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An analysis of the equilibria of neural networks with linear interconnections

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Abstract. In this paper, we analyse the equilibria of neural networks which consist of a set of sigmoid nonlinearities with linear interconnections, *without* assuming that the interconnections are symmetric or that there are no self-interactions. By eliminating these assumptions, we are able to study the effects of imperfect implementation on the behaviour of Hopfield networks. If one views the neural network as evolving on the open n-dimensional hypercube $H = (0, 1)^n$, we have the following conclusions as the neural characteristics become steeper and steeper: (i) There is at most one equilibrium in any compact subset of H, and under mild assumptions this equilibrium is unstable. In fact, the dimension of the stable manifold of this equilibrium is the same as the number of eigenvalues of the interconnection matrix with negative real parts. (ii) There might be some equilibria in the faces of H, and under mild conditions these are always unstable. Moreover, it is easy to compute the dimension of the stable manifold of each such equilibrium. (iii) A systematic procedure is given for determining which corners of the hypercube H contain equilibria, and it is shown that all equilibria in the corners of H are asymptotically stable.

Keywords. Neural networks; linear interconnections; sigmoid nonlinearities.; stability of equilibria.

1. Introduction

Recently there has been a great deal of interest in artificial neural networks, especially of the so-called Hopfield type (Hopfield 1982, 1984; Hopfield & Tank 1985; Tank & Hopfield 1986). These consist of an interconnection of so-called sigmoid nonlinearities which represent the neurons or the switching elements, which are then interconnected through *linear* gains. The equations representing the dynamic behaviour of such networks are of the form

$$C_i\dot{u}_i = -(1/R_i)u_i + \sum_{j=1}^{n} t_{ij}V_j + I_i, \quad i = 1, \ldots, n, \tag{1}$$

where n is the number of neurons, V_i is the neural current and u_i is the neural voltage; I_i is the external current input to the ith neuron, C_i is the membrane capacitance, R_i

is the neural resistance, and t_{ij} is the interconnection term. In Hopfield networks, two further assumptions are made, namely: (i) $t_{ii}=0$ for all i (no self-interactions), and (ii) $t_{ij}=t_{ji}$ for all i, j (symmetric interactions). If one forms the *energy function*

$$E = -\frac{1}{2}\sum_{i=1}^{n}\left[\left(\sum_{j=1}^{n} t_{ij}V_j + 2I_i\right)V_i - (2/R_i)\int_0^{V_i} g_i^{-1}(V)\,\mathrm{d}V\right], \tag{2}$$

then it is straightforward to show that $\dot{E}\leqslant 0$ along the solution trajectories of the network. As a consequence (Hirsch 1987), all solution trajectories of the network approach an equilibrium. It is claimed (see e.g. Hopfield & Tank 1985 and Tank & Hopfield 1986) that neural networks can be used to solve a wide variety of problems in engineering and science by recasting them as minimization problems.

In the analyses of Hopfield (1982, 1984), Hopfield & Tank (1985), Tank & Hopfield (1986) and Hirsch (1987), the assumption that the neural interactions are symmetric is crucial, since otherwise one cannot form the energy function E of (2). The assumption that there are no self-interactions can be dispensed with in some, but not all, of the analyses. In a practical implementation of a neural network, it is often difficult to assure that the interactions are symmetric, since this often requires guaranteeing that two physical quantities (such as resistances or the gains of operational amplifiers) are *exactly* equal. Because of this, the behaviour of artificial neural networks constructed in the laboratory does not always correspond to that of ideal neural networks.

The objective of the present paper is to analyse the number, location, and stability behaviour of neural networks described by (1), *without* the assumptions of no self-interactions and symmetric interactions. On the other hand, since our interest is in studying the effects of imperfect implementations of Hopfield nets, it is assumed that the interconnection matrix is "nearly" symmetric. In other words, it is assumed that there is a *nominal* interconnection matrix T_0 which *is* symmetric, and that the actual interconnection matrix T lies in some ball of radius ε centred at T_0. Thus all the results stated here are of the form "For sufficiently small ε, something or the other is true". It shown that, even in the absence of these assumptions, it is possible to deduce a fairly complete picture of the dynamics of such networks. Specifically, it is shown that if one considers the neural network as evolving on the open hypercube $H=(0,1)^n$ in the "V"-space, then we have the following conclusions as the neural characteristics become steeper and steeper: (i) There is at most one equilibrium in any compact subset of H, and under mild assumptions this equilibrium is unstable. In fact, the dimension of the stable manifold of this equilibrium is the same as the number of eigenvalues of the interconnection matrix with negative real parts. (ii) There might be some equilibria in the faces of H, and under mild conditions these are always unstable. Moreover, it is easy to compute the dimension of the stable manifold of each such equilibrium. (iii) A systematic procedure is given for determining which corners of H contain equilibria, and it is shown that all equilibria in the corners of H are asymptotically stable.

2. Preliminaries

In this section the various assumptions made throughout the paper are briefly summarized.

The input-output relationship of the ith neuron is given by the sigmoid function

$$V_i = g_i(\lambda u_i), \tag{3}$$

where g_i is a given sigmoid function and λ is a scaling constant. The only assumptions made on the sigmoid function are the following.

Assumptions on the sigmoid nonlinearities: $g_i(x)$ is continuously differentiable, strictly increasing, and $g_i(x) \to 1$ as $x \to \infty$, $g_i(x) \to 0$ as $x \to -\infty$. Further, $xg_i'(x) \to 0$ as $|x| \to \infty$.

The assumptions about g_i are quite standard. The assumptions about g_i' are almost a consequence of the fact that $g_i(x)$ has a definite limit as $|x| \to \infty$. Since the function $1/x$ is not integrable over any infinite interval, it follows that

$$g_i(x) \to 1 \text{ as } x \to \infty \Rightarrow \liminf xg_i'(x) = 0 \text{ as } x \to \infty, \tag{4}$$

and similarly as $x \to -\infty$. Note that commonly used sigmoid functions such as $(1 + \tanh x)/2$ and $1/(1 + e^{-x})$ satisfy these assumptions. As $\lambda \to \infty$, the sigmoid becomes steeper and steeper and eventually approaches a switching function. Note that each neuron can have a different switching function, but for simplicity it is assumed that all neurons have the same scaling constant. Now define

$$b_{ij} = t_{ij}/C_i, \; y_i = I_i/C_i, \; i = 1, \dots, n. \tag{5}$$

As mentioned in § 1, no assumptions are made on the symmetry or otherwise of the *actual* interconnection matrix T. But there is a *nominal* interconnection matrix T_0 which is symmetric and satisfies the following conditions:

Assumptions on the nominal interconnection matrix: The matrix T_0 and all of its principal submatrices of size 2×2 or larger are hyperbolic, *i.e.* none of their eigenvalues has a zero real part. Every principal submatrix of T_0 of size 2×2 or larger has at least one eigenvalue with positive real part.

