Proc. Indian Acad. Sci. (Math. Sci.), Vol. 109, No. 1, February 1999, pp. 11–21.

Dolbeault cohomology of compact complex homogeneous manifolds

VIMALA RAMANI and PARAMESWARAN SANKARAN

SPIC Mathematical Institute, 92 G N Chetty Road, Chennai 600 017, India
E-mail: vimala@smi.ernet.in; sankaran@smi.ernet.in

MS received 27 October 1997; revised 9 December 1998

Abstract. We show that if M is the total space of a holomorphic bundle with base space a simply connected homogeneous projective variety and fibre and structure group a compact complex torus, then the identity component of the automorphism group of M acts trivially on the Dolbeault cohomology of M. We consider a class of compact complex homogeneous spaces W, which we call generalized Hopf manifolds, which are diffeomorphic to $S^1 \times K/L$ where K is a compact connected simple Lie group and L is the semisimple part of the centralizer of a one dimensional torus in K. We compute the Dolbeault cohomology of W. We compute the Picard group of any generalized Hopf manifold and show that every line bundle over a generalized Hopf manifold arises from a representation of its fundamental group.

Keywords. Dolbeault cohomology; complex homogeneous manifolds; generalized Hopf manifolds; automorphism groups; Picard group.

1. Introduction

Let M be a compact complex manifold. It is a deep result of Bochner and Montgomery [7] that the group $\mathrm{Aut}(M)$ of all biholomorphic automorphisms of M is a complex Lie group and the action map is complex analytic. If M is a connected homogeneous complex manifold, then $M = G/H$ for some closed subgroup $H \subset G$, where $G = \mathrm{Aut}_1(M)$ denotes the connected component of $\mathrm{Aut}(M)$ that contains the identity map of M. Furthermore, if M is simply connected then a result of Montgomery [17] says that $M = K/L$ for a maximal compact connected subgroup of G and L a closed connected subgroup of K.

It is well known from the work of Wang [19] that if K is any compact semisimple Lie group and L a closed connected Lie subgroup of K whose semisimple part equals the semisimple part of the centralizer of a toral subgroup, then $M = K/L$ admits a K-invariant complex structure compatible with the usual differentiable structure on M provided it is even dimensional. Conversely, if a compact semisimple Lie group K acts transitively and biholomorphically on a simply connected compact complex manifold M then Wang [19] shows that M can be expressed as K/L where L is a closed Lie subgroup of K such that the semisimple part of L coincides with the semisimple part of the centralizer of a toral subgroup of K. If the second Betti number of M is zero, then M admits uncountably many invariant complex structures. As Wang noted, this class of complex manifolds includes the Calabi–Eckmann manifolds, the Stiefel manifold $V_{n,2k}$ when it is even dimensional, and the product $V_{m,2l} \times V_{n,2k}$ when it is even dimensional. Here $V_{n,k}$ denotes the Stiefel manifold of orthonormal k-frames in $\mathbb{R}^n, k < n$.

Recently, Lescure [15] has constructed a compact complex manifold M with a $\mathbb{C}^*$-action such that the induced action on $H^1(M, \mathcal{O}_M)$ is nontrivial. (See Remark 1 in §2 below, and [16].) The earliest example of a (necessarily non-Kähler) manifold which is homogeneous compact complex with the action of $\mathrm{Aut}_1(M)$ on the Dolbeault cohomology of M being nontrivial is due to Kodaira. (See [11].) Akhiezher [1] has proved that if M is compact and is a quotient of a connected complex reductive algebraic group G, then the action of G on $H^q(M, \mathcal{O}_M)$, $q \geq 0$, is trivial. (cf. [2].) When M is a simply connected compact complex homogeneous space, M fibres over a (generalized) flag variety, with fibre and structure group a compact complex torus T, that is, M is the total space of a *Tits bundle*. (A locally trivial holomorphic bundle $\pi : M \longrightarrow X$ with M compact and X a flag variety is called a Tits bundle if $\pi' : M \longrightarrow X'$ is any other such bundle, then there exists a morphism of varieties $q : X \longrightarrow X'$ such that $\pi' = q \circ \pi$. An equivalent condition is that the fibre of π be a parallelizable complex manifold. See [3].)

We shall prove the following

Theorem 1. *Suppose that M is a connected compact complex homogeneous manifold which fibres over a flag variety $X = G/P$, G being a connected complex semisimple Lie group and P a parabolic subgroup, with fibre and structure group a complex torus $T(\cong (S^1)^{2k})$. Then $\mathrm{Aut}_1(M)$ acts trivially on $H^{p,q}(M)$ for all $p, q \geq 0$.*

Our proof of the above theorem makes use of the Borel spectral sequence [8] applied to the bundle $p : M \longrightarrow G/P$ with fibre and structure group a complex torus T. Our proof closely follows the argument in the proof of Lemma 9.3, [8] showing that the group $\mathrm{Aut}_1(M)$ acts trivially on the Dolbeault cohomology of M when M is a Calabi–Eckmann manifold.

We shall introduce the notion of generalized Hopf manifolds which are connected to compact complex homogeneous manifolds and which fibre over a projective variety G/P, where G is a complex simple Lie group and P a maximal parabolic subgroup of G, with fibre and structure group a one dimensional complex torus. Since the second Betti number of a generalized Hopf manifold W vanishes, it is non-Kähler. (See theorem 3.) We establish the vanishing of $H^{1,0}(W)$ and use this to compute the Dolbeault cohomology of a generalized Hopf manifold W. As an example we compute the Dolbeault cohomology of $S^1 \times V_{n,2}$. We also compute the Dolbeault cohomology of $V_{m,2} \times V_{n,2}$. In the last section we compute the Picard group of W when W is a generalized Hopf manifold.

As the fundamental group of a generalized Hopf manifold is infinite cyclic (theorem 3), the class of generalized Hopf manifolds does not form a subclass of the class of compact complex homogeneous manifolds which were classified by Wang.

