FROBENIUS SPLITTING OF CERTAIN RINGS OF INVARIANTS

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Dedicated to Professor Melvin Hochster on the occasion of his sixty-fifth birthday.

Abstract: Let k be an algebraically closed field of characteristic $p > 0$, and V an n-dimensional k-vector space together with a non-degenerate symmetric bilinear form. Let G denote one of the groups $G = \mathrm{SL}(V)$ or $\mathrm{SO}(V)$ where we assume that $p > 2$ if $G = \mathrm{SO}(V)$. Let R denote the coordinate ring of $V_{m,q} := V^{\oplus m} \oplus V^{*\oplus q}$ (resp. $V_m := V^{\oplus m}$) if $G = \mathrm{SL}(V)$ (resp. if $G = \mathrm{SO}(V)$), V^* being the dual of V. The defining representation of G on V induces the diagonal action of G on $V_{m,q}$ (resp. V_m). Let $S = R^G$. In this paper, we show that S is Frobenius split.

1. Introduction

The concept of F-purity was introduced by Hochster-Roberts [6]; the F-purity for a noetherian ring of prime characteristic is equivalent to the splitting of the Frobenius map, when the ring is finitely generated over its subring of p-th powers. It is closely related to the Frobenius splitting property á la Mehta-Ramanathan [10] for algebraic varieties; to make it more precise, the F-split property for an irreducible projective variety X over an algebraically closed field of positive characteristic is equivalent to the F-purity of the ring $\oplus_{n\geq 0} H^0(X; L^n)$ for any ample line bundle L over X (cf.[3],[13],[14]). We feel that it is but appropriate to dedicate this paper to Professor Hochster on the occasion of his sixty-fifth birthday and thus make a modest contribution to this birthday volume.

Let k be an algebraically closed field of characteristic $p > 0$ and let X be a k-scheme. One has the Frobenius morphism (which is only

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an $\mathbb{F}_p$-morphism) $F\colon X\longrightarrow X$ defined as the identity map of the underlying topological space of X, the morphism of structure sheaves $F^{\#}\colon \mathcal{O}_X\longrightarrow\mathcal{O}_X$ being the p-th power map. The morphism F induces a morphism of $\mathcal{O}_X$-modules $\mathcal{O}_X\longrightarrow F_*\mathcal{O}_X$. The variety X is called *Frobenius split* (or *F-split* or, merely, *split*) if there exists a splitting $\varphi\colon F_*\mathcal{O}_X\longrightarrow\mathcal{O}_X$ of the morphism $\mathcal{O}_X\longrightarrow F_*\mathcal{O}_X$. Equivalently, X is Frobenius split if there exists a morphism of sheaves of abelian groups $\varphi\colon \mathcal{O}_X\longrightarrow\mathcal{O}_X$ such that (i) $\varphi(f^pg) = f\varphi(g)$, $f, g \in \mathcal{O}_X$ and (ii) $\varphi(1) = 1$. Basic examples of varieties that are Frobenius split are smooth affine varieties, toric varieties (cf. [1]), generalized flag varieties, and Schubert varieties [10]. Smooth projective curves of genus greater than 1 are examples of varieties that are *not* Frobenius split.

Frobenius splitting is an interesting property to study. If X is Frobenius split, then it is weakly normal (cf. [1], Prop 1.2.5) and reduced (cf. [1], Prop. 1.2.1). Indeed, projective varieties which are Frobenius split are very special. We refer the reader to [1] for further details.

If $X = \mathrm{Spec}(R)$, then X is Frobenius split if and only if the Frobenius homomorphism $R\longrightarrow R$ defined as $a \mapsto a^p$ admits a splitting $\varphi\colon R\longrightarrow R$ such that $\varphi(a^pb) = a\varphi(b)$, and $\varphi(1) = 1$.

If a linearly reductive group G acts on a k-algebra R which is Frobenius split, then the invariant ring R^G is Frobenius split (see [1, Exercise 1.1..E(5)]). To quote Karen Smith [13, p. 571], "The story of F-splitting and global F-regularity for quotients by reductive groups in characteristics p that are not linearly reductive is much more subtle and complicated". We shall show that although the groups $\mathrm{SO}(n), n \geq 3$, and $\mathrm{SL}(n), n \geq 2$, are not linearly reductive, it turns out that certain rings of invariants for these groups *are* Frobenius split.

We state below the main results of this paper.

Let k be an algebraically closed field of characteristic $p > 2$ and V an n-dimensional vector space over k with a symmetric non-degenerate bilinear form. Denote by A_m the the coordinate ring of $V_m := V^{\oplus m}, m \geq 1$, and consider the action of $\mathrm{SO}(V)$ on A_m induced by the diagonal action of $\mathrm{SO}(V)$ on $V^{\oplus m}$. Then

Theorem 1.1. *The invariant ring $A_m^{SO(V)}$ is Frobenius split for all $m \geq 1$.*

The group $\mathrm{SL}(V)$ acts on V, as well as on the dual vector space $V^* = \mathrm{Hom}_k(V, k)$. Now consider the diagonal action of $\mathrm{SL}(V)$ on the vector space $V_{m,q} := V^{\oplus m} \oplus V^{*\oplus q}$, $m, q \geq 1$. This leads to an action of $\mathrm{SL}(V)$ on the coordinate ring $A_{m,q}$ of $V_{m,q}$.