All of the matrices proposed by Tank and Hopfield satisfy these assumptions. Note that, if the nominal interconnection matrix T_0 has zero diagonal elements, then the first assumption implies the second. This is because the trace of a matrix is equal to the sum of its eigenvalues. Thus if the trace of T_0 is zero, and it has no eigenvalues on the imaginary axis, then it must have some eigenvalues with positive real part and others with negative real part. Another important point to note is that the above assumptions are *structurally stable* in the sense that if the nominal (symmetric) interconnection matrix T_0 satisfies them, so do all (nonsymmetric) matrices sufficiently close to T_0.

Define H to be the open hypercube $(0, 1)^n$, and $\bar{H}$ to be the closed hypercube $[0, 1]^n$. The symbol $\mathbf{b}$ denotes the binary set $\{0, 1\}$, and $\mathbf{b}^n$ denotes the set of n-dimensional binary vectors. Note that the set $\mathbf{b}^n$ consists precisely of the 2^n corners of the hypercube $\bar{H}$. The *faces* of the hypercube $\bar{H}$ consist precisely of those vectors $x \in \bar{H}$ with the property that $x_i \in \mathbf{b}$ for some but not all values of i. In other words, a face of $\bar{H}$ is a set of the form

$$\{x \in \bar{H}: x_i \in \mathbf{b} \, \forall i \in I, \; x_i \in [0, 1] \, \forall i \in J\}, \tag{6}$$

where I, J is a nontrivial partition of the set $\{1, \dots, n\}$.

The idea is to analyse the dynamical behaviour of the neural network for different values of the input vector y. It turns out that there are some "exceptional" values of y for which the behaviour is difficult to predict. But this exceptional set is quite small. A subset of R^n is said to be *of type* LV if it is contained in a finite union of linear varieties, each of dimension less than n. Note that a set of type LV has measure zero as a subset of H.

Among other things, we are interested in where the equilibria of (1) can lie as the sigmoid gain $\lambda \to \infty$. Three types of equilibria are identified. (1) If $V \in \bar{H}$ is an equilibrium and $V_i \in (0,1) \forall i$, then the equilibrium is said to be in the *interior* of $\bar{H}$. (2) If all components of V approach either 0 or 1 as $\lambda \to \infty$, then the equilibrium is said to be in a *corner* of $\bar{H}$. (3) If some components of V approach 0 or 1 as $\lambda \to \infty$ while others approach some value in (0, 1), then the equilibrium is said to be in a *face* of $\bar{H}$.

3. Motivation – single-neuron case

Much of what happens in a neural network as the neuron characteristic becomes steeper and steeper can be understood by studying the behaviour of a single neuron. In this case, the neuron dynamics are described by

$$\dot{u} = -u/\alpha + bg(\lambda u) + y, \tag{7}$$

where

$$\alpha = R_1 C_1,\ b = t_{11}/C_1,\ y = I_1/C_1. \tag{8}$$

So the equilibria of this network are at the solutions of

$$(u/\alpha) = bg(\lambda u) + y. \tag{9}$$

Figure 1 shows where the solutions of this equation can lie as $\lambda \to \infty$ when $b > 0$, while figure 2 does the same when $b < 0$. These figures show that as $\lambda \to \infty$, there can be two types of equilibria. First, those where u_{eq} approaches a finite number, and V_{eq} approaches 0 if $u_{eq} < 0$ and 1 if $u_{eq} > 0$; these types of equilibria are labelled as type A in figures 1 and 2. Second, those where $u_{eq} \to 0$ but V_{eq} approaches a number strictly between 0 and 1; this type of equilibrium is labelled as type B in figure 1. In §6 we shall see that, in the case of multiple neurons, it is possible for *some components* of u_{eq} to approach a nonzero value while the remaining components approach zero; in such a case, some components of V_{eq} approach 0 or 1 while the remaining components approach a value strictly between 0 and 1.

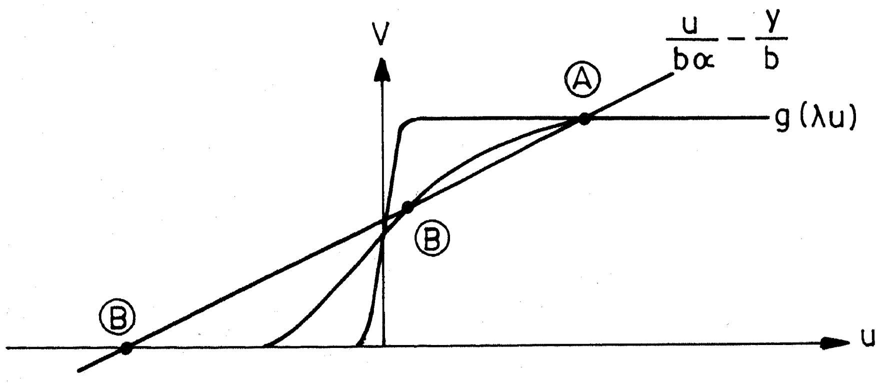

Figure 1. Equilibria in single-neuron case 1: B > 0.

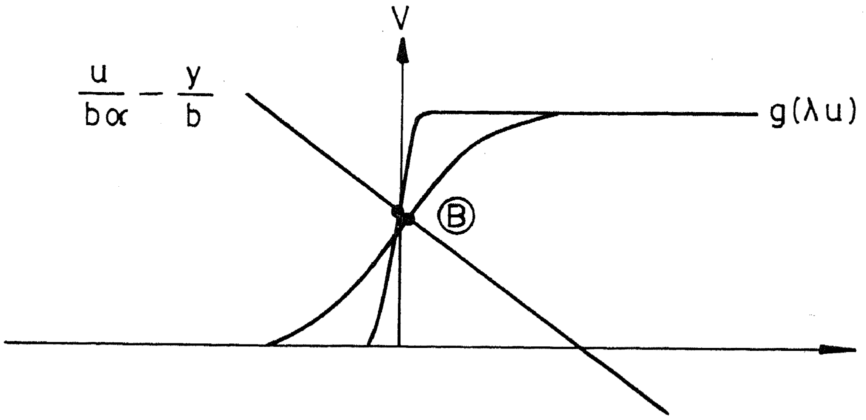

Figure 2. Equilibria in single-neuron case 2: $B < 0$.