2. Action of $\mathrm{Aut}_1(M)$ on Dolbeault cohomology

Let M be a compact connected complex manifold which is the total space of an analytic bundle $p : M \longrightarrow G/P$, where G is a complex semisimple Lie group and P a parabolic subgroup, with fibre and structure group a complex torus T. By [6] one knows that any automorphism of M preserves this fibre bundle structure and so one obtains a homomorphism $\phi : \mathrm{Aut}_1(M) \longrightarrow \mathrm{Aut}_1(G/P)$. Again since G/P is simply connected, using theorem 3 of [6], one sees that the kernel of this homomorphism is T, and so one obtains an exact sequence

$$1 \longrightarrow T \longrightarrow \mathrm{Aut}_1(M) \xrightarrow{\phi} \mathrm{Aut}_1(G/P).$$

The automorphism group of the projective variety G/P has been studied by Onishchik, Tits and others (see [4],[3]). In particular it is known that if G is simple and P is a parabolic subgroup, then $\mathrm{Aut}_1(G/P)$ is the group $\bar{G} = G/Z(G)$ except in three cases. These exceptional cases are $\mathrm{Aut}_1(SO(2n+1)/P_n) = PSO(2n+2)$, $\mathrm{Aut}_1(\mathrm{Sp}(2n)/P_1) = PSL(2n)$, and $\mathrm{Aut}_1(G_2/P_1) = PSO(7)$. (Here we follow the conventions of [9] for indexing maximal parabolic subgroups of a complex simple Lie group.) In fact using theorem 2.2, Chap. 2, Part II of [10], one sees that if X is any of the above three varieties and G is *any* complex Lie group that acts transitively and effectively on X then G has to be one of the two obvious candidates. (For example, when $X = \mathrm{Sp}(2n)/P_1$, one has $G = P\mathrm{Sp}(2n)$ or $PSL(2n)$.)

Suppose $G = \prod_{1 \le i \le k} G_i$ with each G_i simple, and $P = \prod_{1 \le i \le k} P_i$, $P_i \subset G_i$ then $\mathrm{Aut}_1(G/P) = \prod_{1 \le i \le k} \mathrm{Aut}_1(G_i/P_i)$ [6].

Note that any subgroup of $\mathrm{Aut}_1(G/P)$ that acts transitively has to be *semisimple*. Indeed, let $H \subset \mathrm{Aut}_1(G/P)$ be a complex Lie subgroup that acts transitively on the flag variety G/P. Then $G/P \cong H/Q$ for some closed subgroup $Q \subset H$. Since G/P is simply connected and complete, Q must be a parabolic subgroup of H. If $N = \mathrm{rad}(H)$, the radical of H, then $N \subset Q$. Since N is normal in H, the action of N on $X = H/Q$ has to be trivial. Since $H \subset \mathrm{Aut}(G/P)$ the H action on G/P is effective. It follows that $N = 1$ and so H must be semisimple. In particular, the image of $\phi : \mathrm{Aut}_1(M) \longrightarrow \mathrm{Aut}_1(G/P)$ has to be semisimple because the action of $\mathrm{Aut}_1(M)$ on M is transitive. We have proved

PROPOSITION 2

Let M be a compact complex homogeneous manifold which has the structure of a holomorphic principal bundle over a flag variety with fibre and structure group a compact complex torus T. Then $Aut_1(M)$ is an extension of a semisimple Lie group by the torus T. □

Proof of Theorem 1. Our proof mimics the argument in the proof of Lemma 9.3, [8]. The automorphism group $\mathrm{Aut}_1(M)$ acts on the Borel spectral sequence for $p : M \longrightarrow G/P$. Restricting this action to the group $T \subset \mathrm{Aut}_1(M)$ we see that T acts trivially on $H^{p,q}(T)$ as the fibre T is Kählerian and the action of T on the base space G/P is in fact the identity automorphism. Hence T acts trivially on the E_2 diagram of the Borel spectral sequence. It follows that it acts trivially on E_∞ as well. Complete reducibility of T implies that it acts trivially on the Dolbeault cohomology M.

The same argument shows that for any connected compact subgroup $K \subset \mathrm{Im}(\phi)$ the group $\phi^{-1}(K)$ acts trivially on the Dolbeault cohomology of M. Indeed, since the T action on the Borel spectral sequence is trivial, the action of the group $\phi^{-1}(K)$ on the Borel spectral sequence is induced by the action of the group K on the cohomology of G/P. Since K acts trivially on the cohomology of G/P, we see that $\phi^{-1}(K)$ acts trivially on the E_2-diagram of the Borel spectral sequence. Again complete reducibility for the compact group $\phi^{-1}(K)$ implies that the action of $\phi^{-1}(K)$ on the Dolbeault cohomology of M is trivial.

We have shown that the homomorphism $\mathrm{Aut}_1(M) \longrightarrow GL(H^{p,q}(M))$ factors through ϕ and hence yields a homomorphism $\rho : \mathrm{Im}(\phi) \longrightarrow GL(H^{p,q}(M))$. The kernel of this homomorphism is a normal subgroup of the semisimple Lie group $\mathrm{Im}(\phi)$ which contains all compact subgroups of $\mathrm{Im}(\phi)$. It follows that kernel of ρ must be the whole of $\mathrm{Im}(\phi)$ and hence the action of $\mathrm{Aut}_1(M)$ on $H^{p,q}(M)$ is trivial. This completes the proof. □

Remark. (1) Lescure [16] has shown that if a complex Lie group G acts holomorphically on a compact complex manifold M, then the induced representation of G in the Dolbeault cohomology $H^{p,q}(M) = H^q(M; \Omega^p_M)$ of M is holomorphic. Furthermore, given a positive integer q, a semisimple complex Lie group G and a finite dimensional irreducible holomorphic representation R of G, Lescure has constructed a compact complex manifold M with a G action on it such that the induced G action on $H^q(M;\mathcal{O}_M)$ contains R as a sub representation. See § 3 of [16].

(2) The automorphism groups of Calabi–Eckmann manifolds have been computed by A Blanchard [6]. This result is inaccurately quoted in [8]. However, the proof of Lemma 9.3 in [8] is valid with the correct description of the automorphism groups as given in [6].