Theorem 1.2. *The invariant ring $A_{m,q}^{SL(V)}$ is Frobenius split for any $m, q \geq n$.*

We shall now sketch the proofs of the main results (assuming $m, q > n$). Let S be the invariant ring in question. Let R be the ring of invariants under the larger group $\widetilde{G} = \mathrm{GL}(V)$ (resp. $\widetilde{G} = \mathrm{O}(V)$), we have (cf. [2, 8]) that R is the coordinate ring of a certain determinantal variety in $M_{m,q}$, the space of $m \times q$ matrices (resp. $Sym\, M_m$, the space of symmetric $m \times m$ matrices) with entries in k. Now a determinantal variety in $M_{m,q}$ (resp. $Sym\, M_m$) can be canonically identified (cf. [8]) with an open subset in a certain Schubert variety in $G_{q,m+q}$, the Grassmannian variety of q-dimensional subspaces of k^{m+q} (resp. the symplectic Grassmannian variety, the variety of all maximal isotropic subspaces of a $2m$-dimensional vector space over k endowed with a non-degenerate skew-symmetric bilinear form). Hence we obtain that R is Frobenius split (since Schubert varieties are Frobenius split). Let $X = \mathrm{Spec}(S), Y = \mathrm{Spec}(R)$, and $\pi\colon X \longrightarrow Y$, the morphism induced by the inclusion $R \subset S$. When $G = \mathrm{SO}(V)$, we show that π is a double cover which is étale over a 'large' open subvariety – that is a subvariety whose complement has codimension at least 2. When $G = \mathrm{SL}(n)$, we show that restricted to a large open subvariety, π is a $\mathbb{G}_m$ bundle. The main theorems are then deduced using normality of S.

Theorem 1.1 can also be deduced from Hashimoto's work [4], wherein it is shown that if a reductive group G acts on a polynomial ring A over k (of positive characteristic) with good filtration, then the ring A^G of invariants is strongly F-regular. Our Theorem 1.2 does not seem to follow from the results of [4]. Granting the results of [9] and [7]—we don't need all the results of these papers, only some of the relatively easier ones—the arguments used in our proofs are straightforward and quite elementary; the techniques used in [4] are representation theoretic.

Theorem 1.1 will be proved in §2 and Theorem 1.2 in §3.

2. Splitting SO(n)-invariants

Let k be an algebraically closed field of characteristic $p > 0$. Suppose that S is an affine k-algebra which is Frobenius split and that a finite group Γ acts on S as k-algebra automorphisms. Then the invariant ring $R = S^\Gamma$ is Frobenius split provided the order of Γ is not divisible by p (cf. [1, Ex. 1.1.E(5)]). We first obtain a partial converse to this statement in the case when Γ is of order 2.

Assume that $\mathrm{char}(k) > 2$. Let S be an affine k-domain and let $\Gamma = \{1, \gamma\} \cong \mathbb{Z}/2\mathbb{Z}$ act effectively on S. Denote by R the invariant subalgebra S^Γ. Then R is an affine k-algebra and S is quadratic and integral over R. Indeed, any $b \in S$ can be expressed as $b = b_0 + b_1$ where $b_0 = (1/2)(b + \gamma(b)) \in R$ and $b_1 = (1/2)(b - \gamma(b))$ satisfies $\gamma(b_1) = -b_1$. Thus, we can choose generators $u_1, \cdots, u_s$ for the R-algebra S to be in the -1 eigenspace of γ. Clearly $u_i^2 = -u_i\gamma(u_i) =: p_i \in R$ for all $i \leq s$. Furthermore,

$\gamma(u_i u_j) = u_i u_j =: p_{i,j} \in R$ for all $i, j \leq s$ (with $p_{i,i} = p_i$). Observe that $p_{i,j}^2 = p_i p_j$.

We shall assume that S is reduced so that $p_i \neq 0$, for all i. Now let $R_i = R[1/p_i], 1 \leq i \leq n$. Let $S_i = S[1/u_i]$. We claim that $S_i = R_i[u_i]$. To see this, first observe that $R_i[u_i] \subset S[1/u_i]$, since $1/p_i = (1/u_i)^2 \in S[1/u_i]$. To show that $S[1/u_i] \subset R_i[u_i]$, it suffices to show that $u_j \in R_i[u_i]$ for all j and $(1/u_i) \in R_i[u_i]$. Indeed, $1/u_i = u_i/u_i^2 = u_i/p_i \in R_i[u_i]$ and so $u_j = p_{i,j}/u_i \in R_i[u_i]$.

Write $X = \mathrm{Spec}(S), Y = \mathrm{Spec}(R)$ and let $\pi : X \longrightarrow Y$ be the morphism (induced by the inclusion $R \subset S$). As above, let $S_i = S[1/u_i]$, and let $U_i = \mathrm{Spec}(S_i) \subset X$ and let $U := \bigcup_{1\leq i\leq s} U_i$; it is the full inverse image under π of $\bigcup_{1\leq i\leq s} \mathrm{Spec}(R_i)$. It is readily verified that $\pi|U \colon U \longrightarrow \pi(U)$ is étale. Indeed, S_i is a free R_i module with basis $\{1, u_i\}$ and $det(u_i) = -p_i \neq 0$ and so $\pi|U_i$ is étale. Hence $\pi|U$ is étale.