4. Equilibrium in the interior of $\bar{H}$

To analyse the equilibria of the system (1), define

$$\alpha_i = R_i C_i,\ i = 1, \ldots, n;\ A = \mathrm{diag}\{\alpha_1, \ldots, \alpha_n\}, \tag{10}$$

$$b_{ij} = t_{ij}/C_i,\ y_i = I_i/C_i,\ i = 1, \ldots, n; \tag{11}$$

and define the map G: $R^n \to H$ by

$$[G(u)]_i = g_i(u_i). \tag{12}$$

Then the network equations (1) can be rewritten compactly as

$$\dot{u} = -A^{-1}u + BG(\lambda u) + y. \tag{13}$$

Now the equilibria of the system are at the solutions of

$$A^{-1}u = BG(\lambda u) + y. \tag{14}$$

In this section we are interested in equilibria in the interior of $\bar{H}$. If *all* components of V are to stay away from the limits 0 and 1 as $\lambda \to \infty$, then u must approach 0, while λu approaches some well-defined finite limit. Substituting $u = 0$ in (14) and noting that $V = G(\lambda u)$ gives

$$BV_{eq} + y = 0, \text{ or } V_{eq} = -B^{-1}y = -T^{-1}I. \tag{15}$$

Now $-T^{-1}I$ is the only *possible* equilibrium of (1) that can remain in the interior of $\bar{H}$ as $\lambda \to \infty$. Observe however that V_{eq} depends on the external voltage vector I and is thus a variable. Now we can state the main result of this section.

PROPOSITION 4.1.

Let S be a closed subset of the open hypercube H. Then there exists an $\varepsilon > 0$, dependent on S, such that the following statements are true for every interconnection matrix T such that $\|T - T_0\| < \varepsilon$:

(1) *for all sufficiently large λ, the system* (1) *has at most one equilibrium inside S. As $\lambda \to \infty$, this equilibrium* (*if any*) *approaches* $-T^{-1}I$;

(2) *this equilibrium is hyperbolic for all sufficiently large* λ, *and the dimensions of its stable manifold and its unstable manifold* are the same as the number of negative and the number of positive eigenvalues of* T_0, *respectively.*

Proof. (1) It has already been shown that this statement is true *provided* T is nonsingular. But since T_0 is nonsingular, so is T provided ε is sufficiently small.

(2) To prove this statement, linearize the network equations around the equilibrium V_{eq} in V-space, or near $G^{-1}(V_{eq})$ in the u-space. Now

$$\frac{\mathrm{d}}{\mathrm{d}u}[-A^{-1}u+BG(\lambda u)+y]=-A^{-1}+\lambda BJ(\lambda u), \tag{16}$$

where J is the (diagonal) Jacobian matrix of the map G. As $\lambda\to\infty$, the quantity $\lambda u\to G^{-1}(-B^{-1}y)$, as shown earlier. Define

$$\bar{u}=G^{-1}(-B^{-1}y),\ \bar{J}=J(\bar{u}). \tag{17}$$

Then, as $\lambda\to\infty$, the term $\lambda B\bar{J}$ swamps the $-A^{-1}$ term. Therefore the behaviour of the solution trajectories near the equilibrium is determined by the eigenvalues of $B\bar{J}$.

To study these, let us begin with the simple case where $T=T_0$, i.e. the interconnection matrix equals the nominal interconnection matrix and is thus symmetric. It is a well-known result (see e.g. Gantmacher 1951, p. 297) that if M is any nonsingular matrix, then T_0 and M^tT_0M have the same *signature*, that is, the same number of positive, zero, and negative eigenvalues. Since T_0 is hyperbolic and therefore nonsingular by assumption, it follows that neither T_0 nor M^tT_0M has any zero eigenvalues. Now, since

$$T_0MM^t=(M^t)^{-1}(M^tT_0M)M^t, \tag{18}$$

it follows that T_0MM^t has the same eigenvalues as M^tT_0M, and hence the same signature as T_0. Next, letting B_0 denote the matrix B obtained from (11) when $T=T_0$, we have

$$B_0\bar{J}=C^{-1}T_0\bar{J}, \tag{19}$$

where C is the diagonal matrix with C_i on the diagonal. Since C^{-1} and $\bar{J}$ are both diagonal, they commute. Consequently,

$$\bar{J}^{-1/2}C^{-1/2}[C^{-1/2}\bar{J}^{1/2}T_0\bar{J}^{1/2}C^{-1/2}]C^{1/2}\bar{J}^{1/2}=C^{-1}T_0\bar{J}. \tag{20}$$

Hence $B_0=C^{-1}T_0\bar{J}$ is similar to $C^{-1/2}\bar{J}^{1/2}T_0\bar{J}^{1/2}C^{-1/2}$, which in turn has only real eigenvalues and has the same signature as T_0.

Now consider the case where T_0 is replaced by T, which is "near" T_0. Since T is *not* assumed to be symmetric, it is no longer true that $B_0\bar{J}$ has only real eigenvalues. To handle this case we proceed as follows: Given the set S in H, define

$$G^{-1}(S)=\{u\in R^n:G(u)\in S\}, \tag{21}$$

$$\mathbf{J}=\{J(u):\ u\in G^{-1}(S)\}, \tag{22}$$

* See Hirsch & Smale (1974) for definitions of these terms.

and note that **J** is compact. By continuity, for each $J_0 \in \mathbf{J}$ there exist an $\varepsilon > 0$ and a $\delta > 0$ such that, whenever $\| T - T_0 \| < \varepsilon$ and $\| J - J_0 \| < \delta$, the matrices $C^{-1}TJ$ and $C^{-1}T_0J_0$ have the same number of eigenvalues with negative and positive real parts, which in turn is the same as the number of negative and positive eigenvalues of T_0. Of course ε and δ depend on J_0. Now let $B(J_0, \delta)$ denote the ball of radius δ centred at J_0, and let J_0 vary over S. Then the balls $B(J_0, \delta)$ cover S. Since S is compact, a finite subset of these balls is enough to cover S, say $B(J_1, \delta_1), \ldots, B(J_k, \delta_k)$. Pick the corresponding $\varepsilon_1, \ldots, \varepsilon_k$, and let

$$\varepsilon = \min\{\varepsilon_1, \ldots, \varepsilon_k\}. \tag{23}$$

Then, whenever $\| T - T_0 \| < \varepsilon$, we see that $C^{-1}TJ$ has the same number of eigenvalues with negative and positive eigenvalues as does T_0. This completes the proof of the second part.