(3) We do not know if the identity component of the automorphism group of a complex manifold M as in theorem 1 is *algebraic*, assuming that T is an abelian variety. When $M = S^1 \times S^{2n-1}, n > 1$, is the Hopf manifold, one knows that $\mathrm{Aut}_1(M) = GL(n,\mathbb{C})/\Gamma =: G$ where Γ is an infinite cyclic discrete central subgroup of $GL(n,\mathbb{C})$. (cf. [6].) Using the isomorphism $SL(n,\mathbb{C}) \times_Z \mathbb{C}^* = GL(n,\mathbb{C})$, where $Z \cong \mathbb{Z}/n$ is the centre of $SL(n,\mathbb{C})$, and the fact that $\Gamma \cap Z \subset GL(n,\mathbb{C})$ is trivial it follows that $G \cong SL(n,\mathbb{C}) \times_Z E$, where E is the elliptic curve $(\mathbb{C}^*)/\Gamma$. It follows that G is algebraic since it is a quotient of the algebraic group $SL(n,\mathbb{C}) \times E$ by a *finite* central subgroup Z. When M is the Calabi–Eckmann manifold $S^{2n-1} \times S^{2m-1}$, $n, m > 1$, one knows [6] that $\mathrm{Aut}_1(M) = GL(n,\mathbb{C}) \times GL(m,\mathbb{C})/\Gamma$ where $\Gamma \cong \mathbb{C}$ is a certain complex analytic subgroup of the centre $D \cong \mathbb{C}^* \times \mathbb{C}^*$ of $GL(n,\mathbb{C}) \times GL(m,\mathbb{C})$ such that $\Gamma \cap (SL(n,\mathbb{C}) \times SL(m,\mathbb{C}))$ is trivial. Again one shows that $\mathrm{Aut}_1(M) \cong (SL(n,\mathbb{C}) \times SL(m,\mathbb{C})) \times_Z E$ where E is the elliptic curve D/Γ, and Z is the centre of $SL(n,\mathbb{C}) \times SL(m,\mathbb{C})$. Thus $\mathrm{Aut}_1(M)$ is an algebraic group in this case.

3. Generalized Hopf manifolds

Let G be a simply connected complex semisimple Lie group and P be a maximal parabolic subgroup of G containing a fixed Borel subgroup B. We assume that a maximal torus $A \cong (\mathbb{C}^*)^l$ contained in B is fixed. Note that since P is a maximal parabolic, $X = G/P$ is isomorphic to $\bar{G}/\bar{P}$ where $\bar{G}$ is a suitable simple Lie group. So we may (and we do) assume without loss of generality that G itself is simple. Let E denote the total space of the principal $\mathbb{C}^*$ bundle associated to the line bundle $\mathcal{O}_X(-1)$. Let c be any complex number with $|c| > 1$ and $\phi : E \longrightarrow E$ denote the bundle map $e \mapsto e.c$ for any $e \in E$. Then ϕ generates an action of $\mathbb{Z}$ on E which is free and properly discontinuous. The quotient space W_c (or simply W) is a compact connected complex manifold which fibres over X with fibre and structure group the complex torus $T = \mathbb{C}^*/\langle c\rangle \cong \mathbb{C}^*/\mathbb{Z}$ with periods $\{1,\tau\}$ where $\exp(2\pi i\tau) = c$, $\tau \in \mathbb{C}$. Note that $\mathrm{Im}(\tau) \neq 0$ as $|c| > 1$. When $X = SL(n)/P_1 = \mathbb{P}^{n-1}$, one has $E = \mathbb{C}^n \setminus \{0\}$ and W is the Hopf manifold $S^1 \times S^{2n-1}$. For this reason *we call W a generalized Hopf manifold.*

Let V denote the complex vector space which is the dual of the space of all sections of the line bundle $\mathcal{O}_X(1)$. (Here $X = G/P$.) As is well-known, V is the fundamental representation for G with highest weight ϖ, where ϖ is the fundamental weight stabilized by the Weyl group of P. Choosing a basis for V (consisting of weight vectors), one obtains a G-equivariant embedding $f : X \longrightarrow \mathbb{P}(V)$, under which the identity coset is mapped to $\mathbb{C}e$, where $e \in V$ is a highest weight vector. We regard f as an "inclusion". The tautological bundle over $\mathbb{P}(V)$ restricts to the bundle $\mathcal{O}_X(-1)$ and so E is the inverse image of X under the projection $V_0 = V \setminus \{0\} \longrightarrow \mathbb{P}(V)$.

Let K be a maximal compact subgroup of G. We assume that $K \cap A$ is a maximal torus of K. The group K acts on V by restriction. We put a K-invariant hermitian metric on the complex vector space V. Let $e \in E$ be a highest weight vector of norm 1, and let L be the stabilizer of e. The orbit of e under K is thus isomorphic to K/L. Since the action of K on X is transitive, the projection $E \longrightarrow X$ restricts to a surjective map $p : K/L \longrightarrow X$. The map p is the projection of the sphere bundle associated to the real 2-plane bundle underlying the complex line bundle $\mathcal{O}_X(-1)$ over X. Alternatively $p : K/L \longrightarrow X$ may also be regarded as the principal S^1 bundle obtained by reducing the structure group $\mathbb{C}^* \cong S^1 \times \mathbb{R}^+$ of $E \longrightarrow X$ to the group S^1, where $\mathbb{R}^+$ is the multiplicative group of positive reals. In particular this S^1-bundle is canonically oriented.

Theorem 3. *With notation as above, L is the semisimple part of the centralizer of a subgroup S of K isomorphic to S^1. In particular the space K/L is 2-connected. The space $W = E/\langle c \rangle$ is diffeomorphic to $K/L \times S^1$.*

Proof. In the Serre spectral sequence of the sphere bundle $p : K/L \longrightarrow X$ with fibre and structure group S^1, the "positive" generator of $H^1(S^1; \mathbb{Z})$ transgresses to the first Chern class of $\mathcal{O}_X(-1)$ which is a generator of $H^2(X; \mathbb{Z}) \cong \mathbb{Z}$. It follows that $H^2(K/L; \mathbb{Z}) = 0 = H^1(K/L; \mathbb{Z})$. From the homotopy exact sequence of the sphere bundle $p : K/L \longrightarrow X$, one sees that the fundamental group of K/L is cyclic. Since both $H^1(K/L; \mathbb{Z})$ and $H^2(K/L; \mathbb{Z})$ vanish, it follows that K/L must be simply connected. This implies that in the homotopy exact sequence the map $\pi_2(X) \longrightarrow \pi_1(S^1) \cong \mathbb{Z}$ must be surjective since $\pi_2(X) \cong \mathbb{Z}$. It follows that $\pi_2(X) \longrightarrow \pi_1(S^1)$ is an isomorphism. Therefore K/L is 2-connected.