On the other hand, if $y \in Y$ is a closed point such that $p_i(y) = 0$ for all $i \leq s$, then the fibre $f^{-1}(y) = \mathrm{Spec}(S_y \otimes_{R_y} k)$ is the scheme whose coordinate ring is $S_y \otimes_{R_y} k = k[u_1, \cdots, u_s]/\langle u_i^2,\ 1 \leq i \leq s\rangle$. Here R_y is the local ring at y. Thus $f^{-1}(y)$ is non-reduced. It follows that the ramification locus of π *equals* $Y \setminus \pi(U)$. (See [12, §III.10, Theorem 3].)

Proposition 2.1. *Let k be an algebraically closed field of characteristic $p > 2$. Let S be an affine normal domain over k acted on by $\Gamma \cong \mathbb{Z}/2\mathbb{Z}$ and let $R := S^\Gamma$ be Frobenius split. Suppose that the ramification locus of the double cover $\pi \colon Spec(S) \longrightarrow Spec(R)$ has codimension at least 2 in $Spec(R)$. Then, any splitting $\varphi : R \longrightarrow R$ extends uniquely to a splitting $\psi \colon S \longrightarrow S$.*

Proof. We use the notations introduced above.

Since X is normal and the codimension of the the ramification locus of π is at least 2, it suffices to show that $U = \bigcup_{1\leq i\leq s} U_i$ is Frobenius split (cf. [1], Lemma 1.1.7,(iii)).

Let $\varphi : R \longrightarrow R$ be a splitting of $Y = \mathrm{Spec}(R)$. First, we shall extend φ to a splitting $\psi_i \colon S_i \longrightarrow S_i$ of $U_i = \mathrm{Spec}(S_i) (= \mathrm{Spec}(R_i[u_i]))$ for each i and verify that these splittings agree on the overlaps $U_i \cap U_j$ for $1 \leq i, j \leq s$. Thus we will obtain a splitting of $U = \bigcup_{1 \leq i \leq s} U_i$. By normality of X and the hypothesis on the codimension of the ramification locus, we will conclude that this splitting extends to a splitting of X. Next, we shall establish the uniqueness of the extension.

Recall that $\{1, u_i\}$ is an R_i-basis for S_i. Since $u_i = u_i^{-p} p_i^{(1+p)/2}$ on U_i, if $\psi_i \colon S_i \longrightarrow S_i$ is *any* splitting of U_i which extends the splitting φ_i of R_i defined by φ, we must have $\psi_i(au_i) = \psi_i((1/u_i)^p a p_i^{(p+1)/2}) = (1/u_i)\varphi_i(a p_i^{(p+1)/2})$. By additivity, we must have

$$\psi_i(au_i + b) = (1/u_i)\varphi_i(a p_i^{(p+1)/2}) + \varphi_i(b) = (u_i/p_i)\varphi_i(a p_i^{(p+1)/2}) + \varphi_i(b)$$

where $a, b \in R_i$. Thus the extension ψ_i, if it exists, is unique.

We now *define* ψ_i by the above equation and claim that ψ_i is indeed a splitting of S_i. First, observe that $\psi_i(1) = 1$, by the very definition of ψ_i.

Now, for any $x, y, a \in R_i$, we have

$\psi_i((xu_i + y)^p a u_i) = \psi_i(x^p p_i^{(p+1)/2} a + y^p a u_i)$

$= x\varphi(p_i^{(p+1)/2} a) + y\varphi(au_i)$

$= x(p_i/u_i)\psi_i(au_i) + y\varphi(au_i)$

$= xu_i\psi_i(au_i) + y\psi(au_i)$

$= (xu_i + y)\varphi(au_i)$

An entirely similar (and easier) computation shows that

$\psi_i((xu_i + y)^p b) = (xu_i + y)\psi_i(b)$, completing the verification that ψ_i is a splitting.

We verify, by another straightforward computation, that these ψ_i agree on the overlaps $U_i \cap U_j$. Indeed, writing $u_j = u_i p_{i,j}/p_i$, we have

$\psi_i(u_j) = \psi_i(u_i p_{i,j}/p_i) = (u_i/p_i)\varphi((p_{i,j}/p_i) p_i^{(p+1)/2})$

$= (u_i/p_i)\varphi(p_{i,j} p_i^{(p-1)/2})$

Since $p_i = p_{i,j}^2/p_j$ on $U_i \cap U_j$, we have

$\varphi(p_{i,j} p_i^{(p-1)/2}) = \varphi(p_{i,j}^p (p_i/p_{i,j}^2)^{(p-1)/2})$

$= p_{i,j}\varphi(p_j^{(1-p)/2}) = (p_{i,j}/p_j)\varphi(p_j^{(p+1)/2})$

Substituting in the above expression for $\psi_i(u_j)$ we get

$$\psi_i(u_j) = (u_i p_{i,j}/(p_i p_j))\varphi(p_j^{(p+1)/2}) = (u_j/p_j)\varphi(p_j^{(p+1)/2}) = \psi_j(u_j)$$

This implies that the extensions $\{\psi_i\}$ patch to yield a well-defined splitting on U as claimed. As observed above, the normality of X and the hypothesis on the codimension of the ramification locus implies that this splitting extends to a *unique* splitting $\psi\colon S \longrightarrow S$.