Example 4.2. As an illustration of proposition 4.1, consider the A/D converter circuit of Tank & Hopfield (1986). If we study the four-bit converter, then $n = 4$, and

$$T = \begin{bmatrix} 0 & -2 & -4 & -8 \\ -2 & 0 & -8 & -16 \\ -4 & -8 & 0 & -32 \\ -8 & -16 & -32 & 0 \end{bmatrix}, \quad I = \begin{bmatrix} 1 \\ 2 \\ 4 \\ 8 \end{bmatrix} x - \begin{bmatrix} 0{\cdot}5 \\ 2 \\ 8 \\ 32 \end{bmatrix}, \tag{24}$$

where x is the real number which is to be quantized. This neural network evolves on the four-dimensional open hypercube $H = (0, 1)^4$. The objective of the example is to determine the range of values of x for which the network has an equilibrium in the interior or H, and to determine the dimensions of its stable and unstable manifolds.

Taking the second question first, it is easy to verify that T_0 has two negative and two positive eigenvalues. Thus, if the network has an equilibrium in the interior of $\bar{H}$, it is hyperbolic, and its stable and unstable manifolds both have dimension two.

Next, we compute

$$V_{eq} = -\ T^{-1}I = \begin{bmatrix} -2 \\ -0{\cdot}75 \\ -0{\cdot}125 \\ -0{\cdot}1875 \end{bmatrix} + \begin{bmatrix} 1/3 \\ 1/6 \\ 1/12 \\ 1/24 \end{bmatrix} x. \tag{25}$$

It is routine to verify that the above vector belongs to the open hypercube H if and only if $6 < x < 9$. Thus the neural network corresponding to the four-bit A/D converter has an equilibrium in the interior of $\bar{H}$ if and only if x belongs to the open interval $(6, 9)$.

5. Equilibria in the corners

In this section, we study whether any equilibria of the system (1) approach the corners of the hyercube $\bar{H}$ as the sigmoid gain $\lambda \to \infty$. Recall that **b** denotes the Boolean set $\{0, 1\}$, so that $\mathbf{b}^n$ is the set of corners of the closed hypercube $\bar{H}$. Now, since the differential equation (13) evolves on the *open* hypercube H, no vector in $\mathbf{b}^n$ can actually be an equilibrium of this system. However, it is possible that, as $\lambda \to \infty$, some equilibria of (13) *approach* a vector in $\mathbf{b}^n$.

PROPOSITION 5.1.

Let e be an arbitrary vector in $\mathbf{b}^n$. Then an equilibrium of (13) *approaches e as* $\lambda \to \infty$ *if and only if e satisfies the parity condition, defined as follows: Let* $z = Te + I$. *Then*

$$z_i > 0 \text{ if } e_i = 1,\ z_i < 0 \text{ if } e_i = 0. \tag{26}$$

Proof. Put $\dot{u} = 0$ in (13). This gives

$$0 = -A^{-1}u + BG(\lambda u) + y, \tag{27}$$

or

$$u = A[BG(\lambda u) + y]. \tag{28}$$

Now, if we substitute $G(\lambda u) = V = e \in \mathbf{b}^n$, then we get

$$u_{eq} = A[Be + y] = AC^{-1}z. \tag{29}$$

Thus, as $\lambda \to \infty$, $V = G(\lambda u_{eq}) \to e$, *provided*

$$(u_{eq})_i > 0 \text{ if } e_i = 1, (u_{eq})_i < 0 \text{ if } e_i = 0. \tag{30}$$

But since $u_{eq} = AC^{-1}z$, $(u_{eq})_i = R_i z_i$ for all i, and it follows that each component of u_{eq} has the same sign as the corresponding component of z. Hence (30) is equivalent to (26).

PROPOSITION 5.2.

Suppose that an equilibrium of (13) *approaches an element of* $\mathbf{b}^n$ as $\lambda \to \infty$. *Then this equilibrium is exponentially stable for all sufficiently large* λ.

Proof. Linearize (13) around the equilibrium u_{eq} of (29). The Jacobian matrix of the right side of (13) at u_{eq} is

$$-A^{-1} + BJ(\lambda u_{eq})\lambda. \tag{31}$$

By assumption, $\lambda J(\lambda u_{eq}) \to 0$ as $\lambda \to \infty$. Hence the Jacobian approaches $-A^{-1}$, which is a Hurwitx matrix.

Remark. An informal, but informative, way to state the above proposition is: "All equilibria near the corners of $\bar{H}$ are asymptotically stable."

Example 5.3. Consider again the four-bit A/D converter of example 4.2. In Tank & Hopfield (1986) it is claimed that, if x is any real number and if the neural network is started from the zero initial state (i.e. $u_i = 0$ for all i), then eventually the vector V will converge to the correct binary quantization of the real number x. However, for a given x there could be more than one stable equilibrium of the neural network, and depending on the initial condition the solution trajectory of the neural network could converge to an incorrect binary vector. If x is not kept fixed but is changed periodically, then it is necessary to "re-initialize" the network each time x is changed. Otherwise the solution trajectory will converge to an incorrect value. This problem is referred to in Tank & Hopfield (1986) as "hysteresis."

Since the neural network has four neurons, there are $2^4 = 16$ possible binary vectors, or 16 corners to the hypercube $\bar{H}$. By taking each corner in turn, it is possible to determine the values of x for which an equilibrium exists at that corner. This can be done using proposition 5.1. By proposition 5.2, each such equilibrium is asymptotically stable. Hence, for some initial values of u at least, the solution trajectory will converge to that corner.

To illustrate the application of proposition 5.1, consider the corner $e = [1\ \ 0\ \ 1\ \ 1]^t$. Note that the first component represents the lowest or least significant bit whereas the last component represents the highest bit. Hence this vector corresponds to the binary representation of the integer 13. To determine for what values of x an equilibrium exists near this corner, we compute the vector $Be + y$, as per proposition 5.1. This gives

$$Be + y = -\begin{bmatrix} 12{\cdot}5 \\ 28 \\ 44 \\ 56 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ 4 \\ 8 \end{bmatrix} x. \tag{32}$$

Now, in order for an equilibrium to exist near this corner, a necessary and sufficient condition is that

$$-12{\cdot}5 + x > 0, \ -28 + 2x < 0, \ -44 + 4x > 0, \ -56 + 8x > 0. \tag{33}$$

Solving these inequalities shows that an equilibrium exists near this corner if and only if

$$12{\cdot}5 < x < 14. \tag{34}$$

The same process can be repeated at all sixteen binary vectors, and corresponding intervals of x can be computed. This is displayed in table 1. (It is easy to show, using proposition 5.1, that the set of values of x corresponding to a given binary vector is always an interval.) For ease of presentation, the sixteen binary vectors have been shown in terms of the corresponding decimal integer.