Since K is connected and K/L is simply connected we conclude that L must be connected. Since K is semisimple, and K/L is 2-connected, it follows that L is also semisimple. The projective variety X is the quotient of K by a subgroup M which equals the centralizer of a toral subgroup $S \cong S^1$. Since $L \subset M$, it follows that L is contained in the centralizer of S. Clearly L must equal the semisimple part of the centralizer of S since L is semisimple and $M/L \cong S^1$.

Write $c = \exp(2\pi i\tau), \tau = \alpha + i\beta$. The map $\Phi : E \longrightarrow \mathbb{R} \times K/L$ defined by $u \mapsto (t, \exp(-2\pi i\alpha t)u/\|u\|)$ where t is the unique real number such that $\exp(2\pi\beta t) = 1/\|u\|$, is a diffeomorphism. Also it can be verified that Φ is $\mathbb{Z}$ equivariant where the $\mathbb{Z}$ action on $\mathbb{R} \times K/L$ is generated by $(t, v) \mapsto (t+1, v)$. Hence Φ defines a diffeomorphism of W onto $S^1 \times K/L$. □

Conversely, starting with any semisimple Lie group K and a subgroup $L \subset K$ which is the semisimple part of the centralizer of a circle group $S \cong S^1$, one can show that K/L is the total space of a differentiable fibre bundle with base $K/C(S)$, where $C(S)$ is the centralizer of S, having fibre and structure group S^1. Furthermore, the space $X = K/C(S)$ has the structure of a complex projective variety such that K/L is the sphere bundle associated to $\mathcal{O}_X(-1)$. We omit the details.

Example 1. Let X denote the Grassmannian $G_{k,n} = SL(n, \mathbb{C})/P_k$. In this case $V = \Lambda^k(\mathbb{C}^n)$. We put the usual hermitian metric on $\mathbb{C}^n$ which induces a hermitian metric on V. Fixing a standard basis $\{e_i \mid 1 \le i \le n\}$ on $\mathbb{C}^n$ and taking $A \subset SL(n, \mathbb{C})$ to be the group of diagonal matrices, one sees that $\{e_\alpha \mid \alpha \in I_{n,k}\}$ form a basis for V consisting of weight vectors. Here $I_{n,k}$ denotes the set of all sequences $\alpha = 1 \le i_1 < \cdots < i_k \le n$, and $e_\alpha = e_{i_1} \wedge \cdots \wedge e_{i_k}$. The basis vector corresponding to the highest weight is $e_1 \wedge \cdots \wedge e_k$. One has $K = SU(n)$, $L = SU(k) \times SU(n-k)$, and S is the identity component of the group of all transformations t where $t(e_i) = ze_i$, $1 \leq i \leq k$, and $t(e_j) = we_j$, $k < j \leq n$, where $z, w \in U(1)$, $z^k w^{(n-k)} = 1$.

Note that in view of theorem 1, $\mathrm{Aut}_1(W)$ acts trivially on the Dolbeault cohomology of W, where W is any generalized Hopf manifold. Our aim is to compute the Dolbeault cohomology of W. One can apply the Borel spectral sequence for the Tits bundle $\pi : W \longrightarrow X$. The only nontrivial differential to determine is $d_2 : {}^{1,0}E_2^{0,1} \cong H^{1,0}(T) \longrightarrow H^{1,1}(X) \cong {}^{1,1}E_2^{2,0}$. Since both the vector spaces involved here are one-dimensional, we see that this differential is an isomorphism if and only if $H^{1,0}(W) = 0$. We shall establish the following vanishing theorem.

Theorem 4. *Let W be any generalized Hopf manifold. Then $H^{1,0}(W) = 0$.*

Before we can prove the above proposition, we need the following lemma.

Lemma 5. Let U be a dense open subset of W and let $s_1, \cdots s_m$ be global sections of the holomorphic tangent bundle $\mathcal{T}W$ such that for every $u \in U$, the space $\mathcal{T}_u W$ is spanned by the vectors $s_1(u), \ldots, s_m(u)$, and that each s_i vanishes somewhere in W. Then $H^{1,0}(W) = 0$.

Proof. Suppose that ϕ is a global holomorphic 1-form on W. We must show that ϕ must be the zero form.

Clearly $\phi(s_i) =: f_i$ are global holomorphic functions on W and hence constants. Since s_i vanishes somewhere, the same is true of f_i for each i. It follows that each f_i is the zero function. Since the s_i span the holomorphic tangent space at each point of U, it follows that $\phi|U = 0$. Since U is dense in W, we conclude that ϕ itself must be the zero form. □

Our aim is to construct holomorphic vector fields on W satisfying the hypotheses of the above lemma. This will be achieved by constructing $\pi_1(W)$-invariant vector fields on the universal cover E of W. Indeed, G acts on $E \subset V$ and so every one parameter subgroup of G gives rise to a holomorphic vector field on E. We choose the one parameter subgroups in a convenient and natural manner and verify that for these vector fields the hypotheses of the above lemma are valid.

Recall that G is a *simply connected* simple complex Lie group and that P is a maximal parabolic subgroup of G that contains a fixed Borel subgroup B, $A \cong (\mathbb{C}^*)^l$ is a Cartan subgroup contained in B. We let R (resp. R^+) denote the set of roots (resp. positive roots) of Lie(G), and let Δ be the set of simple roots. Denote by $\{X_\beta\}_{\beta \in R} \cup \{H_\gamma\}_{\gamma \in \Delta}$ the Chevalley basis of Lie(G) corresponding to our choice of B, A. We denote the Weyl group of G by W. The 1-parameter subgroup of G generated by X_β, $\beta \in R$, will be denoted as G_β. Let ϖ be the fundamental weight of G stabilized by the Weyl group of P, and let α denote the corresponding simple root.

Let $\{e_\sigma\}_{\sigma \in I}$ denote a basis consisting of weight vectors of $V = V_\varpi$, the irreducible representation of G with highest weight ϖ, and let $\{p_\sigma\}$ denote the dual basis of V^*. Let Λ_ϖ denote the set of weights of V_ϖ. We shall assume that the indexing set I for the chosen basis elements contains Λ_ϖ, and that for $\lambda \in \Lambda_\varpi$, the weight of the element e_λ is λ. Note that if ϖ is not a miniscule weight, then $I \supsetneq \Lambda_\varpi$, and there is no uniqueness in labelling the weight vectors by a weight $\lambda \in \Lambda_\varpi$ which occurs in V_ϖ with multiplicity more than one. Later we will make a more precise choice of the labelling of weight vectors.