Finally, if ψ' is another splitting of X which also extends φ, then ψ' and ψ agree on U_i (for any i) as already observed. As X is irreducible, U_i is dense in X and we conclude that $\psi' = \psi$. □

As a corollary, we obtain the following

Theorem 2.2. *Let $\pi : X \longrightarrow Y$ be a double cover of a Noetherian scheme whose ramification locus has codimension at least 2. Suppose that X is reduced, irreducible and normal and that Y is Frobenius split, then X is Frobenius split.*

Proof. Cover X by finitely many affine patches X_α. Let $Y_\alpha := \pi X_\alpha$. Then each $\pi|_{X_\alpha}$ satisfies the hypotheses of the above proposition. Let φ be a splitting of Y and let ψ_α be the unique splitting of X_α that extends the splitting $\varphi|_{Y_\alpha}$. The ψ_α's agree on overlaps and hence define a unique splitting of X which 'extends' φ. □

We now turn to proof of Theorem 1.1.

Proof of Theorem 1.1. Denote by S the ring of $\mathrm{SO}(V)$-invariants of A_m, where A_m is the coordinate ring of V_m. Let R be the ring of $\mathrm{O}(V)$-invariants.

We shall assume that $m > n$. By [2, 8] we have that $Y := \mathrm{Spec}(R)$ is the determinantal variety $D_n(Sym\, M_m)$ consisting of all matrices in $Sym\, M_m$ (the space of symmetric $m \times m$ matrices with entries in k) of rank at most n; further, we have (cf. [8]) an identification of $D_n(Sym\, M_m)$ with an open subset of a certain Schubert variety in the Lagrangian Grassmannian variety (of all maximal isotropic subspaces of a $2m$-dimensional vector space over k endowed with a non-degenerate skew-symmetric bilinear form). Hence we obtain that Y is Frobenius split (since Schubert varieties are Frobenius split (cf. [10])).

Observe that $\Gamma := \mathrm{O}(n)/\mathrm{SO}(n) \cong \mathbb{Z}/2\mathbb{Z}$ acts on S (the subring of $\mathrm{SO}(V)$-invariants of A_m) and that S^Γ equals R. As above, let

$X := \mathrm{Spec}(S)$, and $\pi : X \longrightarrow Y$ be the morphism induced by the inclusion $R \subset S$. We need only verify the hypotheses of Theorem 2.2. It is well-known that S is an affine normal domain. It remains to verify that the codimension of the branch locus is at least two. This was proved in [7]. In fact, the ramification locus of Y *equals* the singular locus of Y, but this more refined assertion is not relevant here. Since Y is normal it follows that the codimension of the ramification locus is at least 2. Therefore, by Theorem 2.2, X is Frobenius split.

The case $m = n$ is isolated separately as Lemma 2.3 below. When $m < n$, it is easy to see that $S = R$. Again, R is a polynomial algebra over k and hence S is Frobenius split. □

Assume that $m = n$. In this case $R = k[y_{i,j} : 1 \leq i \leq j \leq n]$ is a polynomial ring, being the ring of polynomial functions on the space of $n \times n$ symmetric matrices. As an R-algebra, $S = R[u]/\langle u^2 - f\rangle$, where f denotes the determinant function of the symmetric $n \times n$ matrix whose entry in position (i, j) for $1 \leq i \leq j \leq n$ is $y_{i,j}$.

Lemma 2.3. *Let $m = n$. The ring S of* $\mathrm{SO}(V)$*-invariants is Frobenius split in this case also.*

Proof. There is a natural identification of $\mathrm{Spec}(R)$ with an affine patch of the symplectic Grassmannian and the vanishing locus of f under this identification becomes an open part of a Schubert variety [8, 7]. Thus by [10] (see also [1]), there exists a splitting of $\mathrm{Spec}(R)$ that compatibly splits $\mathrm{Spec}(R/(f))$. Let φ be such a splitting. Continue to denote by φ the restriction of φ to the open part $\mathrm{Spec}(R[1/f])$. Arguing as in the proof of Proposition 2.1 above, we may 'lift' the restriction φ to a splitting (also denoted φ) of $\mathrm{Spec}(S[1/f])$. We claim that φ maps S to S and hence extends to a splitting of $\mathrm{Spec}(S)$. Indeed, a general element of S is of the form $au + b$ with a, b in R, so that $\varphi(au + b) = \varphi(\frac{au^{p+1}}{u^p} + b) = \frac{\varphi(af^{(p+1)/2})}{u} + \varphi(b)$. Since φ compatibly splits the vanishing locus of f, it follows that $\varphi(af^{(p+1)/2})$ belongs to the ideal (f). Writing $\varphi(af^{(p+1)/2}) = cf$, we have $\varphi(au + b) = \frac{cf}{u} + \varphi(b) = cu + \varphi(b) \in S$. □

We conclude this section with the following remarks.