From the table one can see that, corresponding to a given real number x for which it is desired to find a binary quantization, there can be as many as *three* distinct asymptotically stable equilibria. Moreover, these equilibria need not be anywhere close to the correct binary quantization. For example, if $x = 4{\cdot}3$, then there are three asymptotically stable equilibria, at (in decimal representation) $e = 3, 4, 8$. As per the convention of Tank and Hopfield, if $3{\cdot}5 < x < 4{\cdot}5$, then the correct binary quantization is 4. Hence one would hope that the neural network would converge towards the corner $e = 4 = [0\ \ 0\ \ 1\ \ 0]^t$. But since there are two other asymptotically stable equilibria, for suitable initial conditions the neural network will in fact converge

Table 1. Range of equilibria for A/D converters.

e	x	e	x	e	x
0	$x < 0{\cdot}5$	1	$0{\cdot}5 < x < 2$	2	$1 < x < 2.5$
3	$2{\cdot}5 < x < 5$	4	$2 < x < 4{\cdot}5$	5	$4.5 < x < 6$
6	$5 < x < 6{\cdot}5$	7	$6{\cdot}5 < x < 9$	8	$4 < x < 8{\cdot}5$
9	$8{\cdot}5 < x < 10$	10	$9 < x < 10{\cdot}5$	11	$10{\cdot}5 < x < 13$
12	$10 < x < 12{\cdot}5$	13	$12{\cdot}5 < x < 14$	14	$13 < x < 14{\cdot}5$
15	$14{\cdot}5 < x$				

towards the corners $e=3$ or 8. Now one can console oneself that the corner $e=3$ is "only off by one" and call it hysteresis, but one would certainly not be willing to accept an answer of 8 when the desired answer is 4. This brings up the question of whether there is an improved version of an A/D converter which does not exhibit such multiple asymptotically stable equilibria. The answer is "yes", as shown in Vidyasagar (1991).

6. Equilibria in the faces of $\bar{H}$

So far we have studied the existence of equilibria in the interior of $\bar{H}$, and near the corners of $\bar{H}$. In this section, we complete the analysis by studying conditions under which there exist equilibria in the *faces* of $\bar{H}$, i.e. equilibria where some components approach 0 or 1 while the remaining components remain bounded away from 0 and 1 as $\lambda \to \infty$.

We are searching for solutions to

$$A^{-1}u = Be + y = z, \tag{35}$$

where *some components* of e belong to $\{0,1\}$ while the remaining components belong to the open interval (0, 1). Note that if some component of e belongs to (0, 1), then the corresponding component of u_{eq} (and of z) must be zero; otherwise $g(\lambda u) \to 0$ or 1 as $\lambda \to \infty$.

To check for such a solution, suppose without loss of generality that the indices $\{1,\ldots,n\}$ have permuted such that the first k components of z are zero while the remaining $n-k$ are nonzero. Thus the first k components of e belong to (0, 1) while the rest belong to $\{0,1\}$. Define

$$e_a = [e_1 \ldots e_k]^t,\ e_b = [e_{k+1} \ldots e_n]^t, \tag{36}$$

and partition y, z, and B commensurately. Then, in partitioned form, (35) becomes

$$0_k = B_{aa}e_a + B_{ab}e_b + y_a, \tag{37}$$

$$z_b = B_{ba}e_a + B_{bb}e_b + y_b, \tag{38}$$

where $e_a \in H^k$ and $e_b \in \mathbf{b}^{n-k}$. Now (37) can be solved for e_a if B_{aa} is nonsingular. By assumption, all principal submatrices of T_0 of size 2×2 or larger are hyperbolic and thus nonsingular. Hence B_{aa} is nonsingular provided $k \geqslant 2$ and T is sufficiently close to T_0. However, if $k=1$ then B_{aa} is just a scalar which will be zero if there are no self-interactions. In this case (37) reduces to

$$0 = B_{ab}e_b + y_a \tag{39}$$

Since $e_b \in \mathbf{b}^{n-1}$, it can assume one of only 2^{n-1} distinct (vector) values. Hence the set of y_a for which (39) can be satisfied is of type LV. This discussion can be summarized as follows:

PROPOSITION 6.1.

If the neural network has no self-interactions, then for all inputs except for those belonging to a set of type LV, there will be no equilibria which approach an edge of $\bar{H}$ as $\lambda \to \infty$.

Now back to (37)–(38) in the case where $k \geqslant 2$. In this case one can solve (37) to obtain

$$e_a = -B_{aa}^{-1}(B_{ab}e_b + y_a). \tag{40}$$

Now fixing $e_b \in \mathbf{b}^{n-k}$ and allowing e_a to vary over H^k defines a face of $\bar{H}$ of dimension k. Depending on e_b ad y_a, the e_a found from (40) may or may not belong to H^k. This is the first condition to be satisfied. The second condition is obtained by substituting for e_a in (38), which gives

$$z_b = (B_{bb} - B_{ba}B_{aa}^{-1}B_{ab})e_b + y_b - B_{ba}B_{aa}^{-1}y_a. \tag{41}$$

This vector has to satisfy the parity condition that $(z_b)_i > 0$ if $(e_b)_i = 1$ and $(z_b)_i < 0$ if $(e_b)_i = 0$. This discussion can be summarized as follows:

PROPOSITION 6.2.

Pick an $e_b \in \mathbf{b}^{n-k}$. *Then, as* $\lambda \to \infty$, *there is an equilibrium of* (13) *approaching the face of* $\bar{H}$ *defined by letting* e_a *vary over* H^k, if and only if two conditions are satisfied:

(i) e_a *as defined in* (40) *belongs to* H^k, *and*
(ii) *if* z_b *is defined by* (41), *then* z_b *satisfies the parity condition with respect to* e_b; *i.e.* $(z_b)_i > 0$ *if* $(e_b)_i = 1$ *and* $(z_b)_i < 0$ *if* $(e_b)_i = 0$.

In addition, for all inputs except those belonging to a set of type LV, there are only a finite number of equilibria in the faces of $\bar{H}$.

