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On the first cohomology of cocompact arithmetic groups

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Abstract. Results of Matsushima and Raghunathan imply that the first cohomology of a cocompact irreducible lattice in a semisimple Lie group G, with coefficients in an irreducible finite dimensional representation of G, vanishes unless the Lie group is $SO(n,1)$ or $SU(n,1)$ and the highest weight of the representation is an integral multiple of that of the standard representation.

We show here that every cocompact arithmetic lattice in $SO(n,1)$ contains a subgroup of finite index whose first cohomology is non-zero when the representation is one of the exceptional types mentioned above.

Keywords. Cohomology groups; arithmetic groups.

1. Introduction

In this paper we prove some results on the non-vanishing of certain first cohomology groups. We recall that if G is a real semisimple group without compact factors such that G is not locally isomorphic to $SU(n,1)$ or $SO(n,1)$ and $\Gamma \subset G$ is any irreducible *cocompact* lattice and ρ is a finite dimensional irreducible representation of G, then $H^1(\Gamma,\rho)$ is zero. This vanishing theorem is proved in [R1] when ρ is non-trivial, and when ρ is trivial, the vanishing theorem is proved as a consequence of "property T" (in [K1] when G is not locally isomorphic to $Sp(n,1)$ or the real rank one form of F_4 and [Kos] in the remaining cases). See also [Mat] where a large number of groups are covered.

In the remaining cases of $G=SO(n,1)$ or $SU(n,1)$ suppose V is the standard representation of G on $\mathbf{C}^{n+1}$, V^* the dual, $\mathrm{sym}^l(V)$ (respectively $\mathrm{sym}^l(V^*)$) the lth-symmetric power of V (resp. of V^*), Q the quadratic form on V which is preserved by $SO(n,1)$, $Q\mathrm{sym}^{l-2}(V^*)$ the space of elements of $\mathrm{sym}^l(V^*)$ (i.e. polynomials) which are divisible by Q. Let H_l be the quotient space $\mathrm{sym}^l(V^*)/Q\mathrm{sym}^{l-2}(V^*)$. If $G=SU(n,1)$, $\rho\neq \mathrm{sym}^l(V)$ and $\rho\neq\mathrm{sym}^l(V^*)$ for any l, then $H^1(\Gamma,\rho)=0$ for any cocompact lattice $\Gamma\subset SU(n,1)$. If $G=SO(n,1)$ and $\rho\neq H_l$ for any l, then $H^1(\Gamma,\rho)=0$ for any cocompact lattice $\Gamma\subset SO(n,1)$. These two vanishing theorems are proved in [R1].

Thus only the cases (1) $G=SU(n,1)$, $\rho=\mathrm{sym}^l(V)$ or $\mathrm{sym}^l(V^*)$ and (2) $G=SO(n,1)$, $\rho=H_l$, remain to be considered. We note that in these two cases, the representation ρ may be described as the irreducible representation of G whose highest weight is l-times the highest weight of the standard representation.

In this paper we prove the following:

Theorem 1. *Let $n\geqslant 4$ be an integer and assume that $n\neq 7$. Let $\Delta\subset SO(n,1)$ be a cocompact arithmetic lattice. Let $l\geqslant 0$ be an integer, V_{n+1} the standard representation of*

$SO(n,1)$ and Q the quadratic form preserved by $SO(n,1)$ on V_{n+1}. Let H_l be the quotient $\mathrm{sym}^l(V^*_{n+1})/Q\mathrm{sym}^{l-2}(V^*_{n+1})$. *Then there exists a subgroup Δ' of finite index in Δ such that*

$$H^1(\Delta', H_l) \neq 0.$$

Remarks. (1) Theorem 1 holds even when $n=7$, provided Δ comes from an arithmetic structure which is not of the type (in the notation of [T]) ${}^3D_{40}$ or ${}^6D_{40}$.

(2) Theorem 1 is proved for all $n \geqslant 6$ in [L]. The case $n=5$ and $l=0$ is handled in [L-M] and [R-V]. The case of n even and $l=0$ is proved in [M1]. The case n even and l is non-zero is proved in [M2]. Thus, Theorem 1 is new only for $n=5$ and $l \neq 0$. However, our proof, which is a continuation of [R-V], works uniformly for all $n \geqslant 4$ and $l \geqslant 0$ and yields additional information which we describe below in Theorem 2. The main point of interest in our proof is that the non-vanishing of cohomology is obtained as a consequence of a relative congruence subgroup property which was proved in [R-V] and is therefore completely different from the usual proofs involving representation theory.

1.1. *Notation.* In this paper we only consider arithmetic lattices of $SU(n,1)$ which are of the following kind. Assume that K is a totally real number field, L is a totally imaginary quadratic extension of K, V is an $(n+1)$-dimensional vector space over L, $h_0: V \times V \to L$ a bi-additive map which is hermitian with respect to the action of the nontrivial element of the Galois group of L over K. Let G be the K-algebraic group $SU(h_0)$. We assume that $G(K \otimes \mathbf{R})$ is isomorphic to the product of $SU(n,1)$ with a compact group. The groups Γ that we consider are the ones coming from these arithmetic structures on $SU(n,1)$.

1.2. *Notation.* We now describe all the arithmetic lattices of $SO(n,1)$ $(n \geqslant 4)$ except those of $SO(7,1)$ which arise from K-forms of the type ${}^6D_{40}$ or ${}^3D_{40}$. They arise as follows.

Let K be a totally real number-field, D a central simple algebra over K of degree $d \leqslant 2$, V an m-dimensional D-vector space, and ι an involution on D given by $(tr(x)-x)$ if d is 2 and (x) if d is 1, for all x in D. Let $h: V \times V \to D$ be a biadditive map such that for all $\lambda, \mu \in D$ and $v, w \in V$ we have $h(\lambda v, \mu w) = \iota(\lambda) h(v,w)\mu$. Let $H = SU(V,h)$ be the special unitary group of this form h. We assume that h is so chosen that

$$H(K \otimes \mathbf{R}) = SO(n,1) \times \text{a compact group},$$

where $n+1 = md$.

By [T], the only arithmetic lattices in $SO(n,1)$ $(n \geqslant 4,\ n \neq 7)$ arise as $SO(n,1)$-conjugates of arithmetic subgroups of $H(K)$, for some H as above.

1.3. *Notation.* Let $\Delta \subset SO(n,1)$ be an arithmetic lattice as in (1.2). Let L/K be a quadratic extension which is totally imaginary and which splits D (it is easy to see, using weak approximation on K, that such an L exists). Let, for x in L, $x \to \bar{x}$ be the action of the non-trivial element of the Galois group of L/K. Let $D_L = D \otimes_K L$. On D_L, define the involution ι by $\iota(\lambda \otimes a) = \iota(\lambda) \otimes a$. Let $V_L = V \otimes_K L$ and define $h_L: V_L \times V_L \to D_L$ by $h_L(v_1 \otimes a_1, v_2 \otimes a_2) = h(v_1, v_2) \otimes a_1 a_2$ for all $v_1, v_2 \in V$ and $a_1, a_2 \in L$. Consider the K-algebraic group $G = SU(V_L, h_L)$. Then it is easy to see that

$$G(K \otimes_Q \mathbf{R}) = SU(n,1) \times \text{a compact group}.$$

Moreover, the arithmetic structure on $SU(n,1)$ is the same as the one defined in (1.1) for a suitable h_0.

Thus, given an arithmetic lattice Δ of the type (1.2) in $SO(n,1)$, there are natural ways of extending Δ to an arithmetic lattice Γ in $SU(n,1)$, of the kind described in (1.1). We will fix one such.

