_{n,2}) = w_1^{2^s-2} \neq 0$ by Theorem 7. Hence span $((2^s-2)\zeta_{n,2}) = 0$. Other cases follow by the same argument.
ii) Let $(n,k) \neq (2^{s-1}+1, 2)$. As shown in the proof of 4. (i), there exists a map $g = G_{n,k} \to G_{d,1}$ such that $g^*(\zeta_{d,1}) \approx \zeta_{n,k}$. Hence Span $(m\zeta_{n,k}) \geqslant$ Span $(m\zeta_{d,1})$. Similarly Span $(m\zeta_{2^{s-1}+1,2}) \geqslant$ Span $(m\zeta_{d+1,1})$. To show that Span $(d\zeta_{2^{s-1}+1,2}) = 0$ we observe that $d = 2(2^{s-1}-1) = 2^s - 2 = ht(w_1)$ and so $w_d(\zeta_{2^{s-1}+1,2}) = w_1^d \neq 0$.
iii) By 4. (ii), we observe that the following compositions are equivariant.

$$\tilde{G}_{5,2} \overset{i}{\hookrightarrow} \tilde{G}_{6,2} \overset{i}{\hookrightarrow} \tilde{G}_{7,2} \xrightarrow{f} \tilde{G}_{7,1},$$

$$\tilde{G}_{6,3} \overset{i}{\hookrightarrow} \tilde{G}_{7,3} \overset{i}{\hookrightarrow} \tilde{G}_{8,3} \xrightarrow{g} \tilde{G}_{8,1}.$$

Passing to Grassmann manifolds, and pulling back the line bundle $\zeta_{7,1} \approx \gamma_{7,1}$ over $G_{7,1} = \mathbb{R}P^6$ one obtains $\mathrm{Span}(m\zeta_{5,2}) \geqslant \mathrm{Span}(m\zeta_{7,2}) \geqslant \mathrm{Span}(m\zeta_{7,1})$.
From Theorem 1.1 of [7], we obtain $\mathrm{Span}(m\zeta_{7,1})$ to be as stated.
To show that $\mathrm{Span}(m\zeta_{5,2}) = \mathrm{Span}(m\zeta_{7,1})$, we use a Stiefel-Whitney class argument. On $G_{5,2} w_1^6 \neq 0$. Therefore for $1 \leqslant m \leqslant 6$, $w_m(m\zeta_{5,2}) = w_1^m \neq 0$.
The proofs for other cases are similar. □

Remark. The above result enables us to determine the order of $[\zeta_{n,k}] \in \mathrm{KO}(G_{n,k})$ for $n \leqslant 8$ except for the case $n = 8$, $k = 4$. For example, $0([\zeta_{7,4}]) = 0([\zeta_{7,3}]) = 0([\zeta_{8,1}]) = 8$.

Proof of Theorem 6. It is well known [8] that a stable normal bundle for the Grassmannian $G_{n,k}$ is $\lambda^2(\gamma_{n,k}) \oplus \lambda^2(\beta_{n,k})$. On $G_{6,3}$, $\gamma = \gamma_{6,3}$ and $\beta = \beta_{6,3}$ are non-orientable 3-plane bundles.

Hence by 10.3, Ch. 12 of [4], $\lambda^2(\gamma) \approx \gamma \otimes \zeta$, $\lambda^2(\beta) \approx \beta \otimes \zeta$ where $\lambda^3(\gamma) = \zeta$ is the non-trivial line bundle over $G_{6,3}$. Hence a stable normal bundle to $G_{6,3}$ is

$$(\gamma \otimes \zeta) \oplus (\beta \otimes \zeta) \approx (\gamma \oplus \beta) \otimes \zeta \approx 6\varepsilon \otimes \zeta \approx 6\zeta.$$

Since by Theorem 5(iii) 8ζ is trivial it follows that the tangent bundle τ of $G_{6,3}$ is stably equivalent to $2\zeta \oplus 7\varepsilon$. Thus stable span of $G_{6,3}$ is 7. It is known due to Korbaš

[5] that Span $G_{6,3} \geqslant 3$. But from Prop. 20.8 and Corollary 20.5 of [6], one has Span $G_{6,3} = 1$ or Span $G_{6,3}$ = stable span of $G_{6,3} = 7$. It follows that Span $G_{6,3} = 7$.

Acknowledgements

J Korbaš wishes to thank the A von Humboldt foundation and Prof. U. Koschorke for their support during this research. P Sankaran was supported in part by an NSERC grant.

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