SYMMETRIC IDEALS IN GROUP RINGS AND SIMPLICIAL HOMOTOPY

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Abstract. In this paper homotopical methods for the description of subgroups determined by ideals in group rings are introduced. It is shown that in certain cases the subgroups determined by symmetric product of ideals in group rings can be described with the help of homotopy groups of spheres.

1. Introduction

The purpose of this paper is to use the homotopical methods for the description of subgroups determined by certain ideals, here called symmetric ideals, in free group rings.

Let F be a free group, $\mathbb{Z}[F]$ its integral group ring and I a two-sided ideal in $\mathbb{Z}[F]$. The general problem of description of the normal subgroup

$$D(F;I) := F \cap (1+I)$$

of F is very difficult. As an illustration of the complexity of answers for different particular cases we may mention some examples. Let R be a normal subgroup of F, $\mathbf{r} = (R-1)\mathbb{Z}[F]$, and $\mathbf{f}$ the augmentation ideal of $\mathbb{Z}[F]$, then [5]

$$F \cap (1+\mathbf{f}^2\mathbf{r}^2) = \gamma_3(R \cap [F,F])\gamma_4(R),$$
$$F \cap (1+\mathbf{r}\mathbf{f}^2\mathbf{r}) = [R \cap [F,F], R \cap [F,F], R]\gamma_4(R).$$

The subtility of the dimension subgroup problem is well-known; this is the case when $I = \mathbf{f}^n + \mathbf{r}$. For a survey of problems in this area, see [12], [6], [10].

Given a ring A and two-sided ideals $I_1, \dots, I_n$ ($n \geq 2$) in A consider their symmetric product:

$$(I_1 \dots I_n)_S := \sum_{\sigma \in \Sigma_n} I_{\sigma_1} \dots I_{\sigma_n},$$

where Σ_n is the symmetric group of degree n. For example, in the case $n=2$, one has $(I_1I_2)_S = I_1I_2 + I_2I_1$. Observe that while $(I_1 \dots I_n)_S \subseteq I_1 \cap \dots \cap I_n$ always, the reverse inclusion does not hold, in general.

Let F be a free group, and let $R_1, \dots, R_n$ be normal subgroups of F. Consider the induced two-sided ideals in the integral group ring $\mathbb{Z}[F]$ defined as $\mathbf{r}_i = (R_i - 1)\mathbb{Z}[F]$, $i = 1, \dots, n$. The following problems arise naturally:

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(1) Identify the quotient

$$Q(\mathbf{r}_1, \ldots, \mathbf{r}_n) := \frac{\mathbf{r}_1 \cap \cdots \cap \mathbf{r}_n}{(\mathbf{r}_1 \ldots \mathbf{r}_n)_S}.$$

(2) Identify the normal subgroup of F, determined by the ideal $(\mathbf{r_1} \ldots \mathbf{r}_n)_S$, i.e., the subgroup

$$D(F; (\mathbf{r}_1 \ldots \mathbf{r}_n)_S) := F \cap (1 + (\mathbf{r}_1 \ldots \mathbf{r}_n)_S).$$

Let $[R_1, \ldots, R_n]_S$ denote the symmetric commutator subgroup, namely, $\prod_{\sigma\in\Sigma_n}[\ldots [R_{\sigma_1}, R_{\sigma_2}], \ldots, R_{\sigma_n}]$ of the normal subgroups $R_1, \ldots, R_n$:

$$[R_1, \ldots, R_n]_S = \prod_{\sigma\in\Sigma_n}[\ldots [R_{\sigma_1}, R_{\sigma_2}], \ldots, R_{\sigma_n}].$$

Observe that we have always

$$[R_1, \ldots, R_n]_S \subseteq D(F; (\mathbf{r}_1 \ldots \mathbf{r}_n)_S).$$

The fundamental theorem of free group rings (see [6], Theorem 3.1, p. 12) states that, for all $n \geq 1$, the above inclusion is an equality in case $R_i = F$ for $i = 1, \ldots, n$. Apart from this case, there is hardly anything else that seems to be available in the literature about the subgroups $D(F; (\mathbf{r}_1 \ldots \mathbf{r}_n)_S)$, in general. Naturally, one would like to investigate the validity, or otherwise, of the inclusion $D(F; (\mathbf{r}_1 \ldots \mathbf{r}_n)_S) \subseteq [R_1, \ldots, R_n]_S$. Let

$$f_{F;R_1,\ldots,R_n} : \frac{R_1 \cap \cdots \cap R_n}{[R_1, \ldots, R_n]_S} \to \frac{\mathbf{r}_1 \cap \cdots \cap \mathbf{r}_n}{(\mathbf{r}_1 \ldots \mathbf{r}_n)_S},$$

be the natural map defined by

$$f_{F;R_1,\ldots,R_n} : g.[R_1, \ldots, R_n]_S \mapsto g - 1 + (\mathbf{r}_1 \ldots \mathbf{r}_n)_S,\ g \in R_1 \cap \cdots \cap R_n.$$