We have defined, in Theorem 1, the spaces V_{n+1}^* and the spaces H_l. We have the quotient map $\mathrm{sym}^l(V_{n+1}^*)\to H_l$. If G is as in (1.1), and $\sigma\in(\mathrm{Aut}G)(K)$, we have the homomorphism $\sigma(\Gamma)\to\Gamma$, given by $x\to\sigma^{-1}x$. This induces the map

$$H^1(\Gamma,\mathrm{sym}^l(V_{n+1}^*))\to H^1(\sigma(\Gamma),\mathrm{sym}^l(V_{n+1}^*)). \tag{1}$$

We also have the inclusion map $\sigma(\Gamma)\cap H\subset\sigma(\Gamma)$ and the quotient map above which induce the map

$$H^1(\sigma(\Gamma),\mathrm{sym}^l(V_{n+1}^*))\to H^1(\sigma(\Gamma)\cap H,H_l)). \tag{2}$$

The composite of (1) and (2) yields a map, which we denote by

$$\mathrm{Res}_\sigma: H^1(\Gamma,\mathrm{sym}^l(V_{n+1}^*))\to H^1(\sigma(\Gamma)\cap H,H_l).$$

Denote by Res the product map $\Pi_{\sigma\in(\mathrm{Aut}G)(K)}\mathrm{Res}_\sigma$. Then we have

Theorem 2. *If $n\geqslant 6$, and $n+1$ is even, then the map*

$$\mathrm{Res}: H^1(\Gamma,\mathrm{sym}^l(V_{n+1}^*))\to \prod_{\sigma\in(\mathrm{Aut}G)(K)} H^1(\sigma(\Gamma)\cap H,H_l)$$

is injective.

This paper is organized as follows. In § 2, we prove a proposition (see (2.4)) which relates the congruence subgroup kernels of two groups G and H with the injectivity of certain restriction maps closely related to those that occur in Theorems 2 and 3. We also show that the assumptions of (2.4) on G and H are satisfied for a large class of groups G and H (see 2.7). In § 3, we prove Theorem 2 using (2.7) and (2.4). In § 4, we prove Theorem 2. A more involved version of (2.4), namely Proposition (2.5) is used. We also prove an analogue of Theorem (2.7) in (4.4). In § 5, we deduce Theorem 1 from Theorem (4.6), and a Theorem in [B-W] for certain cocompact arithmetic lattices in $SU(n,1)$.

2. The congruence subgroup kernel and H^1

2.1. *Notation.* Let G be a linear algebraic simply connected semisimple group defined over a number field K. Assume that G is absolutely almost simple. Let H be a simply connected semi-simple group over K, $i: H\to G$ a morphism of algebraic groups over K with finite kernel. If A is an algebra over K, we denote by $G(A)$ the group of A-rational points of G. We assume that $\Pi_{v\in\infty}G(K_v)$ and $\Pi_{v\in\infty}H(K_v)$ are both noncompact. The group $G(K)$ may be given the structure of a topological group whose topology (called *arithmetic* (resp. *congruence*) *topology*) is obtained by designating *arithmetic* (resp. *congruence*) subgroups of $G(K)$ as open. The completion $G(a)$ (resp. $G(c)$) of $G(K)$, with respect to the arithmetic (resp. congruence) topology is called the *arithmetic* (resp. *congruence*) *completion* of G (note that by assumption $\Pi_{v\in\infty}G(K_v)$ is not compact and therefore, by strong approximation, $G(c) = G(\mathbf{A}_f)$, where $\mathbf{A}_f$ denotes the ring of finite adeles over K). The identity map $G(K) \to G(K)$ induces a natural continuous homomorphism $G(a) \to G(c)$ whose kernel, as may be easily checked, is a compact profinite group. We have thus an exact sequence of groups given by $1 \to C(G) \to G(a) \to G(c) \to 1$, where $C(G)$ is called the *congruence subgroup kernel of* G. The map $i: H \to G$ induces natural continuous homomorphisms $i(a): H(a) \to G(a)$ and $i(c): H(c) \to G(c)$ such that $i(a)(C(H)) \subset C(G)$. Moreover the rectangles in the following diagram commute:

$$\begin{array}{ccccccccc} 1 & \to & C(G) & \to & G(a) & \to & G(c) & \to & 1 \\ & & \uparrow i(a) & & \uparrow i(a) & & \uparrow i(c) & & . \\ 1 & \to & C(H) & \to & H(a) & \to & H(c) & \to & 1 \end{array}$$

By taking $H = G$ and $f: G \to G$ an automorphism of G over K, we see that the group $(\mathrm{Aut}G)(K)$ acts on $C(G)$. Moreover $G(a)$ normalizes $C(G)$. We denote by $C(H, G, i)$ the closed subgroup of $C(G)$ generated by the collection $\{\sigma(i(a)(C(H))); \sigma \in (\mathrm{Aut}G)(K)\}$ of subgroups of $C(G)$. This group is normalized by $G(K) \subset G(a)$ and since $G(K)$ is dense in $G(a)$, we see that $C(H, G, i)$ is normalized by $G(a)$. We denote by $C_{H,G,i}$ (or by C when there is no ambiguity about H and i) the quotient group $C(G)/C(H, G, i)$. Let $\hat{G}$ be the quotient group $G(a)/C$. Then we have surjections $G(a) \to \hat{G} \to G(c)$ and $G(K)$ is a subgroup of $\hat{G}$. We write $G(K) \cap \hat{G}$ for the topological space obtained by the relative topology on $G(K)$ in $\hat{G}$. It is then easy to see that an arithmetic group Δ is open in $G(K) \cap \hat{G}$ if and only if $\Delta \cap \sigma(H)$ is a congruence subgroup of $\sigma(H)$ for all $\sigma \in (\mathrm{Aut}G)(K)$.

We now consider (mainly for handling the case of $SO(4,1)$ and $SO(5,1)$), a quotient of $\hat{G}$. Let Δ be an arithmetic group in $G(K)$ which satisfies the condition $(*)$ below:

$(*)$ there exists a congruence subgroup Γ of $G(K)$ such that for all $\sigma \in H(K)$ we have: $\sigma(H) \cap \Delta \supset \sigma(H) \cap \Gamma$.

In particular, $\sigma(H) \cap \Delta$ is a congruence subgroup of $G(K)$, i.e. Δ is open in $G(K) \cap \hat{G}$, by the remark made above. The completion of $G(K)$ with respect to the topology generated by designating Δ which satisfy $(*)$ to be open, is denoted by G^*. We clearly have surjections $\hat{G} \to G^* \to G(c)$. Write C^* for the image of C in G^*. Note that $C^* = C^*_{H,G,i}$ depends on i and H.

It is immediate from the definitions that $C = \lim(\bar{\Delta}/\Delta)$ where Δ runs through arithmetic subgroups which are open in $G(K) \cap \hat{G}$ and $\bar{\Delta}$ is the smallest congruence subgroup of $G(K)$ which contains Δ. It is also immediate that $C^* = \lim(\bar{\Delta}/\Delta)$ where Δ now runs through arithmetic subgroups which are open in $G^* \cap G(K)$ and Δ is as before (here, lim denotes the inverse limit). In particular, we observe that C^* is contained in the closure of Δ in G^* for every Δ, i.e. the closure Γ^* (of any *congruence* subgroup Γ of $G(K)$) contains C^*.