Note that $E \subset V$, the total space of the principal $\mathbb{C}^*$-bundle associated to G/P, is invariant under the action of G on V. We denote the holomorphic vector field generated by the action of G_β, by abuse of notation, by X_β. Thus, for $v \in V$, $X_\beta(v) = \frac{d}{dt}|_{t=0}$ $(\exp(tX_\beta)(v))$. It is obvious that X_β is $\mathbb{C}^*$-invariant. In particular, it is invariant under the action of $\langle c\rangle \subset \mathbb{C}^*$, and hence defines a holomorphic tangent vector field on $W = E/\langle c\rangle$ which we again denote by X_β, $\beta \in R$.

Note that $X_\beta(e_\sigma)$ is zero unless $\mathrm{wt}(\sigma) + \beta$ is a weight of V_ϖ. Here $\mathrm{wt}(\sigma)$ stands for the weight of e_σ. If $\mathrm{wt}(\sigma) + \beta$ is indeed a weight of V, then $X_\beta(e_\sigma) = \sum k^\tau_{\beta,\sigma} e_\tau$, where $k^\tau_{\beta,\sigma} \in \mathbb{C}$ and the sum runs over all τ such that $\mathrm{wt}(\tau) = \mathrm{wt}(\sigma) + \beta$. It is known that, for $\beta \in R^+$, among all $\tau \in I$ for which $\mathrm{wt}(\tau) = \varpi - \beta$, there is at least one τ such that $k^\tau_{-\beta,\omega}$ is nonzero. See Theorem 6.1, Chap. XVII, [12]. By relabelling the $(\varpi - \beta)$-weight vectors if necessary, we may assume $k^\tau_{-\beta,\varpi} \neq 0$ for $\tau = \varpi - \beta$. A straightforward verification leads to the following expression for the vector field X_β on E in terms of $\{(\partial/\partial p_\sigma)\}$:

$$X_\beta = \sum_\tau \left(\sum_\sigma p_\sigma k^\tau_{\beta,\sigma}\right)(\partial/\partial p_\tau).$$

Let $S \subset R^+$ be the set of complementary roots of G with respect to P. Note that $\#S = \dim G/P$. It is well-known that $\varpi - \beta$ is a weight of $V = V_\varpi$ for each $\beta \in S$.

We put a partial order on the indexing set I of the weight vectors of V by declaring that $\sigma \leq \tau$ if either $\sigma = \tau$ or $\mathrm{wt}(\sigma) < \mathrm{wt}(\tau)$.

Lemma 6. *With the above notation, the set of vector fields* $\{X_{-\beta} \mid \beta \in S\} \cup \{X_\alpha\}$, *span the holomorphic tangent bundle* $\mathcal{T}U$ *of* $U = \{[v] \in W | v \in E,\ p_\varpi(v) \neq 0, p_{\varpi-\alpha}(v) \neq 0\} \subset W$.

Proof. Note that, on E,

$$X_{-\beta} = p_\varpi k^{\varpi-\beta}_{-\beta,\varpi}(\partial/\partial p_{\varpi-\beta}) + \text{other terms},$$

where the "other terms" on the right involve $(\partial/\partial p_\tau)$ with τ either incomparable or strictly less than $\varpi - \beta$ in the partial order. Similarly,

$$X_\alpha = p_{\varpi-\alpha} k^\varpi_{\alpha,\varpi-\alpha}(\partial/\partial p_\varpi) + \text{lower terms}.$$

Recall that $k^{\varpi-\beta}_{-\beta,\varpi} \neq 0$, for $\beta \in S$, and that $k^\varpi_{\alpha,\varpi-\alpha} \neq 0$. This shows that, for $v \in E$ with $p_\varpi(v) \neq 0$ and $p_{\varpi-\alpha}(v) \neq 0$ the tangent vectors $X_{-\beta}(v)$, $\beta \in S$, $X_\alpha(v)$ are related by means of an upper triangular matrix with nonzero diagonal entries to the vectors $(\partial/\partial p_{\varpi-\beta})|_v$, $\beta \in S$, $(\partial/\partial p_\varpi)|_v \in \mathcal{T}_v V$. Therefore $X_{-\beta}(v)$, $\beta \in S$, $X_\alpha(v)$ are linearly independent. Since $\dim_{\mathbb{C}} E = \dim G/P + 1 = \#S + 1$, they span $\mathcal{T}_v E$. Since the vector fields involved are invariant under the action of $\langle c\rangle$ on E, it follows that $\{X_{-\beta} \mid \beta \in S\} \cup \{X_\alpha\}$, regarded as vector fields on W, span the holomorphic tangent bundle of the dense open set $U \subset W$, and the lemma follows. □

Proof of Theorem 4. In view of lemmas 5 and 6, we need only show that each of the vector fields $X_{-\beta}, \beta \in S, X_\alpha$, vanish somewhere on W. One merely notes that $X_{-\beta}([e_\lambda]) = 0, \beta \in S$, where $\lambda = w_0(\varpi)$, w_0 being the longest element of the Weyl group of G, so that $e_\lambda \in I$ is the lowest weight vector. Also, $X_\alpha([e_\varpi]) = 0$. This completes the proof. □

We shall now compute the Dolbeault cohomology of generalized Hopf manifolds W using Borel spectral sequence for the Tits bundle $\pi : W \longrightarrow X$ with fibre and structure group $T = \mathbb{C}^*/\langle c\rangle$. Let $R = H^*(X; \mathbb{C}) = \oplus_{p\geq 0} H^{p,p}(X; \mathbb{C})$, and let A denote the annihilator ideal of $y \in H^2(X; \mathbb{Z}) \subset H^{1,1}(X; \mathbb{C}) \cong \mathbb{C}$. We shall denote the ring $R/\langle y\rangle$ by S. Note that S

is a graded ring coming from the grading on the cohomology algebra of X, and that A is a graded S-module. We denote by $\tilde{A}$ the graded S-algebra which is just $S \oplus A$ as the underlying S-module and the multiplication in $\tilde{A}$ is defined by setting $u.v = 0$ for $u, v \in \tilde{A}$; the gradation on $\tilde{A}$ is as follows: if $u \in A$ is of type (p,p) then its type in $\tilde{A}$ is declared to be $(p+1,p)$. We are now ready to state our theorem.