Example 6.3. (Three-bit A/D converter) As an illustration of proposition 6.2, consider the same Tank and Hopfield A/D converter circuit, but this time with only three neurons, so that it does a three-bit quantization of a given real number. In this case,

$$T = B = \begin{bmatrix} 0 & -2 & -4 \\ -2 & 0 & -8 \\ -4 & -8 & 0 \end{bmatrix}, \; I = y = -\begin{bmatrix} 0{\cdot}5 \\ 2 \\ 4 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} x. \tag{42}$$

Let $x = 3{\cdot}2$; we show that it is possible to obtain a complete characterization of all equilibria of the neural network.

First, compute

$$V_{eq} = -B^{-1}y = \begin{bmatrix} 0{\cdot}35 \\ 0{\cdot}425 \\ 0{\cdot}4625 \end{bmatrix} \tag{43}$$

Since $V_{eq} \in (0,1)^3$, there is indeed an equilibrium at this point as $\lambda \to \infty$, i.e. as the neural characteristics approach those of an ideal switch. Next, let us check for equilibria in the corners of $\bar{H}^3$. Using the procedure of proposition 5.1 as illustrated in example 5.2, one finds that there are (asymptotically stable) equilibria only at $e = [0 \;\, 0 \;\, 1]^t = 4$ and at $e = [1 \;\, 1 \;\, 0]^t = 3$. Finally, let us check for solutions of (38) in the faces of $\bar{H} = [0, 1]^3$. First, since all diagonal elements of B are zero, it follows from proposition 6.1 that there are no equilibria along the edges of the cube $\bar{H}$. Next we try setting one component of e equal to zero and solving for the other two. If we set $e_1 = 0$, then solving (37) gives

$$\begin{bmatrix} e_2 \\ e_3 \end{bmatrix} = \begin{bmatrix} 0{\cdot}6 \\ 0{\cdot}55 \end{bmatrix} \in (0, 1)^2. \tag{44}$$

Thus it can be concluded that, as $\lambda \to \infty$, there will be an equilibrium near $V = [0 \;\; 0{\cdot}6 \;\; 0{\cdot}55]'$. Similarly it can be verified that there will be another equilibrium near $[1 \;\; 0{\cdot}1 \;\; 0{\cdot}3]'$, and that these are the only equilibria along the faces of $\bar{H}^3$.

Now let us study the stability of the equilibria in the faces. Once the integer k and the vector $e_b \in \mathbf{b}^{n-k}$ are fixed, proposition 6.2 allows one to determine whether there exists an equilibrium in the corresponding face of $\bar{H}$. To study the stability of this equilibrium, define

$$\Lambda_k = \begin{bmatrix} \lambda I_k & 0 \\ 0 & I_{n-k} \end{bmatrix}, \; u_k = \Lambda_k u. \tag{45}$$

Then from (13) it follows that

$$\dot{u}_k = \Lambda_k \dot{u} = -\Lambda_k A^{-1} u + \Lambda_k BG(\lambda u) + \Lambda_k y \tag{46}$$

$$= -A^{-1} u_k + \Lambda_k BG(\lambda \Lambda_k^{-1} u_k) + \Lambda_k y. \tag{47}$$

Here we have used the obvious fact that

$$\Lambda_k A^{-1} \Lambda_k^{-1} = A^{-1}, \tag{48}$$

since all matrices are diagonal. Now let $\lambda \to \infty$. Then the fact that the conditions of proposition 6.2 are satisfied ensures that

$$G(\lambda \Lambda_k^{-1} u_k) \to \begin{bmatrix} e_a \\ e_b \end{bmatrix} = e. \tag{49}$$

Define

$$u_{k*} = G^{-1}(e). \tag{50}$$

Now linearize (13) around the equilibrium in u_k-space. The Jacobian matrix is

$$-A^{-1} + \Lambda_k BJ(\lambda \Lambda_k^{-1} u_{k*}) \lambda \Lambda_k^{-1}. \tag{51}$$

Now consider separately the matrix

$$M = J(\lambda \Lambda_k^{-1} u_{k*}) \lambda \Lambda_k^{-1}. \tag{52}$$

This is a diagonal matrix; moreover

$$m_{ii} = g_i'[(u_k)_i], \text{ for } 1 \leqslant i \leqslant k, \tag{53}$$

$$m_{ii} = g_i'[\lambda u_i)] \to 0 \text{ as } \lambda \to \infty, \text{ for } k+1 \leqslant i \leqslant n. \tag{54}$$

Hence, as $\lambda \to \infty$,

$$M \to \begin{bmatrix} M_a & 0 \\ 0 & 0 \end{bmatrix}, \tag{55}$$

$$-A^{-1} + \Lambda_k BM \to \begin{bmatrix} -A_a^{-1} + \lambda B_{aa} M_a & 0 \\ B_{ba} M & -A_b^{-1} \end{bmatrix}. \tag{56}$$

As $\lambda \to \infty$, the matrix becomes block-triangular. Moreover, in the upper left block, the $-A_a^{-1}$ term becomes insignificant in comparison to the $\lambda B_{aa}M_a$ term. Thus the spectrum of the linearized system approaches

$$\text{spec}(\lambda B_{aa}M_a) \cup \text{spec}(-A_b^{-1}). \tag{57}$$

Of course the spectrum of $-A_b^{-1}$ is just $-\{\alpha_{k+1}^{-1}, \ldots, -\alpha_n^{-1}\}$. Now, as in the proof of proposition 4.1, it is easy to show that the matrix $B_{aa}M_a$ has the same number of eigenvalues with negative and positive real parts as does the matrix $(T_0)_{aa}$, provided $\| T - T_0 \|$ is sufficiently small. This discussion can be summarized as follows:

PROPOSITION 6.4.

Let $J = \{j_1, \ldots, j_k\}$ be a subset of $\{1, \ldots, n\}$, with $2 \leqslant k \leqslant n-1$. Pick a value of either 0 *or* 1 *for each $e_j, j \notin J$, and consider the corresponding k-dimensional face of $\bar{H}$ defined by*

$$e_j \in (0, 1) \text{ for } j \in J. \tag{58}$$

Let S be any closed subset of this face, and define the $k \times k$ principal submatrix of T_0:

$$(T_0)_{JJ} = [(T_0)_{ij}, i, j \in J]. \tag{59}$$

Suppose $(T_0)_{JJ}$ has v positive eigenvalues. Then there exists an $\varepsilon > 0$ such that the following statements are true whenever $\| T - T_0 \| < \varepsilon$.