2.2. *Notation.* Let $\rho: (\mathrm{Aut}G)(K) \to GL(E)$ be a rational representation on a finite dimensional complex vector space E. Suppose that $\Gamma \subset G(K)$ (resp. $\Delta \subset H(K)$) is a congruence subgroup of G (resp. H). We have homomorphisms $\pi: G(K) \to (\mathrm{Aut}G)(K)$ and $\pi \circ i: H(K) \to (\mathrm{Aut}G)(K)$. Consider the cohomology group $H^1(\Gamma, \rho \circ \pi)$ (resp. $H^1(\Delta, \rho \circ \pi \circ i)$). We have the homomorphism $\sigma: \sigma^{-1}(\Gamma) \to \Gamma$ for all $\sigma \in (\mathrm{Aut}G)(K)$ which induces a map $H^1(\sigma): H^1(\Gamma, \rho) \to H^1(\sigma^{-1}(\Gamma), \rho \circ \sigma)$ which is given on a one-cocycle $Z(\gamma)$

by $H^1(\sigma)(Z(\gamma)) = \rho(\sigma)^{-1}(Z(\sigma(\gamma))$. Thus we have a map $H^1(\Gamma,\rho) \xrightarrow{H^1(\sigma)} H^1(\sigma^{-1}(\Gamma),\rho)$. We also have the "restriction" map $H^1(\sigma^{-1}(\Gamma),\rho) \xrightarrow{H^1(i)} H^1(i^{-1}\sigma^{-1}(\Gamma),\rho)$. The composition of these two maps will be denoted $\mathrm{Res}_\sigma : H^1(\Gamma,\rho) \to H^1(i^{-1}\sigma^{-1}(\Gamma),\rho)$. We denote by Res the product map $\Pi_{\sigma\in(\mathrm{Aut}G)(K)}\mathrm{Res}_\sigma$ where

$$\mathrm{Res}: H^1(\Gamma,\rho) \to \prod_{\sigma\in(\mathrm{Aut}G)(K)} H^1(i^{-1}\sigma^{-1}(\Gamma),\rho).$$

The representation ρ is defined over the algebraic closure $\bar{K}$ of K (and hence on a finite extension K' of K), since $(\mathrm{Aut}G)$ is a reductive algebraic group over K. Therefore $E = E_{K'} \otimes_{K'} \mathbf{C}$ where $E_{K'}$ is a K'-vector space, and $\rho(G(K)) \subset \rho(G(K')) \subset GL(E_{K'})$.

Let l be a non-archimedian local field containing K'. Let K_v and $\mathbf{Q}_p$ be the closures of K and $\mathbf{Q}$ in l. Write $E_l = E_{K'} \otimes_{K'} l$. Let $\Omega \subset GL(E_l)$ be a compact subgroup. Then it is easily checked that there exists a compact open subgroup $\mathscr{E}(\Omega)$ of E_l such that $\mathscr{E}(\Omega)$ is stable under multiplication by the integers O_l in l, and under the action of the group Ω.

If $\Gamma \subset G(K)$ is an arithmetic group, we have $\rho(\Gamma) \subset \rho(G(l)) \subset GL(E_l)$ and since Γ is contained in a compact subgroup of $G(l)$, it is clear that $\rho(\Gamma)$ lies in a compact subgroup Ω of $GL(E_l)$. Let $\mathscr{E} = \mathscr{E}(\Omega)$ as above. Then $\mathscr{E}$ is stable under the action of Γ on E_l. We have

$$H^1(\Gamma,\rho) = H^1(\Gamma,E) = H^1(\Gamma,E_{K'}) \otimes_{K'} \mathbf{C},$$

$$H^1(\Gamma,E_{K'}) \otimes_{K'} l = H^1(\Gamma,E_l) = H^1(\Gamma,\mathscr{E}) \otimes_{O_l} l.$$

Given $\xi \in H^1(\Gamma,\rho)$, we may restrict the class of ξ to the subgroup $\Gamma \cap \sigma(H)$ for every $\sigma \in (\mathrm{Aut}G)(K)$. We thus obtain a map

$$(**)\ H^1(\Gamma,\rho) \to \prod_\sigma H^1(\Gamma \cap \sigma(H),\rho)$$

which we again denote by Res.

2.3. *Lemma. Suppose that C^* is finite. Then, the kernel of the map*

$$\mathrm{Res}: H^1(\Gamma,\mathscr{E}) \to \prod_\sigma H^1(\Gamma \cap \sigma(H),\mathscr{E})$$

is a torsion group. Here Γ is a congruence subgroup of $G(K)$ and Res *is the restriction map got by replacing ρ by $\mathscr{E}$ in* (**).

Proof. Suppose $\xi \in H^1(\Gamma,\mathscr{E})$ is such that $\mathrm{Res}(\xi) = 0$. Let Z be the one-cocycle representing ξ, $Z:\Gamma \to \mathscr{E}$. We may view Z as a homomorphism τ_Z of Γ into the semi-direct product $GL(\mathscr{E}) \ltimes \mathscr{E}$:

$$\tau_Z(\gamma) = \begin{pmatrix} \rho(\gamma) & Z(\gamma) \\ 0 & 1 \end{pmatrix}.$$

Since $\mathrm{Res}(\xi) = 0$, we have, for each σ, and $\gamma \in \sigma(H) \cap \Gamma$, a vector $v \in \mathscr{E}$ such that

$$\tau_Z(\gamma) = \begin{pmatrix} 1 & -v \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \rho(\gamma) & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix}.$$

Let $\mathscr{U}$ be an open subgroup of $GL(\mathscr{E})\ltimes\mathscr{E}$. It is easy to see that $\mathscr{U}$ contains an open subgroup $\mathscr{U}'$ of the form $\mathscr{U}_1\ltimes\mathscr{U}_2$ where $\mathscr{U}_1\subset GL(\mathscr{E})$ is an open torsion free subgroup and

$$\mathscr{U}_2\supset\{(\rho(g)-1)(w); g\in\mathscr{U}_1, w\in\mathscr{E}\}.$$

It is easy to see that

$$\tau_Z^{-1}(\mathscr{U})\cap\sigma(H)\cap\Gamma\supset\rho^{-1}(\mathscr{U}_1)\cap\sigma(H)\cap\Gamma.$$

The group $\rho^{-1}(\mathscr{U}_1\cap\Gamma)$ is a congruence subgroup since the map $\rho: G(K)\to GL(E_l)$ extends to a map

$$\rho: G(K_v)\to G(l)\to GL(E_l).$$

Therefore, $\tau_Z^{-1}(\mathscr{U})$ is open in $G^*\cap G(K)$. This shows that τ_Z extends to a continuous homomorphism $\tau_Z^*:\Gamma^*\to GL(\mathscr{E})\ltimes\mathscr{E}$. Then $G(K)\cap\rho^{-1}(\mathscr{U}_1)=\Gamma_1$ is a congruence subgroup of $G(K)$ and $\tau_Z^*(\Gamma_1^*)\subset\mathscr{U}_1\ltimes\mathscr{E}$. Since $\Gamma_1^*\supset C^*$, and by assumption, C^* is finite, the group $\tau_Z^*(C^*)$ is a finite subgroup of the torsion-free group $\mathscr{U}_1$ and is therefore trivial. Thus, we get a homomorphism

$$\tau_Z^*:\Gamma(c)\to GL(\mathscr{E})\ltimes\mathscr{E}\subset GL(E_l)\ltimes E_l,$$

where $\Gamma(c)$ is the (congruence) closure of Γ in $G(c)$. It is easily shown that $H^1(\Gamma(c),E_l)=0$. Therefore, $Z(\gamma)=\rho(\gamma)w-w$ for some w in E_l for all $\gamma\in\Gamma$. Since $\mathscr{E}$ is an open subgroup of E_l, there exists an integer $M\geqslant 0$ such that $p^M w\in\mathscr{E}$ (recall that $l\supset\mathbf{Q}_p$). Hence $p^M\xi=0$ in $H^1(\Gamma,\mathscr{E})$ i.e. ξ is a torsion.

