VIRTUAL BETTI NUMBERS OF COMPACT LOCALLY SYMMETRIC SPACES

T. N. VENKATARAMANA

Abstract. We show that the virtual Betti number of a compact locally symmetric space with arithmetic fundamental group is either 0 or else is infinite.

1. Introduction

Let G be a connected non-compact linear Lie group with finite centre, such that G is simple modulo its centre. Let Γ be a torsion free cocompact arithmetic (not necessarily congruence) subgroup in G and let $i \geq 0$ be an integer. Consider the direct limit cohomology group

$$\mathcal{H}^i = lim H^i(\Delta, \mathbb{C})$$

where the direct limit is over all finite index subgroups Δ in Γ; we emphasize that Γ is only assumed to be an arithmetic subgroup of G and is not assumed to be a congruence subgroup of G. The dimension of the direct limit $\mathcal{H}^i$ as a $\mathbb{C}$-vector space is called the **virtual** i-th Betti number of Γ.

Theorem 1. *If the direct limit $\mathcal{H}^i$ is finite dimensional, then $\mathcal{H}^i = H^i(G_u/K, \mathbb{C})$ where G_u/K is the compact dual of the symmetric space G/K of G.*

As a special case we recover the following result of Cooper, Long and Reid (see [CLR]).

Corollary 1. *If M is a compact arithmetic hyperbolic 3-manifold with non-vanishing first Betti number, then M has infinite virtual first Betti number.*

Proof. Take $G = SL_2(\mathbb{C})$ in Theorem 1, and observe that the compact dual $G_u/K = S^3$ has vanishing first cohomology. □

1991 *Mathematics Subject Classification.* Primary 11F75; Secondary 22E40, 22E41
T. N. Venkataramana, School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, Bombay - 400 005, INDIA. venky@math.tifr.res.in.

The present note was motivated by the recent preprint [CLR] of Cooper, Long and Reid, where they prove Corollary 1, by using crucially, the fact that M is a hyperbolic 3-manifold. We show that this is true in greater generality. The point of Theorem 1 is that the group Γ is not assumed to be a congruence subgroup; if Γ is a congruence subgroup, this is a result of A.Borel (see [B]).

2. Proof of Theorem 1

Let $K \subset G$ be a maximal compact subgroup; write $\mathfrak{k}$ and $\mathfrak{g}$ for the complexified Lie algebras of K and G. We have the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Note that Γ (and hence the finite index subgroup Δ) is torsion-free and cocompact in G. We then get by the Matsushima-Kuga formula (see [BoW]),

$$H^i(\Delta, \mathbb{C}) = Hom_K(\wedge^i \mathfrak{p}, \mathcal{C}^\infty(\Delta \backslash G)(0).$$

In this formula, $\mathcal{C}^\infty(\Delta \backslash G)(0)$ denotes the space of complex valued smooth functions on the manifold $\Delta \backslash G$ which are annihilated by the Casimir of $\mathfrak{g}$ (the latter space in the Matsushima-Kuga formula may be identified with the space of **harmonic** differential forms of degree i on $\Delta \backslash G/K$ with respect to the G-invariant metric on the symmetric space G/K).

Taking direct limits in the Matsushima -Kuga formula yields the equality

$$\mathcal{H}^i = lim H^i(\Delta, \mathbb{C}) = Hom_K(\wedge^i \mathfrak{p}, \bigcup_{\Delta \subset \Gamma} \mathcal{C}^\infty(\Delta \backslash G)(0)).$$

Here, Δ runs through finite index subgroups of Γ. Consider the space

$$\mathcal{F} = \bigcup_{\Delta \subset \Gamma} \mathbb{C}^\infty(\Delta \backslash G)(0).$$

On the space $\mathcal{F}$, G acts on the right (since the Casimir commutes with the G-action).