(i) As $\lambda \to \infty$, at the most one equilibrium of (13) approaches S, and it can be found using proposition 6.2.
(ii) This equilibrium is hyperbolic. Its stable and unstable manifolds have dimensions $n - v$ and v respectively.

Remark. Once the index set J is fixed, there are 2^{n-k} different combinations of values that can be assigned to the components $e_j, j \notin J$. Proposition 6.4 makes it clear that equilibria in *each* of these faces, if any, have the same "signature." Put another way, equilibria in "opposite" faces of $\bar{H}$ have the same signature.

Example 6.5. Let us continue example 6.3. The analysis previously carried out shows that there is an equilibrium at $V_{eq} = [0{\cdot}35 \;\; 0{\cdot}425 \;\; 0{\cdot}4625]^t$. Now the matrix T of (42) has one negative and two positive eigenvalues. Accordingly, from proposition 4.1, this equilibrium has a stable manifold of dimension one and an unstable manifold of dimension two. Next, there are asymptotically stable equilibria at $e_1 = [0 \;\; 0 \;\; 1]^t$ and $e_2 = [1 \;\; 1 \;\; 0]^t$. Now consider equilibria in the faces. Letting $J = \{2, 3\}$ and assigning $e_1 = 0$ leads to the equilibrium at $V_1 = [0 \;\; 0{\cdot}6 \;\; 0{\cdot}55]^t$, whereas assigning $e_1 = 1$ leads to the equilibrium $V_2 = [1 \;\; 0{\cdot}1 \;\; 0{\cdot}3]^t$. These equilibria are in opposite faces of the three-dimensional cube $[0, 1]^3$. Now

$$(T_0)_{JJ} = \begin{bmatrix} t_{022} \, t_{023} \\ t_{032} \;\; t_{033} \end{bmatrix} = \begin{bmatrix} 0 & -8 \\ -8 & 0 \end{bmatrix}. \tag{60}$$

This matrix has one positive eigenvalue. This shows that both V_1 and V_2 have stable manifolds of dimension two and an unstable manifold of dimension one.

The most important point to note about this example is that *all the above conclusions remain valid even if the interconnection matrix is perturbed slightly from its original* *symmetric value*. Of course the actual values of the various equilibria will change slightly in a continuous fashion, but the dimensions of the various stable and unstable manifolds will not change.

7. Rate of convergence of trajectories

In this section some preliminary results are given about the rate at which the equilibria of the system (13) approach the corners of $\bar{H}$, and the rate at which the solution trajectories approach these equilibria.

Suppose e is a vector in $\mathbf{b}^n$, i.e. suppose e is a corner point of the hypercube $\bar{H}$. Then proposition 5.1 states that the system (13) has an equilibrium approaching e if and only if the vector $z = Te + I$ has the same "parity" as e. Thus, in proposition 5.1, only the signs of the various components of z are pertinent, and their magnitudes do not play any role in determining whether or not *there exists* an equilibrium near a particular corner. Now it is shown that the magnitudes of the components of z do determine the *speed of convergence* of the equilibrium to e as the sigmoid gain $\lambda \to \infty$.

To be specific, suppose all the neural characteristics are identical, and are given by

$$g_i(x) = 1/(1 + e^{-x}), \quad \text{for } i = 1, \ldots, n. \tag{61}$$

Suppose $e \in \mathbf{b}^n$ and that $z = Te + I$ has the same parity as e. In accordance with (26), define

$$u_{eqi} = (\alpha_i/\beta_i) z_i = R_i z_i, \quad \text{for } i = 1, \ldots, n. \tag{62}$$

Now define

$$V_{eq} = G(\lambda u_{eq}), \tag{63}$$

and let $\lambda \to \infty$, i.e. let the sigmoid characteristics become steeper and steeper

PROPOSITION 7.1.

Let all symbols be as defined above. Then

$$\lim_{\lambda \to \infty} \frac{\ln |e_i - v_{eqi}|}{\ln |e_j - v_{eqj}|} = \frac{|u_{eqi}|}{|u_{eqj}|}. \tag{64}$$

Proof. Suppose first that $u_i > 0$, $e_i = 1$ (by the parity condition). Then, as $\lambda \to \infty$, we have

$$v_{eqi} = 1/[1 + \exp(-\lambda u_{eqi})] \approx 1 - \exp(-\lambda u_{eqi}), \tag{65}$$

$$\ln(1 - v_{eqi}) \approx -\lambda u_{eqi}. \tag{66}$$

Now suppose $u_i < 0$, $e_i = 0$. Then, as $\lambda \to \infty$, we have

$$\ln v_{eqi} = -\ln[1 + \exp(-\lambda u_{eqi})] \approx -\ln[\exp(-\lambda u_{eqi})] = \lambda u_{eqi}. \tag{67}$$

The relationship (64) now follows readily from (66) and (67).

Proposition 7.1 addresses the issue of the rapidity with which V_{eq} approaches the corner e as $\lambda \to \infty$. Basically, the larger the value of $|u_{eqi}|$, the more rapidly v_{eqi} approaches e_i. One can also explore the time behaviour of the solution trajectories of (13) for a fixed "large" value of λ.

PROPOSITION 7.2.

Let all symbols be as defined above. Then

$$\lim_{\lambda \to \infty} \lim_{t \to \infty} \frac{\ln|v_i(t) - v_{eqi}|}{\ln|v_j(t) - v_{eqj}|} = \frac{\alpha_j}{\alpha_i}. \tag{68}$$

Proof. Suppose λ is "large" and that the initial condition $u_i(0)$ is "near" u_{eqi}. Then it follows from (31) that

$$u_i(t) \approx u_{eqi} + [u_i(0) - u_{eqi}]\exp(-t/\alpha_i). \tag{69}$$

Suppose $u_{eqi} > 0$. Then, in analogy with (66), we have

$$\begin{aligned} v_i(t) &= 1/\{1 + \exp[-\lambda u_i(t)]\} \approx 1 - \exp[-\lambda u_i(t)] \\ &\approx 1 - \exp\{\lambda[u_{eqi} + (u_i(0) - u_{eqi})e^{-t/\alpha_i}]\} \\ &= 1 - \exp(\lambda u_{eqi})\cdot\exp[\lambda(u_i(0) - u_{eqi})e^{-t/\alpha_i}] \\ &\approx 1 - \exp(\lambda u_{eqi})\cdot[1 + \lambda(u_i(0) - u_{eqi})e^{-t/\alpha_i}] \\ &\approx v_{eqi} - \lambda \exp(\lambda u_{eqi})[u_i(0) - u_{eqi}]\exp(-t/\alpha_i), \end{aligned} \tag{70}$$

$$\ln[|v_i(t) - v_{eqi}|] \approx -(t/\alpha_i) + \lambda u_{eqi} + \ln \lambda |u_i(0) - u_{eqi}|. \tag{71}$$

As $t \to \infty$ for a fixed λ, the first term on the right side dominates the rest. A similar approximation applies when $u_{eqi} < 0$. The desired result (68) now follows readily.