2.4. PROPOSITION

If C^* is finite, then

$$\mathrm{Res}: H^1(\Gamma,\rho)\to\prod_\sigma H^1(\Gamma\cap\sigma(H),\rho)$$

is injective.

Proof. Consider the commutative diagram

$$\begin{array}{ccc} H^1(\Gamma,\mathscr{E}) & \xrightarrow{a} & \prod_\sigma H^1(\Gamma,\mathscr{E}) \\ \downarrow c & & \downarrow b \\ H^1(\Gamma,E_l) & \xrightarrow{d} & \prod_\sigma H^1(\Gamma,E_l), \end{array}$$

where a, d are restriction maps; b, c are induced by the inclusion of $\mathscr{E}$ in E_l and σ runs through all the elements of $(\mathrm{Aut}G)(K)$.

Let F be the kernel of d. Then there exists a finite subset Σ of $(\mathrm{Aut}G)(K)$ such that if we let σ run through only the elements of Σ in the commutative diagram above but denote the maps by the same letters, then kernel of d is again F. This is a consequence of the well-known fact that for the arithmetic group Γ, the space $H^1(\Gamma,E_l)$ is finite dimensional over l. We will therefore assume that in the above diagram the σ's lie in Σ.

Clearly, $\ker(b)$ and $\ker(c)$ are torsion groups and by Lemma (1.3), $\ker(a)$ is also torsion. Then,

$$\ker(d)\cap\mathrm{Im}(c)=c(c^{-1}(\ker(d)))=c(\ker(dc))=c(\ker(ba))$$

is a torsion group. Since $H^1(\Gamma, E_l)$ is torsion-free it follows that $\ker(d) \cap \mathrm{Im}(c) = 0$. But $\mathrm{Im}(c) \otimes_{O_l} l$ is $H^1(\Gamma, E_l)$ and therefore $\mathrm{Im}(c)$ is an open subgroup of $H^1(\Gamma, E_l)$. Hence $0 = \ker(d) \cap \mathrm{Im}(c)$ is an open subgroup of the l-vector space $\ker(d)$ i.e. $\ker(d) = 0$. This proves the proposition.

We now reformulate Proposition (2.4) slightly differently, replacing $\Gamma \cap \sigma(H)$ by the group $\sigma(\Gamma) \cap H$ and considering the restriction map defined at the beginning of (2.2). This is because, we prefer to work with a fixed H and varying groups $\sigma(\Gamma)$. Thus, we have

2.4. PROPOSITION (reformulated)

Let (G, H, i) *be as in* (2.1). *Assume that* C *is finite.* ρ, Γ *are as in* (2.2), *then the map*

$$\mathrm{Res}: H^1(\Gamma, \rho) \to \prod_{\sigma \in (\mathrm{Aut}G)(K)} H^1(i^{-1}\sigma^{-1}(\Gamma), \rho)$$

is injective. Consequently if $H^1(\Gamma, \rho) \neq 0$ *for some congruence subgroup* Γ *of* $G(K)$ *then* $H^1(\Delta, \rho) \neq 0$ *for some congruence subgroup* Δ *of* $H(K)$.

2.5. *Notation.* In place of H, we may even take infinitely many K-groups H_m, *with* K-homomorphisms $i_m: H_m \to G$ with finite kernel. Instead of $C(G, H, i)$ consider the closed subgroup B of $C(G)$ generated by the collection $\{C(G, H_m, i_m)\}$ of subgroups of $C(G)$. The quotient group $C(G)/B$ is still denoted C. We define a new topology on $G(K)$ by designating an arithmetic subgroup $\Delta \subset G(K)$ to be open if there exists a *congruence* subgroup Γ of $G(K)$ such that for all $\sigma \in (\mathrm{Aut}G)(K)$, and all ϕ, we have

$$\sigma(H_\phi) \cap \Delta \supset \sigma(H_\phi) \cap \Gamma.$$

Then, the completion of $G(K)$ with respect to this topology (we must check that a left Cauchy sequence is right Cauchy; this can be readily checked; then, one can form the completion; cf. [S1]) is again denoted by G^*. We have, as before, a surjection $\hat{G} \to G^*$ and the image of C under this map is again denoted by C^*. As in (2.2) we may consider the restriction maps

$$H^1(\Gamma, \rho) \to \prod_{\sigma \in (\mathrm{Aut}G)(K)} H^1(i_m^{-1}\sigma^{-1}(\Gamma), \rho)$$

for each m. We may then take the product of all these maps and obtain the big restriction map

$$H^1(\Gamma, \rho) \to \prod_m \prod_{\sigma \in (\mathrm{Aut}G)(K)} H^1(i_m^{-1}\sigma^{-1}(\Gamma), \rho).$$

We then obtain the following.

2.6. PROPOSITION

Let (G, H_m, i_m) *be as in* (2.4). *Assume that* C^* *is finite.* ρ, Γ *are as in* (2.2), *then the map*

$$H^1(\Gamma, \rho) \to \prod_m \prod_{\sigma \in (\mathrm{Aut}G)(K)} H^1(i_m^{-1}\sigma^{-1}(\Gamma), \rho)$$

is injective. Consequently if $H^1(\Gamma, \rho) \neq 0$ *for some congruence subgroup* Γ *of* $G(K)$, *then* $H^1(\Delta, \rho) \neq 0$ *for some congruence subgroup* Δ *of* $H_m(K)$ *for some* m.

Proof. The proof is exactly the same as that of Proposition (2.4), except that instead of one group H, we have to consider infinitely many groups H_m. We will need this proposition to handle the case of $SO(5,1)$ and $SO(4,1)$.

2.7. *Notation.* Suppose K is a totally real number field, D a central simple algebra over K such that every Archimedian completion K_v of K splits D. Let W be a finite dimensional right D-module equipped with a D-valued form h on $W \times W$ such that h is Hermitian with respect to the standard involution $\iota(\iota(x) = \mathrm{tr}(x) - x$ on D if the degree of D over K is 2 and $\iota(x) = x$ if $D = K)$. Let $H = \mathrm{Spin}(h)$ be the spin group of the Hermitian form h. We assume that h is so chosen, that for a fixed Archimedean completion ∞ of K, the group $H(K_\infty)$ is isomorphic to $\mathrm{Spin}(2m\text{-}1, 1)$ and for all other archimedean completions v of K, the group $H(K_v)$ is isomorphic to $\mathrm{Spin}(2m)$ (here, m is the dimension of W over D). We now choose a totally imaginary quadratic extension L over K such that L splits D. Let $a \to \bar{a}$ be the nontrivial Galois automorphism of (L/K). Write $W_L = W \otimes_K L$, $D_L = D \otimes_K L$. Given $\lambda = d \otimes a \in D_L$ with $d \in D$ and $a \in L$, let $\bar{\lambda} = \iota(d) \otimes \bar{a} \in D_L$. Then we get an involution $x \to \bar{x}$ of the second kind on D_L. On the free D_L module $W_L \times W_L$ define a K-biinvariant map by writing, for $w_1, w_2 \in W$ and $\lambda, \mu \in D_L, h_L(w_1 \otimes \lambda, w_2 \otimes \mu) = \bar{\lambda} h(w_1, w_2)\mu$. Let $G = SU(h_L)$ be the special unitary group of the form h_L on W_L which is Hermitian with respect to the involution on D_L defined above. We have then a K-homomorphism $\iota: H \to G$ with finite kernel.