Theorem 7. *We keep the notation of the previous paragraph. Let W be a generalized Hopf manifold. Then as a bigraded algebra the Dolbeault cohomology of W is isomorphic to the algebra $S[x]/\langle x^2\rangle \otimes_S \tilde{A}$.*

Proof. Since by theorem 4, $H^{1,0}(W) = 0$, the generator $x_{1,0} \in H^{1,0}(T, \mathbb{C}) \cong \mathbb{C}$ transgresses to zy for some *nonzero* complex number z. This shows that in the E_2 diagram, everything in the ideal generated by y is in the image of d_2 and hence $E_\infty^{*,0} = H^*(X;\mathbb{C})/\langle y\rangle =: S \subset H^*_{\bar\partial}(W)$. Furthermore each class $u \otimes x_{1,0}$ with $u \in H^*(X;\mathbb{C})$ such that $uy = 0$ is a permanent cocycle and yields a class in the Dolbeault cohomology of W. By degree consideration the generator $x_{0,1} \in H^{0,1}(T;\mathbb{C}) \cong \mathbb{C}$ transgresses to zero. By the multiplicative property of the spectral sequence, we see that the subalgebra of $H^*_{\bar\partial}(W)$ generated over S by the image of the classes $u \otimes x_{1,0}$ in E_2 as u varies in the annihilator ideal A of y is isomorphic to $\tilde{A}$. Since the Dolbeault cohomology of the fibre is isomorphic to the exterior algebra over $\mathbb{C}$ on the classes $x_{0,1}, x_{1,0}$, it follows that $H^*_{\bar\partial}(W)$ is isomorphic to $S[x]/\langle x^2\rangle \otimes_S \tilde{A}$. □

Example 2. Let $X = Q_n = SO(n,\mathbb{C})/P_1 \cong SO(n,\mathbb{R})/(SO(2,\mathbb{R}) \times SO(n-2,\mathbb{R}))$, $n \geq 5$ which is the complex quadric $\{[z] \in \mathbb{P}^{n-1} | q(z) = \sum_{1\leq i\leq n} z_i^2 = 0\}$. In this case the $W = E/\langle c\rangle$ is the product $S^1 \times V_{n,2}$, where $V_{n,2}$ is the real Stiefel manifold $SO(n,\mathbb{R})/\ SO(n-2,\mathbb{R})$. Let $\pi : W_n \longrightarrow Q_n$ denote the projection of the associated Tits bundle.

Recall that the cohomology algebra $H^*(Q_n;\mathbb{C})$ is isomorphic to $\mathbb{C}[x, e_t]/\sim$ where $t = [(n-2)/2]$ and the relations are given by $x^{n-1} = 0, e_t = 0$ if n is odd and $e_t^2 = (-1)^t x^{n-2}$ if $n = 2t+2$ is even; x and e_t are of type $(1,1)$ and (t,t). One can take x to be the first Chern class of $\mathcal{O}_{Q_n}(-1)$ and e_t to be the Euler class of a certain oriented real $n-2$-plane bundle which is complementary to the real 2-plane bundle underlying the complex line bundle $\mathcal{O}_{Q_n}(-1)$; we will refer to x and e_t as the canonical generators of $H^*(Q_n;\mathbb{C})$.

Case (i). Let n be odd. In this case $H^*(Q_n;\mathbb{C})$ is generated by $y \in H^2(Q_n;\mathbb{C}) \cong \mathbb{C}$, and so in the notation of the above theorem, one has $S \cong \mathbb{C}$, $A = H^{2d}(Q_n;\mathbb{C}) = \mathbb{C}y^d$, where $d = \dim_{\mathbb{C}} Q_n = n-2$. Hence $H^*_{\bar\partial}(W_n;\mathbb{C}) = \mathbb{C}[x,u]/\sim$ where the relations are $x^2 = 0,\ u^2 = 0$, $xu = -ux$; x and u are of type $(0,1)$ and $(d+1,d)$ respectively.

Case (ii). Let $n = 2t+2 > 5$ be even. Then a straightforward computation in $H^*(Q_n;\mathbb{C})$ shows that $S = \mathbb{C}[e_t]$, where $e_t \in H^{n-2}(Q_n;\mathbb{C})$, and $e_t^2 = 0$ in S. Also, $A = \mathbb{C}e_t \oplus \mathbb{C}e_t^2$ and $\tilde{A} = S \oplus A = S[u]/\langle u^2\rangle$, where u is of type $(t+1,t)$. Hence $H^*_{\bar\partial}(W_n;\mathbb{C}) = S[x,u]/\sim = \mathbb{C}[x, e_t, u]/\sim$ where the relations are $x^2 = 0 = u^2 = e_t^2$, $xe_t = e_tx$, $ue_t = e_tu$, $xu = -ux$ and x, e_t, u are of type $(0,1), (t,t), (t+1,t)$ respectively.

4. Dolbeault cohomology of $V_{m,2} \times V_{n,2}$

In this section we compute the Dolbeault cohomology of the space $W_{m,n} = V_{m,2} \times V_{n,2} = (SO(m,\mathbb{R})/SO(m-2,\mathbb{R})) \times (SO(n,\mathbb{R})/SO(n-2,\mathbb{R})) = (SO(m,\mathbb{R}) \times SO(n,\mathbb{R}))/(SO(m-2,\mathbb{R}) \times SO(n-2,\mathbb{R}))$. Let $E_{m,n} = E_m \times E_n$ denote the total space of the principal