Remark 2.4. (i) The condition on codimension of U in Proposition 2.1 will be satisfied if S is generated over R by two or more elements u_i such that there exist u_i, u_j such that the supports D_i and D_j of the reduced scheme defined by $u_i = 0$ and $u_j = 0$ do not have any component in common.

(ii) Theorem 2.2 is not valid when the hypothesis on the codimension of the ramification locus is not satisfied. For example, if $\Gamma \cong \mathbb{Z}/2\mathbb{Z}$ is generated by the involution of a hyperelliptic curve X of genus $g \geq 2$, then the quotient is a smooth projective curve which is Frobenius split. However, X is not split since ω_X is ample but $H^1(X;\omega) \cong k$, whereas higher cohomologies for ample line bundles over Frobenius split projective varieties vanish.

(iii) We do not know if Theorem 2.2 remains valid if Γ is *any* finite group whose order is prime to the characteristic p of k, even in the case when Γ is cyclic.

(iv) One has an isomorphism of SL(2) with SO(3) such that the SO(3) action on $V = k^3$ corresponds to the conjugation action of SL(2) on trace zero 2×2 matrices. In this case the Frobenius splitting property of A_m was proved by Mehta-Ramadas [11, Theroem 6]. It should be noted that when $\dim V = 3$, the completion of the stalk at the origin in A_m is isomorphic to the completion of the stalk at the point corresponding to the class of the trivial rank 2 vector bundle in the moduli space of equivalence classes of semi-stable, rank 2 vector bundles with trivial determinant on a smooth projective curve of genus $m > 2$ (see [11]).

3. Splitting SL(n) invariants

In this section we shall establish Theorem 1.2. Let V be an n dimensional vector space over an algebraically closed field k of characteristic $p \geq 2$ and let V^* denote its dual. Let $V_{m,q} := V^{\oplus m} \oplus V^{*\oplus q}$, and let A denote the ring of regular functions on $V_{m,q}$. By fixing dual bases for V and V^*, we shall view elements of V and V^* as row and column vectors respectively, so that $V^{\oplus m}$ (resp. $V^{*\oplus q}$) is identified with the space $M_{m,n}$ of $m \times n$ matrices (resp. the space $M_{n,q}$ of $n \times q$ matrices) over k; further, $GL(V)$ gets identified with $\mathrm{GL}_n(k)$ (the group of invertible $n \times n$ matrices over k). In the sequel, we shall denote $\mathrm{GL}_n(k)$ by just GL(n). Then the action of GL(V) on $V^{\oplus m}$ gets identified with the multiplication on the right of $M_{m,n}$ by GL(n). Similarly, the action of $g \in \mathrm{GL}(V)$ on $V^{*\oplus q}$ gets identified with the multiplication on the left of $M_{n,q}$ by g^{-1}. The diagonal action of GL(V) on $V^{\oplus m} \oplus V^{*\oplus}$ is therefore defined as $(u,\xi).g = (ug, g^{-1}\xi)$ where $g \in \mathrm{GL}(n)$ and $(u,\xi) \in \mathcal{M}_{m,q} := M_{m,n} \times M_{n,q}$. We identify A with the coordinate ring of $\mathcal{M}_{m,q}$.

We denote by R and S the rings of invariants $A^{\mathrm{GL}(n)}$ and $A^{\mathrm{SL}(n)}$ respectively. Let $Y = \mathrm{Spec}(R)$ and $X = \mathrm{Spec}(S)$. Note that Y and X are the GIT quotients $\mathcal{M}_{m,q}/\!\!/\mathrm{GL}(n)$ and $\mathcal{M}_{m,q}/\!\!/\mathrm{SL}(n)$ respectively.

Let $m, q \geq n$. By [2, 8] we have that Y is the determinantal variety $D_n(M_{m,q})$ consisting of all matrices in $M_{m,q}$ (the space of $m \times q$ matrices with entries in k) of rank at most n; further, we have (cf. [8])

an identification of $D_n(M_{m,q})$ with an open subset of a certain Schubert variety in the Grassmannian variety (of q-dimensional subspaces of k^{m+q}). Hence we obtain that Y is Frobenius split (since Schubert varieties are Frobenius split). The multiplication map $\mu\colon \mathcal{M}_{m,q} \longrightarrow M_{m,q}$ factors through Y; further, under $\pi\colon X \longrightarrow Y$ (induced by the inclusion $R \subset S$), we have, $\pi([u,\xi]) = u\xi \in M_{m,q}$ where $[u,\xi] \in X$ is the image of $(u,\xi) \in \mathcal{M}_{m,q}$ under the GIT quotient $\mathcal{M}_{m,q} \longrightarrow X$.

Let $I(n,m)$ denote the set of all n-element subsets I of $\{1,2,\cdots,m\}$. Any such I determines a regular function $u_I\colon \mathcal{M}_{m,q} \longrightarrow k$ which maps (u,ξ) to the determinant of the $n\times n$ submatrix $u(I)$ of $u \in M_{m,n}$ with column entries given by I. Clearly u_I is invariant under the action of $\mathrm{SL}(n)$ on $\mathcal{M}_{m,q}$ and hence yields a regular function u_I on X.