2.8. *Theorem. In the notation of* (1.1), *if the degree of* D *over* K *is* 2, $K \neq \mathbf{Q}$ *and* $\dim_D(W) \geqslant 3$, *then* $C(G, H, i)$ *is finite.*

We refer to [R-V] for the proof. We also note that if $K = \mathbf{Q}$, but the other assumptions of (2.8) hold, then, by the Hasse–Minkowski theorem for quadratic forms, it follows that the $\mathbf{Q}$-ranks of H and G are both 1. Thus, the arithmetic lattice is not cocompact.

3. Proof of Theorem 2

3.1. *Notation.* Let Ω be an abstract group. We consider the category $\mathscr{C} = \mathscr{C}(\Omega)$ of finite dimensional completely reducible complex representations of Ω. If $E_1 \subset E_2 \in \mathscr{C}$, then $H^1(\Omega, E_1)$ is a direct summand of $H^1(\Omega, E_2)$, since $E_2 = E_1 \oplus E'_1$ for some $E'_1 \in \mathscr{C}$ and (1) $H^1(\Omega, E_2) = H^1(\Omega, E_1) \oplus H^1(\Omega, E'_1)$. Suppose that $f: \Omega \to \Omega'$ is a homomorphism of groups such that if ρ is a semisimple representation of Ω', then the composite of ρ and f is a semisimple representation of Ω. Let E' be a semisimple representation of Ω'. Let E'' be an Ω invariant subspace of E'. We may write $E' = E'' \oplus E$ as Ω modules. There is a canonical restriction map (2) $\mathrm{Res}: H^1(\Omega', E') \to H^1(\Omega, E')$. In view of (1), we get a projection map from $H^1(\Omega, E')$ into $H^1(\Omega, E'')$. We denote again by Res, the composite of this projection with the restriction map.

3.2. *Notation.* Suppose m, n are integers with $1 \leqslant m < n$. Let V_n be an n-dimensional vector space, and $\mathrm{sym}^l(V_n^*)$ the space of polynomials of degree l on V_n. If $0 \leqslant i$, we have, $\mathrm{sym}^i(V_n/V_m)^* \subset \mathrm{sym}^l(V_n^*)$ and

$$\mathrm{sym}^l(V_n^*) = \oplus_{0 \leqslant i \leqslant l} \mathrm{sym}^i(V_m^*) \otimes \mathrm{sym}^{l-i}(V_n/V_m)^*.$$

Let Q be a non-degenerate quadratic form on V_n. In particular, we have $Q \in \mathrm{sym}^2(V_n^*)$. We have

$$Q\mathrm{sym}^{l-2}(V_n^*) = \{P \in \mathrm{sym}^l(V_n^*);\ Q \text{ divides } P\}.$$

With these notations, we have

3.3. *Lemma. For all $i \geqslant 0$, and $m \leqslant n$, we have*

$$Q\mathrm{sym}^{l-2}(V_n^*) \cap [\mathrm{sym}^i(V_m^*) \otimes \mathrm{sym}^{l-i}(V_n/V_m)^*] = 0.$$

Proof. Suppose that $Qf = \Sigma A_r B_r$, where f, A_r, B_r belong respectively to the spaces $\mathrm{sym}^{l-2}(V_n^*)$, $\mathrm{sym}^i(V_m^*)$, and $\mathrm{sym}^{l-i}((V_n/V_m)^*)$. We may write $V_n = V_m \oplus W_m$ for some subspace W_m of V_n and for g in $\mathrm{sym}^k(V_n^*)$, we may write $g = g(x, y)$ for x in V_m and y in W_m. Then, for all complex numbers a, we have

$$Q(ax, y) f(ax, y) = \sum A_r(ax) B_r(y) = a^i \sum A_r B_r = a^i Qf.$$

This shows that Qf is divisible by $Q(ax, y)$ for all $a \neq 0$. This is impossible by the unique factorization of polynomials, unless $Qf = 0$.

3.4. *Notation.* We continue the notation of (3.2). By Lemma (3.3) we may assume that $E_i = \mathrm{sym}^i(V_m^*) \otimes \mathrm{sym}^{l-i}(V_n/V_m)^*$ is a subspace of

$$H_l = \mathrm{sym}^l V_n^* / Q\mathrm{sym}^{l-2} V_n^*.$$

Now let Ω be a reductive subgroup of $SO(Q)$ which maps V_m into itself. Then the spaces E_i, H_l are all semisimple Ω-modules, and therefore, $H_l = E_i \oplus E_i'$ for an Ω-stable submodule E_i' of H_l. We thus get the following inclusion of Ω-modules

$$\mathrm{sym}^l V_n^* = \oplus E_i \subset \oplus H_l \tag{1}$$

where the last sum is taken $(l+1)$-times

3.5. COROLLARY of (2.8)

Let $n+1$ be even and suppose that $n \geqslant 7$. Then, the map

$$\mathrm{Res}: H^1(\Gamma, \mathrm{sym}^l V_{n+1}^*) \to \prod_{\sigma \in (\mathrm{Aut}G)(K)} H^1(\sigma(\Gamma) \cap H, \mathrm{sym}^l(V_{n+1}^*))$$

is injective.

Proof. Observe that $K \neq \mathbf{Q}$ (cf. the remark after the statement of Theorem (2.8). The corollary follows from Theorem (2.8) and Proposition (2.4).

3.6. *Notation.* Assume that $n+1$ is even and that $n \geqslant 7$. Let H be as in (2.2). By the Morita theory (see [Sch]) we may assume that the degree of D is always 2 (since $(n+1)$-dimensional quadratic spaces may be thought of as $((n+1)/2)$-dimensional Hermitian spaces over $D = M_2(K)$). We write, as in (2.2), $H = \mathrm{Spin}(W, h)$. As is easily seen, we can find a basis $e_1, \ldots, e_k$ of W over D such that if $W' = e_1 D \oplus \cdots \oplus e_{k-1} D$, then $\mathrm{Spin}(W', h) = \Omega$ is a K-algebraic group with $\Omega(K \otimes \mathbf{R}) = \mathrm{Spin}(n-2, 1) \times K'$ where K' is compact. Furthermore, $W = W' \oplus e_k D$ is an orthogonal direct sum.

3.7. *Lemma. Let* $(n+1)$ *be even with* $n \geqslant 7$. *Let* H *and* Ω *be as in* (3.6). *Then the restriction map*

$$\mathrm{Res}: H^1(\Delta, \mathrm{sym}^l V^*_{n+1}) \to \prod_{\tau \in (\mathrm{Aut} H)(K)} H^1(\tau(\Delta) \cap \Omega, \mathrm{sym}^l V^*_{n+1})$$

in injective.

Proof. If $K = \mathbf{Q}$, then $\mathbf{Q}$-rank$(H) = \mathbf{Q}$-rank$(\Omega) = 1$ and therefore the arithmetic group is not cocompact. Therefore, $K \neq \mathbf{Q}$ and, by the main theorem of [R-V], the group $C_H/C(H, \Omega)$ is finite (this is where we need $n \geqslant 7$) and by Proposition (2.4), the lemma follows.

Theorem 3. *Let* H, G *be as in* (2.2) *with* $n+1$ *even and* $n \geqslant 7$. *Then, for every congruence subgroup* Γ *of* $G(K)$, *the map*

$$\mathrm{Res}: H^1(\Gamma, \mathrm{sym}^l(V^*_{n+1})) \to \prod_{\sigma \in (\mathrm{Aut} G)(K)} H^1(\sigma(\Gamma) \cap H, H_l)$$

is injective.