$\mathbb{C}^* \times \mathbb{C}^*$ bundle associated to the holomorphic 2-plane bundle $\mathcal{O}_{Q_m}(-1) \oplus \mathcal{O}_{Q_n}(-1)$ on $Q_m \times Q_n$. Indeed $E_{m,n}$ may be identified with the subspace $\{(z,w) \in \mathbb{C}^m \times \mathbb{C}^n | \sum_{1\leq i\leq m} z_i^2 = 0 = \sum_{1\leq j\leq n} w_j^2\} \setminus \{0\}$. The action of $\mathbb{C}^* \times \mathbb{C}^*$ on $E_{m,n}$ is then given by $(\lambda,\mu).(z,w) = (\lambda.z, \mu.w)$. Let τ be any complex number such that $\mathrm{Im}(\tau) \neq 0$. Then the map $\alpha \mapsto (\exp(2\pi i\tau\alpha), \exp(2\pi i\alpha))$ is an analytic imbedding of $\mathbb{C}$ into $\mathbb{C}^* \times \mathbb{C}^*$. The group $\mathbb{C}$ acts on $E_{m,n}$ via this imbedding. The quotient of $E_{m,n}$ by this action of $\mathbb{C}$ is nothing but $W_{m,n}$ which gives a complex structure on $W_{m,n}$. The complex structure so obtained depends on τ. One sees that the map $\pi : W_{m,n} \longrightarrow Q_m \times Q_n$ is the projection of an analytic bundle with fibre and structure group the one dimensional torus $T = \mathbb{C}^* \times \mathbb{C}^* / \mathbb{C}$. It is straightforward to see that T is isomorphic to the complex torus with periods $\{1, \tau\}$. Also, the restriction of the T-bundle $W_{m,n} \longrightarrow Q_m \times Q_n$ to $Q_m \times \{[w]\}$, $[w] \in Q_n$ is isomorphic to the T-bundle $W_m \longrightarrow Q_m$.

We denote by y_1, e_s and y_2, e_t the canonical generators of $H^*(Q_m \times Q_n; \mathbb{C}) \cong H^*(Q_m; \mathbb{C}) \otimes H^*(Q_n; \mathbb{C})$.

Lemma 8. *In the Borel spectral sequence of the T-bundle* $\pi : W_{m,n} \longrightarrow Q_m \times Q_n$, *the generator* $x_{1,0} \in H^{1,0}(T)$ *transgresses to* $u = \alpha y_1 + \beta y_2$, *with* $\alpha, \beta \in \mathbb{C}^*$. *In particular* $H^{1,0}(W_{m,n}) = 0$.

Proof. We need only show that α, β are nonzero. The inclusion map j of the T-bundle $W_m \longrightarrow Q_m$ into $W_{m,n} \longrightarrow Q_m \times Q_n$ induces a map j^* between the Borel spectral sequences associated to the two bundles. Using naturality of the Borel spectral sequence and the fact that transgression commutes with j^* we see that, in view of Example 2, we must have $\alpha \neq 0$. Similarly $\beta \neq 0$. □

Note. The numbers α and β are determined by τ.

Let $5 \leq m \leq n$, $s = [(m-2)/2]$, $t = [(n-2)/2]$. Let $u_1 = \alpha y_1$, and $u_2 = \beta y_2$, so that $u = u_1 + u_2 \in H^2(Q_m \times Q_n; \mathbb{C})$. Denote by S the ring $H^*(Q_m \times Q_n; \mathbb{C})/\langle u \rangle$. A simple calculation shows that $S \cong \mathbb{C}[u_1]/\langle u_1^{m-1}\rangle[e_s, e_t]/\sim$ where the relations are as follows: (i) $e_s = 0$ (resp. $e_t = 0$) if m (resp. n) is odd, (ii) $u_1 e_s = 0$, $u_1 e_t = 0$, (iii) $e_s^2 e_t^2 = 0$, (iv) $e_s^2 = (-1)^s \alpha^{2-m} u_1^{m-2}$ and, (v) $e_t^2 = 0$ if $m < n$, and $e_t^2 = (-1)^m \alpha^{m-2}\beta^{2-m} e_s^2$ if $m = n$. The elements e_s and e_t are of type (s,s) and (t,t) respectively. The annihilator ideal $A = \mathrm{ann}(u) \subset H^*(Q_m \times Q_n; \mathbb{C})$ is generated, as an algebra over $\mathbb{C}$, by $e_s e_t, e_t u_1^{m-2}, e_s u_2^{n-2}$, $v_0, v_1 := u_1 v_0, \ldots, v_{m-2} := u_1^{m-2} v_0$ where $v_0 = u_2^{n-2} - u_2^{n-3} u_1 + \cdots + (-1)^{m-2} u_2^{n-m} u_1^{m-2}$. Note that $v_0^2 = 0$ if $m < n$ and $v_0^2 = (-1)^m 2t v_{m-2} = (-1)^m 2t u_1^{m-2} u_2^{n-2}$ if $m = n$; $v_i v_j = 0$, if $0 \leq i \leq j$, $j \geq 1$; and $e_s^2 e_t^2 = (-1)^{s+t} \alpha^{m-2}\beta^{n-2} u_1^{m-2} u_2^{n-2} = (-1)^{s+t}\alpha^{m-2}\beta^{n-2} u_1^{m-2} v_0$. Furthermore, as an S-module the relations in A can be easily obtained from the relations in $H^*(Q_m \times Q_n; \mathbb{C})$. For example, $u_1 e_s e_t = 0$, $e_s.(e_s e_t) = (-1)^s \alpha^{2-m} e_t u_1^{m-2}$, $e_t.v_i = 0$ unless $m = n$ and $i = 0$, in which case one has $e_t.v_0 = (-1)^m e_t u_1^{m-2}$, $u_1.v_i = v_{i+1}$, $0 \leq i < m-2$, $u_1.v_{m-2} = 0$.

We define $\tilde{A}$ to be the graded S-algebra $S \oplus A$ where the multiplication is defined by setting $a.b = 0$ for all $a, b \in A$. The gradation in $\tilde{A}$ is got by declaring the bidegree of an element of type (r,r) in A to be $(r+1, r)$. The proof of the following theorem is completely analogous to that of theorem 7 and is therefore omitted.

Theorem 9. *Let* $5 \leq m \leq n$, *and let* $W_{m,n} = V_{m,2} \times V_{n,2}$. *With the above notation the Dolbeault cohomology of* $W_{m,n}$ *is isomorphic as an algebra to* $S[x]/\langle x^2\rangle \otimes_S \tilde{A}$, *where x is of type* $(0,1)$. □

Remark. (1). The above method can be used to compute the Dolbeault cohomology of the spaces $S^{2m-1} \times V_{n,2}$, $m \geq 2$ and $V_{n,4}$.

(2). Let $X = G/P$ where G is a complex simple algebraic group and P is a maximal parabolic subgroup. The annihilator ideal of y, the generator of $H^2(X, \mathbb{C}) \cong \mathbb{C}$, is the same as the ideal generated by the cohomology classes defined by *primitive* algebraic cycles. In particular, $\mathrm{ann}(y) \cap H^k(X, \mathbb{C})$ is zero for $k < \dim_{\mathbb{C}} X$.