Proposition 7.2 shows that, in the case where all neurons are identical (i.e. $\alpha_i = \alpha$ for all i), the trajectory in the V-space converges to the equilibrium at essentially the same rate in all components.

Example 7.3. Consider again the four-bit A/D converter of examples 4.2 and 5.3. Suppose the input x equals, say, 5.4. In this case, from table 1, one sees that there are three equilibria, namely at $e = [1\ 0\ 1\ 0]^t = 5$, $[0\ 1\ 1\ 0]^t = 6$, and $[0\ 0\ 0\ 1]^t = 8$. Table 2 shows the corresponding values of $z = Te + I$.

From the table one can see that, in two out of the three cases (in fact the two which represent the best digital approximations to the given input x, the components of z are smallest in magnitude corresponding to the least significant bits, and largest in magnitude corresponding to the most significant bits. Thus, as the sigmoid nonlinearities

Table 2. Rate of convergence for A/D converters.

e	z			
$[1\ 0\ 1\ 0]^t$	[0·9	−1·2	9·6	$27{\cdot}2]^t$
$[0\ 1\ 1\ 0]^t$	[−1·1	0·8	5·6	$-36{\cdot}8]^t$
$[0\ 0\ 0\ 1]^t$	[−3·1	−7·2	−18·4	$11{\cdot}2]^t$

become steeper and steeper ($\lambda \to \infty$), one would expect that the most significant bits converge meet rapidly to the "correct" values. The same phenomenon can be observed for almost all values of the input variable x. The details are routine and are left to the reader.

8. Existence of equilibria in the corners

Proposition 5.1 states that *if* the system (13) has any equilibria near the corners of $\bar{H}$, then these are asymptotically stable. But, under certain circumstances, there might be *no* equilibria near the corners of $\bar{H}$.

First a positive result.

PROPOSITION 8.1.

Suppose the interconnection matrix T satisfies the following conditions: (i) *T is symmetric, and all of its diagonal elements are zero;* (ii) *Every principal submatrix of T of size 2×2 or larger, including T itself, is hyperbolic and has at least one positive eigenvalue. Under these conditions, for all inputs I expect those belonging to a set of type* LV, *there exists at least one binary vector $e \in \mathbf{b}^n$ such that $Te + I$ has the same parity as e.*

Proof. The assumptions ensure that the neural network exhibits total stability, i.e. every solution trajectory converges to an equilibrium (Hirsch 1987). Propositions 4.1, 6.1 and 6.2 show that there can be no asymptotically stable equilibria except near the corners of $\bar{H}$, while proposition 6.2 guarantees that there can only be a finite number of equilibria in the faces of $\bar{H}$. All these facts plus total stability lead one to conclude that there must exist at least one asymptotically stable equilibrium near a corner of $\bar{H}$. By proposition 5.1, this is equivalent to the parity condition being satisfied at some corner of $\bar{H}$. This is the desired conclusion.

Now an example to show that proposition 8.1 is *not* valid if the interconnection matrix T is perturbed.

Example 8.2. Consider a two-neuron network with the interconnection matrix

$$T = \begin{bmatrix} -\varepsilon & -1 \\ -1 & \varepsilon \end{bmatrix}. \tag{72}$$

Then, by applying proposition 5.1, one can verify that if

$$0 < i_1 < \varepsilon, i_2 < -\varepsilon, \tag{73}$$

then *none of the four vectors* in $\mathbf{b}^2$ satisfies the parity condition. But by applying proposition 6.2, one can see that there is an equilibrium near

$$e_1 = (i_1/\varepsilon), e_2 = 0. \tag{74}$$

To determine the signature of this equilibrium, let

$$u_1 = g_1^{-1}(e_1), m_{11} = g_1'(u_1) > 0. \tag{75}$$

Then, by (57) the spectrum of the linearized system around the equilibrium is asymptotically equal to

$$\{-\lambda m_{11}, -\alpha_2\}. \tag{76}$$

Hence this equilibrium is asymptotically stable.

The point of proposition 8.1 and example 8.2 is the following: Under ideal conditions, there is (almost) always an asymptotically stable equilibrium near a corner of $\bar{H}$. Since the parity condition of proposition 5.1 is just an algebraic relationship, it is easy to see that, *for each fixed input vector I*, there is a small allowed perturbation of T such that there continues to exist an equilibrium near some corner of $\bar{H}$. But example 8.2 shows that the order of the quantifiers cannot be interchanged: It is *not* true that there exists a small allowed perturbation of T for which there continues to exist an equilibrium near some corner of $\bar{H}$.

As a final comment, observe that the proof of proposition 8.1 is quite round-about and unsatisfactory. The parity condition involves only linear algebra, and as such one would expect to be able to find a proof of the proposition based purely on linear algebra.

9. Conclusions

In this paper, we have given a complete analysis of the equilibria of neural networks with linear interconnections, in the case where the interconnection matrix is "nearly" symmetric, but *without* assuming that the interconnection matrix is actually symmetric. This allows one to analyse, among other things, the effects of imperfect implementation on the behaviour of Hopfield neural networks.

If a neural network has symmetric interconnections, then (Hirsch 1987) the network exhibits *total* stability, i.e. all solutions approach an equilibrium. This means, for example, that there are no limit cycles, i.e. there are no nontrivial periodic solutions. This conclusion depends heavily on the ability to construct a total Lyapunov or energy function, and the energy function of Hirsch (1987) is only valid if the interconnection matrix is symmetric. Thus it is still an open question as to whether a network with "nearly" symmetric interconnections can exhibit limit cycles, and if so, under what conditions.

Another issue which is as yet unresolved, even in the symmetric interconnections case, is that of calculating (or at least estimating) the basin or domain of attraction of each asymptotically stable equilibrium, which we now know can only lie in the corners of the hypercube $\bar{H}$ if the interconnection matrix has zero diagonal elements. This is a topic for further research.

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