We define $\xi(J)$ and ξ_J for $J \in I(n,q)$ analogously; ξ_J is also an $\mathrm{SL}(n)$-invariant.

We have, $u_I\xi_J =: p_{I,J} \in R$ for all $I \in I(n,m), J \in I(n,q)$; indeed $p_{I,J}([u,\xi])$ is just the determinant of the $n\times n$ submatrix of $u\xi \in M_{m,q}$ with row and column indices given by I and J respectively. It is shown in [9], among other things, that S is generated as an R-algebra by $u_I, \xi_J,\ I \in I(n,m), J \in I(n,q)$, the ideal of relations being generated by $u_I\xi_J - p_{I,J},\ I \in I(n,m), J \in I(n,q)$ together with certain quadratic relations among the u_I's and certain quadratic relations among the ξ_J's. Further, in [9], a standard monomial basis is constructed for S; as a particular consequence, we have that each u_I (resp. ξ_J) is algebraically independent over R for $I \in I(n,m)$ (resp. $J \in I(n,q)$).

For $K \in I(n,m), L \in I(n,q)$, let

$$R_{K,L} = R[1/p_{K,L}], Y_{K,L} = \mathrm{Spec}(R_{K,L})$$

For a given $I \in I(n,m), J \in I(n,q)$, let

$$Y_I = \bigcup_{J' \in I(n,q)} Y_{I,J'},\ Y_J = \bigcup_{I' \in I(n,m)} Y_{I',J}$$

Note that for $I \in I(n,m)$, any $Y_{I,J'}$ is contained in Y_I; similarly, for $J \in I(n,q)$, any $Y_{I',J}$ is contained in Y_J.

Set $X_I = \pi^{-1}(Y_I) \subset X$ and $X_J = \pi^{-1}(Y_J) \subset X$. Note that u_I (resp. ξ_J) is non-zero on X_I (resp. X_J). Denote by $f_I\colon X_I \longrightarrow Y_I \times k^*$ the morphism $f_I = (\pi|X_I, u_I|X_I)$, and by $f_J\colon X_J \longrightarrow Y_J \times k^*$ the morphism $f_J = (\pi|X_J, \xi_J|X_J)$.

Lemma 3.1. *The morphisms $f_I\colon X_I \longrightarrow Y_I \times k^*$ and $f_J\colon X_J \longrightarrow Y_J \times k^*$ are isomorphisms for any $I \in I(n,m), J \in I(n,q)$.*

Proof. We shall prove that f_I is an isomorphism, the proof in the case of f_J being the same.

Let $X_{I,J} = \pi^{-1}(Y_{I,J})$; then $X_{I,J}$ equals $\mathrm{Spec}(S_{I,J})$ (where $S_{I,J} = S[1/p_{I,J}]$) and $X_{I,J}$ is contained in X_I. The morphism $f_{I,J}\colon X_{I,J}\longrightarrow Y_{I,J}$ defined by the restriction of f_I is induced by the $R_{I,J}$-algebra map $f^*_{I,J}\colon R_{I,J}[t,t^{-1}]\longrightarrow S_{I,J}$ which maps t to u_I. Note that $p_{I,J} = u_I\xi_J$ implies that u_I is invertible in $S_{I,J}(= S[1/p_{I,J}])$.

We must show that

(1) $f^*_{I,J}$ is an isomorphism of k-algebras
(2) $f_{I,J}$ and $f_{I,J'}$ agree on the overlap $X_{I,J} \cap X_{I,J'}$ for any two $J, J' \in I(n,q)$.

(1). Note that $u_{I'} = u_I u_{I'}\xi_J/p_{I,J} = u_I p_{I',J}/p_{I,J} = f^*_{I,J}(p_{I',J}/p_{I,J}t)$. Hence $u_{I'}$ is in the image of $f^*_{I,J}$ for any $I' \in I(n,m)$. Similarly $\xi_{J'}$ is also in the image of $f^*_{I,J}$ for any $J' \in I(n,q)$. Therefore $f^*_{I,J}$ is surjective. Now suppose that $P(t) \in R_{I,J}[t,t^{-1}]$ is in the kernel of $f^*_{I,J}$. We may assume that $P(t)$ is a polynomial in t and that the coefficients of $P(t)$ are actually in R. Then $0 = f^*_{I,J}(P(t)) = P(u_I)$. Since $X_{I,J}$ is open in X, which is irreducible, the we see that the equation $P(u_I) = 0$ must hold in S. This contradicts the fact that u_I is algebraically independent over R (cf. [9],Theorem 6.06,(3)). Hence $f^*_{I,J}$ is an isomorphism.