Proof. Let $\xi \in H^1(\Gamma, \mathrm{sym}^l(V^*_{n+1}))$ be such that $\mathrm{Res}(\xi)$ is zero. Then, for every $\tau \in (\mathrm{Aut} H)(K)$, we see that the restriction of ξ, as an element of $H^1(\tau(\sigma(\Gamma) \cap H) \cap \Omega, H_l)$ is zero. In the notation of (3.4) this means that the restriction of ξ, as an element of $H^1(\tau(\sigma(\Gamma) \cap H) \cap \Omega, E_i))$ is zero. But, by (1) of (3.4), it follows that $\mathrm{Res}(\xi)$ as an element of $H^1(\tau(\sigma(\Gamma) \cap H) \cap \Omega, \mathrm{sym}^l(V^*_{n+1})$ is zero. By Lemma (3.7), $\mathrm{Res}(\xi)$ as an element of $H^1(\sigma(\Gamma) \cap H, \mathrm{sym}^l(V^*_{n+1}))$ is zero. Now (3.5) shows that ξ is zero.

3.8. *Notation.* We will now assume that $n+1$ is odd, with $n+1 \geqslant 9$. In the notation of (2.2), we have $D = K$ and Q is a quadratic form on an $n+1$-dimensional K-vector space W; set $H_1 = \mathrm{Spin}(W, Q)$. Let $V = W \oplus e_{n+1}K$ be an $(n+2)$-dimensional K-vector space. We may define a new quadratic form h on V by writing, for w in W and $\lambda \in K$

$$h(w + \lambda, w + \lambda) = Q(w, w) + \lambda^2 \theta$$

for some scalar $\theta \in K$ such that the group $H = \mathrm{Spin}(V, h)(K \otimes_{\mathbf{Q}} \mathbf{R})$ is isomorphic to the product of $\mathrm{Spin}(n+1, 1)$ with a compact group. We may also find a subspace W' of codimension 1 in W such that (a) $W = W' \oplus e_{n+1}K$ is an orthogonal direct sum for some vector e_{n+1} and (b) if h' denotes the restriction of h to W', then the group $H' = \mathrm{Spin}(W', h')(K \otimes_{\mathbf{Q}} \mathbf{R})$ is isomorphic to the product of $\mathrm{Spin}(n-1, 1)$ with a compact group. Let $\Delta \subset H(K)$ be a congruence subgroup. Then, by Lemma (3.7) (notice that $n+2$ is even!), we have

$$\mathrm{Res}\, H^1(\Delta, \mathrm{sym}^l V^*_{n+2}) \to \prod_{\sigma \in (\mathrm{Aut} H)(K)} H^1(\sigma(\Delta) \cap H_1, \mathrm{sym}^l V^*_{n+2}) \tag{1}$$

is injective.

Since $V = W \oplus e_{n+1}K$, we see that $V_{n+2} = V_{n+1} \oplus e_{n+2}\mathbf{C}$. By (1) of (3.3), we get the following relations of H_1-modules

$$\mathrm{sym}^l V^*_{n+1} = \oplus E_i \subset \oplus H_l \tag{2}$$

where (∗) $H_l = \mathrm{sym}^l V^*_{n+1}/Q\mathrm{sym}^{l-2} V^*_{n+1}$. We also note that $h = Q$ on W.

3.9. Theorem. *Suppose that $n+1$ is odd. We use the notation of* (2.2) *and* (3.9). *The map*

$$\mathrm{Res}: H^1(\Gamma, \mathrm{sym}^l V_{n+2}^*) \to \prod_{\sigma \in (\mathrm{Aut} G)(K)} H^1(\sigma(\Gamma) \cap H_1, H_l)$$

is an injection, provided $n \geqslant 8$.

Proof. Let $W'' = e_{n+1}K \oplus e_{n+2}K$. For every $\tau \in \mathrm{Spin}(W'', h)(K)$ we consider the map

$$\mathrm{Res}\, H^1(\Gamma, \mathrm{sym}^l V_{n+2}^*) \to \prod_{\sigma \in (\mathrm{Aut} G)(K)} H^1(\sigma(\Gamma) \cap \tau(H_1), \tau(H_l))$$

where $\tau(H_l)$ is the quotient space defined in $(*)$ with V_{n+1} replaced by $\tau(V_{n+1})$.

Consider now the map Res of the Theorem. Suppose that for some $\xi \in H^1(\Gamma, \mathrm{sym}^l V_{n+2}^*)$, we have $\mathrm{Res}(\xi) = 0$. Considering the restriction to the smaller group H', $\mathrm{Res}_\sigma: H^1(\Gamma, \mathrm{sym}^l V_{n+2}^*) \to H^1(\sigma(\Gamma) \cap H', H_l)$ we obtain: $\mathrm{Res}_\sigma = 0$ in $H^1(\sigma(\Gamma) \cap H', H_l)$, for all $\sigma \in \mathrm{Aut} G(K)$. Now (2) shows that $\mathrm{sym}^l V_{n+1}^*$ is an H' submodule of $\oplus H_l$. Hence, by the remarks in (3.1), $\mathrm{Res}_\sigma = 0$ in $H^1(\sigma(\Gamma) \cap H', \mathrm{sym}^l V_{n+1}^*)$. By the naturality of the restriction map, we may replace V_{n+1} by the space $\tau(V_{n+1})$ and obtain for all σ and τ: $\mathrm{Res}(\xi) = 0 \in H^1(\sigma(\Gamma) \cap \tau(H'), \mathrm{sym}^l \tau(V_{n+1})^*)$. But, as an H'-module, the sum $\Sigma_\tau \mathrm{sym}^l \tau V_{n+1})^*$ is the same as the sum $\Sigma_i \mathrm{sym}^i V_n^* \otimes \mathrm{sym}^{l-i}(\tau(V_{n+1})/V_n)^*$. This, in turn, is $\mathrm{sym}^l V_{n+2}^*$. We now apply the remarks in (3.1) to conclude that ξ lies in the kernel of the map

$$\mathrm{Res}: H^1(\Gamma, \mathrm{sym}^l V_{n+2}^*) \to \prod_{\sigma \in (\mathrm{Aut} G)(K)} H^1(\sigma(\Gamma) \cap H_1, \mathrm{sym}^l V_{n+2}^*)$$

and from (1), we get $\mathrm{Res}(\xi) = 0$ in $\Pi_{\sigma \in (\mathrm{Aut} G)(K)} H^1(\sigma(\Gamma) \cap H, \mathrm{sym}^l V_{n+2}^*)$. Now Theorem 2 (replace n by $n+1$ there) shows that $\xi = 0$.

4. The cases $n = 4$ and 5

4.1. *Notation* We now assume that $n = 4$ or 5. We use the same notation in these two different cases in order to treat them simultaneously.

If $n = 4$, then we start with an arithmetic lattice in $SO(4,1)$, which, as we remarked in (2.2), comes from a quadratic form Q_5 over K. We now choose $\theta \in K^*$ as in (3.1) so that if (W_5, Q_5) is the quadratic space on which $SO(Q_5)$ operates, then $W_5 \oplus e_6 K = W_6$ has the quadratic form $Q_6 = Q_5 \oplus \theta \lambda^2$ on it. Write $E = W_5$. Moreover, θ is so chosen that if $H_6 = \mathrm{Spin}(W_6, Q_6)$, then $H_6(K \otimes_{\mathbf{Q}} \mathbf{R})$ is the product of Spin(5, 1) with a compact group. Clearly, H_6 contains $H_5 = H$ as a K-subgroup. We may write W_5 as an orthogonal direct sum $W_4 \oplus e_5 k$ with respect to Q_5 such that, if Q_4 denotes the restriction of Q_5 to W_4 and $H_4 = \mathrm{Spin}(W_4, Q_4)$, then $H_4(K_\infty) = \mathrm{Spin}(3,1)$. We may also choose a two-dimensional space E' over K with a quadratic form $\mathbf{Q}'$ so that if $W_8 = W_6 \oplus E'$, $Q_8 = Q_6 \oplus Q'$, and $H_8 = \mathrm{Spin}(W_8, Q_8)$, then $H_8(K \otimes_{\mathbf{Q}} \mathbf{R})$ is isomorphic to the product of Spin (7, 1) with a compact group. Choose a quadratic extension (L/K) as in (2.7) and let $G_8 = SU(W_8 \otimes L, Q_8 \otimes L)$ as in (2.7).