5. Picard groups

We compute here the Picard groups of generalized Hopf manifolds. Picard groups of Hopf manifolds $S^1 \times S^{2n-1}$ have been computed by Ise [14]. Our main result of this section is

Theorem 10. *Let W be a generalized Hopf manifold. Then* $\mathrm{Pic}^0(W) = \mathrm{Pic}(W) \cong \mathbb{C}^*$. *Every line bundle over W arises from a representation of the fundamental group $\pi_1(W) \cong \mathbb{Z}$ and hence admits an integrable holomorphic connection.*

Proof. Consider the exact sequence of sheaves

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O}_W \longrightarrow \mathcal{O}_W^* \longrightarrow 1.$$

In the induced long exact sequence in cohomology one has

$$H^1(W, \mathbb{Z}) \longrightarrow H^1(W, \mathcal{O}_W) \longrightarrow H^1(W, \mathcal{O}_W^*) \longrightarrow H^2(W, \mathbb{Z}).$$

The left most homomorphism is easily seen to be $1-1$, and one has isomorphisms $H^1(W, \mathbb{Z}) \cong \mathbb{Z}$, $H^2(W, \mathbb{Z}) \cong 0$ as can be seen using theorem 3. As $H^2(W, \mathbb{Z}) = 0$, we conclude that $\mathrm{Pic}(W) = \mathrm{Pic}^0(W)$. Also, $H^1(W, \mathcal{O}_W) = H^{0,1}(W) \cong \mathbb{C}$ by using the Borel spectral sequence. By definition one has $H^1(W, \mathcal{O}_W^*) \cong \mathrm{Pic}(W)$. It follows that $\mathrm{Pic}(W) \cong \mathbb{C}/\mathbb{Z} \cong \mathbb{C}^*$.

Choose a base point in $w \in W$ and fix a generator $t \in \pi_1(W) \cong \pi_1(S^1) \cong \mathbb{Z}$. Let $\alpha \in \mathbb{C}^*$ and consider the homomorphism $\pi_1(W) \longrightarrow \mathbb{C}^*$ which maps t to α. Then $L_\alpha := E \times_{\mathbb{Z}} \mathbb{C}$ is the total space of a line bundle over $W = E/\mathbb{Z}$, where E is the universal cover of W and $\mathbb{Z}$ acts on $\mathbb{C}$ by $n.z = \alpha^n.z$. As is well-known, this leads to an analytic homomorphism of groups $\phi : \mathbb{C}^* \cong \mathrm{Hom}(\pi_1(W), \mathbb{C}^*) \longrightarrow \mathrm{Pic}(W) \cong \mathbb{C}^*$. Let T be the fibre of the Tits bundle associated to W that contains $w \in W$. Choosing α to be in the unit circle $\{z \in \mathbb{C}^* \mid |z| = 1\}$, $\alpha \neq 1$, one sees that $\phi(\alpha)$ restricts to a nontrivial line bundle on T (see §2 of [18]). Hence ϕ is not the trivial homomorphism. By dimension consideration it follows that ϕ is surjective. As is well-known (see [5]), it follows that every line bundle over W admits an integrable holomorphic connection.

Remark. Line bundles over simply connected compact complex homogeneous manifolds have been studied by Ise [13].

Acknowledgement

We thank A Borel for bringing to our attention the work of D Akhiezer [1]. We thank D Akhiezer for sending us a reprint of [1]. We thank the referees for their comments.

References

[1] Akhiezer D, Sur les représentations de groupes de Lie dans les espaces de cohomologie de Dolbeault, *C R Acad. Sci. Paris* **321** (1995) 1583–1586

[2] Akhiezer D, Group actions on the Dolbeault cohomology of homogeneous manifolds, *Math. Zeit.* **226** (1997) 607–621

[3] Akhiezer D, Lie group actions in complex analysis, *Aspects Math.* **E 27** (Vieveg) (1995)

[4] Akhiezer D, Homogeneous complex manifolds, in *Several Complex Variables-IV, Encycl. Math. Sci.* **10** Springer-Verlag, 1990

[5] Atiyah M F, Complex analytic connections in fibre bundles, *Trans. Am. Math. Soc.* **85** (1957) 181–207

[6] Blanchard A, Automorphismes des variété fibrée analytiques complexes, *C R Acad. Sci. Paris* **233** (1951) 1337–1339

[7] Bochner S and Montgomery D, Groups on analytic manifolds, *Ann. Math.* **48** (1947) 659–669

[8] Borel A, A spectral sequence for complex analytic bundles, Appendix-II to F. Hirzebruch, *Topological methods in algebraic geometry*, Grundlehren **131** (Third Edition), Springer-Verlag, 1976

[9] Bourbaki N, *Groupes et algèbre de Lie*, (Ch. IV–VI) Hermann, Paris, 1968

[10] Gorbatsevich V V and Onishchik A L, Lie transformation groups, in Lie groups and Lie algebras-I, *Encycl. Math. Sci.* **20**, Springer-Verlag, 1993

[11] Gugenheim V K A M and Spencer D C, Chain homotopy and the de Rham theory, *Proc. Am. Math. Soc.* **7** (1956) 144–152

[12] Hochschild G P, *Basic theory of algebraic groups and Lie algebra*, GTM-**75**, Springer-Verlag, 1981

[13] Ise M, Some properties of complex analytic vector bundles over compact complex homogeneous spaces, *Osaka Math. J.* **12** (1960) 217–252

[14] Ise M, On the geometry of Hopf manifolds, *Osaka Math. J.* **12** (1960) 387–402

[15] Lescure F, Action nontriviale sur le premier groupe de cohomologie de Dolbeault, *C R Acad. Sci. Paris* **316** (1993) 823–825

[16] Lescure F, Example d'actions induites non résolubles sur la cohomologie de Dolbeault, *Topology* **35** (1996) 561–581

[17] Montgomery D, Simply connected homogeneous spaces, *Proc. Am. Math. Soc.* **1** (1950) 467–469

[18] Mumford D, *Abelian Varieties*, Oxford University Press, Bombay, 1974

[19] Wang H.-C., Closed manifolds with homogeneous complex structures, *Am. J. Math.* **76** (1954) 1–32