(2). It is evident that $f^*_{I,J}(t) = u_I \in S_{I,J}$ and $f^*_{I,J'}(t) = u_I \in S_{I,J'}$ both restrict to the same regular function, namely $u_I|X_{I,J} \cap X_{I,J'}$, on the overlap $X_{I,J} \cap X_{I,J'} = \mathrm{Spec}(S[1/p_{I,J}, 1/p_{I,J'}])$. It follows that $f_{I,J}$ and $f_{I,J'}$ agree on $X_{I,J} \cap X_{I,J'}$. This completes the proof that f_I is an isomorphism. □

Observe that, if $J, J' \in I(n,q)$, then $\xi_J/\xi_{J'} \in S[1/\xi_{J'}]$ defines a regular function on $Y_{J'}$. This is because, on $Y_{I,J'}$, $\xi_J/\xi_{J'} = (u_I\xi_J)/(u_I\xi_{J'}) = p_{I,J}/p_{I,J'}$. It is immediately seen that, on $Y_{I,J'} \cap Y_{I',J'}$, the two regular functions $p_{I,J}/p_{I,J'}$ and $p_{I',J}/p_{I',J'}$ agree. Therefore we conclude that $\xi_J/\xi_{J'}$ is a well-defined regular function on $Y_{J'}$. Clearly it is invertible on $Y_J \cap Y_{J'}$. Similar statements concerning $u_I/u_{I'}$ hold for any $I, I' \in I(n,m)$.

Notation: Let $m, q \geq n$.

Denote by $\mathcal{I}$ the disjoint union $I(n,m)\coprod I(n,q)$. We set

$$\lambda_{\beta,\alpha} = \begin{cases} u_\alpha/u_\beta & if \quad \alpha,\beta \in I(n,m), \\ \xi_\beta/\xi_\alpha & if \quad \alpha,\beta \in I(n,q), \\ p_{\alpha,\beta} & if \quad \beta \in I(n,q), \alpha \in I(n,m), \\ 1/p_{\beta,\alpha} & if \quad \beta \in I(n,m), \alpha \in I(n,q). \end{cases}$$

Consider the covering $\{Y_\alpha\}_{\alpha\in\mathcal{I}}$ of the open subvariety $Y_0 := \bigcup_{\alpha\in\mathcal{I}} Y_\alpha \subset Y$. The cocycle condition $\lambda_{\alpha,\beta}\lambda_{\beta,\gamma} = \lambda_{\alpha,\gamma}$ is readily verified for any $\alpha,\beta,\gamma \in \mathcal{I}$. Thus we obtain a $\mathbb{G}_m$-bundle over Y_0; call it $\mathcal{E}$.

Let $X_0 := \bigcup_{\alpha\in\mathcal{I}} X_\alpha$.

Lemma 3.2. *Assume that $m, q \geq n$. With the above notations, the total space of the $\mathbb{G}_m$-bundle $\mathcal{E}$ over Y_0 is isomorphic to the open subvariety $X_0 := \bigcup_{\alpha\in\mathcal{I}} X_\alpha \subset X$.*

Proof. The total space of the $\mathbb{G}_m$-bundle corresponding to $\mathcal{D}$ is $\coprod_{\alpha\in\mathcal{I}} Y_\alpha \times k^*/\sim$ where $(\pi([u,\xi]),t) \in Y_\alpha \times k^*$ is identified with $(\pi([u;\xi]), \lambda_{\beta,\alpha}(\pi([u,\xi]).t)) \in Y_\beta \times k^*$ whenever $\pi([u,\xi]) \in Y_\alpha \cap Y_\beta$. One has the following commuting diagram for any $\alpha,\beta \in \mathcal{I}$:

$$\begin{array}{ccccccc}
Y_\alpha \times k^* & \supset & (Y_\alpha \cap Y_\beta) \times k^* & \xrightarrow{\lambda_{\beta,\alpha}} & (Y_\beta \cap Y_\alpha) \times k^* & \subset & Y_\beta \times k^* \\
f_\alpha \uparrow & & f'_\alpha \uparrow & & \uparrow f'_\beta & & \uparrow f_\beta \\
X_\alpha & \supset & X_\alpha \cap X_\beta & == & X_\beta \cap X_\alpha & \subset & X_\beta
\end{array}$$

where f'_α is the restriction of f_α. Since, by Lemma 3.1, the f_α's are isomorphism of varieties, it follows that that the total space of the $\mathbb{G}_m$ bundle over Y_0 is isomorphic to the union $X_0 := \bigcup_\alpha X_\alpha \subset X$. □

We shall now compute the codimension of $Z := X - X_0$. We give the reduced scheme structure on Z. It is evident that Z is defined by the equations $p_{I,J} = 0, \forall I \in I(n,m), J \in I(n,q)$. We claim $Z = Z_u \cup Z_\xi$ where Z_u is the closed subvariety with reduced scheme structure defined by the equations $u_I = 0, \forall I \in I(n,m)$ and Z_ξ, by the equations $\xi_J = 0, \forall J \in I(n,q)$. Clearly $Z_u \cup Z_\xi \subset Z$. On the other hand, if $[u,\xi]$ is not in $Z_u \cup Z_\xi$, then $u_I([u,\xi]) \neq 0$ for some I and $\xi_J([u,\xi]) \neq 0$ for some J. This implies that $p_{I,J}([u,\xi] \neq 0$. Hence $[u,\xi] \in X_0$. Thus $Z_u \cup Z_\xi = Z$.

Lemma 3.3. *Let $m > n$ (resp. $q > n$). Then the codimension of Z_u (resp. Z_ξ) in X is at least* 2.