Now, Γ is an arithmetic subgroup of G. That is, there is a semi-simple (simply connected) algebraic group $\mathbf{G}$ defined over $\mathbb{Q}$ and a smooth surjective homomorphism $\pi : \mathbf{G}(\mathbb{R}) \to G$ with compact kernel such that $\pi(\mathbf{G}(\mathbb{Z}))$ is commensurable to Γ. We define $G(\mathbb{Q})$ simply to mean the image group $\pi(\mathbf{G}(\mathbb{Q}))$. It follows from weak approximation ([PR]) that $G(\mathbb{Q})$ is dense in G.

Now, there is an action on $\mathcal{F}$ by $G(\mathbb{Q})$ on the left (which therefore commutes with the right G action), as follows. Given a function $\phi \in \mathcal{F}$ and given an element $g \in G(\mathbb{Q})$, the function ϕ is left Δ-invariant for some finite index subgroup Δ in Γ. Consider the function $g(\phi) = x \mapsto \phi(g^{-1}x)$. This function is left-invariant under $g\Delta g^{-1}$ and hence under $\Gamma \cap g\Delta g^{-1}$; since $g \in G(\mathbb{Q})$, it follows that g commensurates Γ and hence that the subgroup $\Gamma \cap g\Delta g^{-1}$ is of finite index in Γ. Therefore, $g(\phi)$ lies in $\mathcal{F}$. This defines an action of $G(\mathbb{Q})$ on the direct limit $\mathcal{H}^i$. Note that under this action, the action of Δ on the cohomology group $H^i(\Delta, \mathbb{C})$ is trivial.

Suppose that $\mathcal{H}^i$ is finite dimensional. Since $\mathcal{H}^i$ is a direct limit of finite dimensional vector spaces, it follows that it coincides with one of them. Therefore there exists a finte index subgroup Δ of Γ such that

$$\mathcal{H}^i = H^i(\Delta, \mathbb{C}).$$

The last sentence of the foregoing paragraph says that while $G(\mathbb{Q})$ acts on $H^i(\Delta, \mathbb{C})$, the action by Δ is trivial. Hence the action by the normal subgroup N generated by Δ in $G(\mathbb{Q})$ is also trivial. The density of $G(\mathbb{Q})$ in G is easily seen to imply the density of the normal subgroup N in G. Thus the image of $\wedge^i \mathfrak{p}$ under any element of $\mathcal{H}^i$ (viewed via the Matsushima-Kuga formula as a (K-equivariant) homomorphism of $\wedge^i \mathfrak{p}$ into $\mathcal{F}$), goes into G invariant functions in $\mathcal{C}^\infty(\Delta \backslash G)$, i,e, the constant functions. But $Hom_K(\wedge^i \mathfrak{p}, \mathbb{C})$ is the space of harmonic differential forms on the compact dual G_u/K, and is therefore isomorphic to $H^i(G_u/K, \mathbb{C})$.

This proves Theorem 1.

Remark. If Γ and all the subgroups Δ are **congruence** subgroups, then one sees at once from strong approximation, that the above $G(\mathbb{Q})$ action on the direct limit translates into the action of the "Hecke Operators" $G(\mathbb{A}_f)$ ($\mathbb{A}_f$ are the ring of finite adeles) and amounts to the proof of Borel in [B]. In this sense, the proof of Theorem 1 is an extension of Borel's proof to the non-congruence case.

Acknowledgement: The author gratefully acknowledges the hospitality of the Forschungsinstitut für Mathematik, ETH, Zurich in November 2006, where this note was written.

References

[CLR] D.Cooper, D.D. Long and A.W.Reid, On the virtual Betti numbers of arithmetic hyperbolic 3-manifolds, preprint October 2, 2006.

[B] A.Borel, Cohomologie de sous-groupes discrets et representations de groupes semi-simples, Asterisque, **32-33** (1976), 73-112.

[BoW] A. Borel and N. R. Wallach, Continuous cohomology, discrete subgroups and representations of reductive groups, Annals of Math. Studies, **94**, Princeton Univ. Press, Princeton, 1980.

[PR] V.P.Platonov and A.Rapinchuk, Algebraic Groups and Number Theory, 1990.

School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, Bombay - 400 005, INDIA.

E-mail address: `venky@math.tifr.res.in`