If $n = 5$, then an arithmetic lattice in $SO(n,1)$ comes from a Hermitian form over a central simple algebra D (of degree 2 over K) as in (1.2). Let $E = E_3$ be a free module over D of rank 3 and h_3 a D-valued Hermitian form on E_3 with respect to the standard involution on D described in (2.7). Set $H_6 = H = \mathrm{Spin}(E_3, h_3)$. Let h_2 denote the restriction of H_3 to E_2. Write $H_4 = \mathrm{Spin}(E_2, h_2)$. Furthermore, we may choose E_2 so that $H_4(K_\infty) = \mathrm{Spin}(3,1)$. We also choose a 1-dimensional space E_1 over D, and a Hermitian form h' on it such that if $h_4 = h_3 \oplus h'$ is the Hermitian form on $E_4 = E_3 \oplus E_1$ and $H_8 = \mathrm{Spin}(E_4, h_4)$, then $H_4(K \otimes_{\mathbf{Q}} \mathbf{R})$ is the product of Spin(7, 1) with a compact group. Let (L/K) denote a quadratic extension as in (3.1) and write, as in (2.2), $G_8 = SU(W_4 \otimes L, h_4 \otimes L)$.

We note once again that H_4, H_6, H, E, G_8 stands for two different things depending on the cases $n = 4$ or 5. From now on n is 4 or 5. Note that $G_8(K \otimes_{\mathbf{Q}} \mathbf{R})$ is the product of $SU(7,1)$ and a compact group. Consider the standard action of $SU(7,1)$ on $\mathbf{C}^8 = V_8$. If $k = 3, 4$ or 5, then for the action of $H_{k+1}(K)$ on V_8 via $H_{k+1}(K_\infty) = \mathrm{Spin}(k,1)$, we have $V_8 = V_{k+1} \oplus \mathbf{C}^{8-k-1}$ where $\mathbf{C}^{8-k-1}$ is the trivial $\mathrm{Spin}(k,1)$ module of dimension $(8-k-1)$. We have thus surjective homomorphisms $V_8^* \to V_{k+1}^*$ and the restriction maps of (3.1) may be defined. We also denote by Q_{n+1} the quadratic form preserved by H_{n+1} on V_{n+1} and write $H_l = \mathrm{sym}^l V_{n+1}^* / Q_{n+1} \mathrm{sym}^{l-2} V_{n+1}^*$.

4.2. *Lemma. The restriction map*

$$\mathrm{Res}: H^1(\Gamma_8, \mathrm{sym}^l(V_8^*)) \to \prod_{\sigma \in (\mathrm{Aut} G_8)(K)} H^1(\sigma(\Gamma_8) \cap H_6, \mathrm{sym}^l(V_8^*))$$

is injective.

This is just a restatement of Lemma (3.7).

4.3. *Notation.* Let $\{H_\phi\}$ denote the collection of simply connected K-subgroups of H_6 such that for all ϕ, (1) $H_\phi(K_\infty) = \mathrm{Spin}(3,1)$ and (2) $H_\phi = \mathrm{Spin}(E_\phi, h_\phi)$ where E_ϕ is a D-submodule of E such that $E = E_\phi \oplus E'_\phi$ as an orthogonal direct sum, H_ϕ is the restriction of h to E_ϕ. Let $C(H_6\{H_\phi\})$ denote the group defined in (2.5). We also write $H_\phi = H(E_\phi)$.

4.4. PROPOSITION

The group $(C(H_6)/C(H_6, \{H_\phi\}))^*$ is finite.

Proof. Let Δ be an open subgroup of $H_6(K) \cap H^*$. Let Ω be a congruence subgroup such that for all σ and all ϕ we have

$$\Delta \cap \sigma(H_\phi) \supset \sigma(H_\phi \cap \Omega).$$

Let Δ^* denote the smallest congruence subgroup of $H_6(K)$ containing Δ. Let v be a vector in $W(K)$ such that vD is isotropic over K_∞. Let r_v be the reflection with respect to v in $\mathrm{Aut}H(K)$. Let Ω' be a congruence subgroup such that $r_v \Omega' r_v^{-1} \subset \Omega$. For $\delta \in \Delta^* \cap \Omega'$ we have, $r_v \delta r_v^{-1} \delta^{-1}$ lies in the group $H(vD + \delta(v)D) \cap \Omega$ and the latter lies in Δ. But any element of Δ^* may be represented by an element of $\Delta^* \cap \Omega'$ modulo Δ since $\Delta\Omega' \supset \Delta^*$. This shows that on the quotient group Δ^*/Δ, the element r_v acts trivially. Since the group $(C(H_6)/C(H_6, \{H_\phi\}))^*$ is the inverse limit of the groups Δ^*/Δ (see (2.1)), it follows that r_v acts trivially, and hence elements of $H(K)$ of the form $r_v x r_v^{-1} x^{-1}$ act trivially on the group $(C(H_6)/C(H_6, \{H_\phi\}))^*$. The proposition now follows by the projective simplicity of $H_6(K)$ (see [To]).

4.5. COROLLARY

Let Δ_6 be a congruence subgroup of $H_6(K)$. Then the restriction map (for every finite dimensional representation ρ of $SO(5,1)$)

$$\mathrm{Res}: H^1(\Delta_6, \rho) \to \prod_{\phi} \prod_{\tau \in (\mathrm{Aut} H_6)} H^1(\tau(\Delta_6) \cap H_\phi, \rho)$$

is injective.

This is immediate from Proposition (4.4) and Proposition (2.6).

4.6. Theorem. *Let $\Gamma_8 \subset G_8(K)$ be a congruence subgroup. Then, in the notation of* (4.1), *the restriction map*

$$\mathrm{Res}: H^1(\Gamma_8, \mathrm{sym}^l(V_8^*)) \to \prod_{\sigma \in (\mathrm{Aut}\, G_8)(K)} H^1(\sigma(\Gamma_8) \cap H, H_l)$$

is injective.

Proof. Suppose that $\xi \in H^1(\Gamma_8, \mathrm{sym}^l(V_8^*))$ is in the kernel of Res. Restricting further to $\sigma(\Gamma_8 \cap H_\phi)$, we obtain, for each σ and ϕ, $\mathrm{Res}(\xi) = 0$ in $H^1(\sigma(\Gamma_8 \cap H_\phi, H_l)$. Corresponding to the inclusion $E_\phi \subset E$, we get a subspace V_ϕ of V_{n+1} and surjections $V_{n+1}^* \to V_\phi^*$. Hence, as an H_ϕ module, $\mathrm{sym}^l(V_{n+1}^*)$ decomposes as $\oplus E_\phi(i)$ where $E_\phi(i) = \mathrm{sym}^i(V_\phi^*) \otimes \mathrm{sym}^{l-i}(V_{n+1}/V_\phi)^*$. By Lemma (3.3), each $E_\phi(i)$ is an H_ϕ stable subspace of H_l and therefore, $\mathrm{Res}(\xi) = 0$ in $H^1(\sigma(\Gamma_8 \cap H_\phi, E_\phi(i))$ for all i.