Proof. Consider the closed subvariety $M_u := D_{n-1}(M_{m,n}) \times M_{n,q} \subset \mathcal{M}_{m,q}$ (with reduced scheme structure). We have,

$dim\, M_u = (n-1)(m+1) + nq$ (note the dimension of the determinantal variety $D_t(M_{r,s})$ (consisting of $r \times s$ matrices of rank at most t) equals $t(r+s-t)$ (cf. [8])). Clearly M_u is stable under the SL(n)-action and $M_u /\!\!/ \mathrm{SL}(n) = Z_u$. We shall find an open subset $Z_{u,0}$ of Z_u such that SL(n) acts *freely* on the inverse image of $Z_{u,0}$ under the quotient morphism $\eta\colon M_u \longrightarrow Z_u$ and $\eta^{-1}(Z_{u,0}) /\!\!/ \mathrm{SL}(n) = Z_{u,0}$. It would then follow that $\dim(Z_u) = \dim(\eta^{-1}(Z_u)) - \dim(\mathrm{SL}(n)) = (n-1)(m+1) + nq - (n^2-1) = (m+n)q - (n^2-1) - (m-n+1) = \dim(X) - (m-n+1) \leq \dim(X) - 2$ (note that $dim\, X = (m+n)q - (n^2-1)$ (cf. [9])).

Define

$$W_u = D_n(M_{m,n}) \times M^0_{n,q}$$

where $M^0_{n,q} := \{\xi \in M_{n,q} \mid \xi_J(\xi) \neq 0, \text{ for some } J \in I(n,q)\}$. Then W_u is the inverse image of

$$Z_{u,0} := \{[u,\xi] \mid \xi_J(\xi) \neq 0\}$$

under the quotient morphism $\eta\colon M_u \longrightarrow Z_u$. The assertion that the $\mathrm{SL}(n)$-action is free on W_u follows from the fact that the $\mathrm{SL}(n)$-action on $M^0_{n,q}$ is free.

An entirely similar argument shows that Z_ξ has codimension at least 2, and consequently codimension of Z in X is at least 2. □

We are now ready to prove Theorem 1.2.

Proof of Theorem 1.2. Let $m, q > n$. As already observed, we have that $Y = D_n(M_{m,q})$ can be identified with an open subset of a certain Schubert variety in the Grassmann variety $\mathrm{SL}(m+q)/P_q$ of q-dimensional vector subspaces in k^{m+q}. Since Schubert varieties in the Grassmann variety are Frobenius split, it follows that Y is Frobenius split. Since Y_0 is open in Y, it follows that it is also Frobenius split. The variety X_0, being the total space of a $\mathbb{G}_m$-bundle over Y_0, is Frobenius split by [1],Lemma 1.1.11. Now X being normal and codimension of X_0 in X being at least 2, it follows that X is Frobenius split (cf. [1],Lemma 1.1.7,(iii)).

If $m, q < n$, then $X = Y = M_{m,q}$ and hence Frobenius split. The case $m = n$ is isolated separately as Lemma 3.4 below. □

Assume that $q = n = m$. In this case $Y = M_{n,n}$. Denote the (i,j)-th coordinate function on Y by $y_{i,j}$. The set $I(n,m)$ and $I(n,q)$ are singletons and so $S = R[u,\xi]/(u\xi - f)$ where f is the determinant function on $Y = M_{m,q}$.

Lemma 3.4. *Let $q = n = m$. The ring S of $\mathrm{SL}(V)$-invariants is Frobenius split in this case also.*

Proof. Let φ be a splitting of $\mathrm{Spec}(R)$. Continue to denote by φ the restriction of φ to the open part $\mathrm{Spec}(R[1/f])$. We can 'lift' φ to the $\mathbb{G}_m$-bundle $\mathrm{Spec}(R[1/f][u,u^{-1}])$ (over $\mathrm{Spec}(R[1/f])$) as follows: define $\varphi(a + \sum b_i u^i + \sum c_j u^{-j}) := \varphi(a) + \sum \varphi(b_i) u^{i/p} + \sum \varphi(c_j) u^{-j/p}$, where the summations are over positive integers and $u^{i/p}$ (respectively $u^{-j/p}$) is interpreted to be 0 unless i (respectively $-j$) is an integral multiple of p. Observe that $R[1/f][u,u^{-1}] = S[1/f]$, so we have a splitting of $\mathrm{Spec}(S[1/f])$ which we still denote φ. We claim that φ maps S to S and hence extends to a splitting of $\mathrm{Spec}(S)$. Indeed, a general element s of S

is of the form $a+\sum b_i u^i+\sum c_j\xi^j$ with a, b_i, and c_j in R, so that $\varphi(s) = \varphi(a+\sum b_i u^i+\sum c_j f^j u^{-j}) = \varphi(a)+\sum\varphi(b_i)u^{i/p}+\sum\varphi(c_j f^j)u^{-j/p}$. Rewriting $\varphi(c_j f^j)u^{-j/p}$ as $\varphi(c_j)f^{j/p}u^{-j/p}=\varphi(c_j)\xi^{j/p}$, we see that $\varphi(s)$ belongs to S. □

Remark 3.5. In the case when one of $\{m, q\}$ being $< n$, and the other $\geq n$, we expect the ring of invariants to be Frobenius split though at the moment, we do not have a proof of this assertion!

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