We now replace σ by $\tau\sigma$ where $V_8 = V_\phi \oplus \mathbf{C}^4$ for the action of H_ϕ (and $\mathbf{C}^4$ is the trivial H_ϕ module), and $\tau \in G_8$ leaves the space V_ϕ pointwise fixed. By using the naturality of the restriction map, it follows that $\mathrm{Res}(\xi) = 0$ in $H^1(\sigma(\Gamma_8 \cap H_\phi, \tau^{-1}(E_\phi(i))$ for all i, τ, σ. But, τ is so chosen that $\tau(H_\phi) = H_\phi$ and $\tau(V_4) = V_4$. Hence, for all τ, σ, i we obtain, $\mathrm{Res}(\xi) = 0$ in

$$H^1(\sigma(\Gamma_8 \cap H_\phi, \mathrm{sym}^i V_4^* \otimes \mathrm{sym}^{l-i}(\tau(V_6)/V_4)^*)).$$

But, as is easily checked, the τ's act irreducibly on $\mathrm{sym}^l(\mathbf{C}^4) = \mathrm{sym}^l(V_8/V_4)$ and hence the sum over all i of $\Sigma_\tau \mathrm{sym}^i V_4^* \otimes \mathrm{sym}^{l-i}(\tau(V_6)/V_4)^*)$ is the sum over all i of $\mathrm{sym}^i V_4^* \otimes \mathrm{sym}^{l-i}(V_8/V_4)^*)$ which is $\mathrm{sym}^l V_8^*$. Hence, $\mathrm{Res}(\xi) = 0$ in the group $H^1(\sigma(\Gamma_8 \cap H_\phi, \mathrm{sym}^l V_8^*)$ for all σ, ϕ.

Therefore, if Δ_8 denotes any one of the groups $\sigma(\Gamma_8) \cap H$ and $\xi_\Delta = \mathrm{Res}(\xi) \in H^1(\Delta_8, \mathrm{sym}^l V_8^*)$, then $\mathrm{Res}(\xi) \in H^1(\Delta_8 \cap H_\phi, \mathrm{sym}^l V_8^*)$ for every ϕ. By corollary (4.5), this means that $\xi_\Delta = 0$. Now (4.2) shows that $\xi = 0$ and the Theorem is proved.

5. Proof of Theorem 1

5.1. *$n+1$ is even and greater than* 8. By [T], every arithmetic lattice Δ in $SO(n,1)$ arises as one of those in (1.2) with the degree of D over K being 2. We may extend Δ (by replacing it with a subgroup of finite index, if necessary) to an arithmetic lattice Γ of $SU(n,1)$ as in (1.3). By replacing Δ and Γ by subgroups of finite index if necessary, we may assume, by [B-W], Ch. [8], Theorem (5.9), that

$$H^1(\Gamma, \mathrm{sym}^l(V^*)) \neq 0. \tag{1}$$

Now, Theorem 2 shows that if $n \geqslant 9$, then, the map

$$\mathrm{Res}: H^1(\Gamma, \mathrm{sym}^l(V^*)) \to \prod_\sigma H^1(\sigma(\Gamma) \cap \mathrm{Spin}(n,1), H_l) \tag{2}$$

is injective. In particular, we see that the map

$$\mathrm{Res}: H^1(\Gamma, \mathrm{sym}^l(V^*)) \to \prod_\sigma H^1(\sigma(\Gamma) \cap SO(n,1), H_l) \tag{3}$$

is injective. Now (1) and (3) imply that for one of the groups $\Delta' = \sigma(\Gamma) \cap SO(n,1)$ we have

$$H^1(\Delta', H_l) \neq 0.$$

This proves Theorem 1 for odd $n \geqslant 9$.

5.2. *$n+1$ is odd and greater than* 7. By [T], every arithmetic lattice Δ in $SO(n,1)$ is one of the form described in (3.8). Let Γ be an arithmetic lattice in $G = SU(n+1,1)$ as in (3.8). As in (5.1) we may assume that

$$H^1(\Gamma, \mathrm{sym}^l(V^*)) \neq 0. \tag{1}$$

Now, Theorem (3.9), and arguments similar to those in (5.1) to replace Spin$(n,1)$ by $SO(n,1)$ show that

$$H^1(\Delta', H_l) \neq 0,$$

where $\Delta' = \sigma(\Gamma) \cap SO(n,1)$ for some $\sigma \in (\mathrm{Aut}G)(K)$. This proves Theorem 1 for even $n \geqslant 8$.

5.3. *$n = 4$ or* 5. We have already described the arithmetic lattices of $SO(n,1)$ in (4.1). Let Γ_8 be an arithmetic subgroup of $SU(7,1)$ as in (4.6). We may assume that

$$H^1(\Gamma_8, \mathrm{sym}^l(V^*)) \neq 0 \tag{1}$$

by replacing it by a subgroup of finite index if necessary. Now Theorem (4.6) implies that

$$H^1(\Delta', H_l) \neq 0,$$

where $\Delta' = \sigma(\Gamma_8) \cap SO(n,1)$ for some $\sigma \in (\mathrm{Aut}G_8)(K)$. This proves Theorem 1 for $n = 5$ or 4.

5.4. $n = 6$. As observed before, arithmetic lattices in $SO(6,1)$ arise as unit groups of quadratic forms Q_7 in 7 variables over K. Let W_7, Q_7 be the quadratic space over K. We may write $W_7 = W_6 \oplus e_7 K$, and denote by Q_6 the restriction of Q_7 to W_6, so that

$$SO(W_6, Q_6)(K \otimes_{\mathbf{Q}} \mathbf{R}) = SO(5,1) \times K_6,$$

where K_6 is a compact group. We may also find a quadratic space $W_8 = W_7 \oplus e_8 K$ with a quadratic form Q_8 whose restriction to W_7 is Q_7 and

$$SO(W_8, Q_8)(K \otimes_{\mathbf{Q}} \mathbf{R}) = SO(7,1) \times K_8,$$

where K_8 is a compact group. Write

$$H_l = \mathrm{sym}^l(W_7^*)/Q_7 \mathrm{sym}^{l-2}(W_7^*)$$

and

$$H_l(6) = \mathrm{sym}^l(W_6^*)/Q_6 \mathrm{sym}^{l-2}(W_6^*).$$

Then, as a representation of $H_6 = \mathrm{Spin}(5,1)$, $H_l(6)$ is a direct summand of H_l. By Theorem (4.4), the map

$$\mathrm{Res}: H^1(\Gamma_8, \mathrm{sym}^l(W_8^*)) \to \prod_\sigma H^1(\sigma(\Gamma_8) \cap \mathrm{Spin}(5,1), H_l(6))$$

is injective. Therefore

$$\mathrm{Res}: H^1(\Gamma_8, \mathrm{sym}^l(W_8^*)) \to \prod_\sigma H^1(\sigma(\Gamma_8) \cap \mathrm{Spin}(5,1), H_l)$$

is injective. In particular

$$\mathrm{Res}: H^1(\Gamma_8, \mathrm{sym}^l(W_8^*)) \to \prod_{\sigma} H^1(\sigma(\Gamma_8) \cap \mathrm{Spin}(6,1), H_l) \tag{1}$$

is injective. We may choose, by Theorem 2 (and by [B-W], Theorem (5.9), Ch. (8)), Γ_8 such that

$$H^1(\Gamma_8, \mathrm{sym}^l(W_8^*) \neq 0. \tag{2}$$

Thus, Theorem 1 for $n = 6$, follows from (1) and (2).

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