The main idea of this paper is based on the fact that, for a certain choice of groups $F, R_1, \ldots, R_n$, there exists a space X, such that the map $f_{F;R_1,\ldots,R_n}$ is the $(n-1)$st Hurewicz homomorphism:

$$\begin{tikzcd}
\frac{R_1\cap\ldots\cap R_n}{[R_1,\ldots,R_n]_S} \arrow[r, "f_{F;R_1,\ldots,R_n}"] \arrow[d, equal] & \frac{\mathbf{r}_1\cap\cdots\cap\mathbf{r}_n}{(\mathbf{r}_1\ldots\mathbf{r}_n)_S} \arrow[d, equal] \\
\pi_{n-1}(X) \arrow[r] & H_{n-1}(X)
\end{tikzcd}$$

In that case, the quotient $\frac{D(F;(\mathbf{r}_1\ldots\mathbf{r}_n)_S)}{[R_1,\ldots,R_n]_S}$ is exactly the kernel of Hurewicz homomorphism and we are able to use arguments from simplicial homotopy for the computation of subgroups determined by symmetric product of ideals. Our analysis also yields an example where the inclusion

$$D(F; (\mathbf{r}_1 \ldots \mathbf{r}_n)_S) \supseteq [R_1, \ldots, R_n]_S$$

is proper.

In Section 2 we prove certain technical results needed for our investigation. Our main results are Theorems 3.1 and 3.2 (see Section 3).

2. Technical results

We need some preparation for proving our main results. Given a group G, let $\Delta(G)$ denote the augmentation ideal of its integral group ring $\mathbb{Z}[G]$. The following result is well-known.

Lemma 2.1. *If N is a normal subgroup of a group G, then $N\cap(1+\Delta(N)\Delta(G)) = [N,\, N]$.*

For the case of two normal subgroups in the free group F, we have the following

Proposition 2.1. *Let $F = R_1R_2$. Then the map*

$$f_{F;\,R_1,\,R_2} : \frac{R_1\cap R_2}{[R_1,\, R_2]} \to \frac{\mathbf{r}_1\cap\mathbf{r}_2}{\mathbf{r}_1\mathbf{r}_2+\mathbf{r}_2\mathbf{r}_1}$$

is an isomorphism. In particular, $D(F;\, (\mathbf{r}_1\mathbf{r}_2)_S) = [R_1,\, R_2]$.

Proof. Let $T = \{t_i\}_{i\in I} \subseteq R_1$ be a left transversal for R_2 in F. Then every element $f \in F$ can be written uniquely as $f = ts$ with $t \in T$ and $s \in R_2$; in particular, if $f \in R_1$, then $s \in R_1 \cap R_2$. Let $\varphi : \mathbb{Z}[F] \to \mathbb{Z}[R_2]$ be the $\mathbb{Z}$-linear extension of the map $F \to R_2$ given by $f \mapsto s$. Observe that $\mathbf{r}_1\mathbf{r}_2 = \Delta(R_1)\Delta(R_2)$ and $\mathbf{r}_2\mathbf{r}_1 = \Delta(R_2)\Delta(R_1)$ since $F = R_1R_2$. Furthermore,

$$\varphi(\mathbf{r}_1\mathbf{r}_2 + \mathbf{r}_2\mathbf{r}_1) \subseteq \Delta(R_1\cap R_2)\Delta(R_2) + \Delta([R_1,\, R_2])\mathbb{Z}[R_2].$$

Consider the map

$$\theta : R_1\cap R_2 \to \frac{\mathbf{r}_1\cap\mathbf{r}_2}{\mathbf{r}_1\mathbf{r}_2+\mathbf{r}_2\mathbf{r}_1}, \quad f \mapsto f - 1 + \mathbf{r}_1\mathbf{r}_2 + \mathbf{r}_2\mathbf{r}_1.$$

Clearly θ is a homomorphism and $[R_1,\, R_2] \subseteq \ker\theta$. Let $f \in R_1 \cap R_2$ be an element in $\ker\theta$. We then have

$$f - 1 = \varphi(f-1) \in \Delta(R_1\cap R_2)\Delta(R_2) + \Delta([R_1,\, R_2])\mathbb{Z}[R_2]$$

in the group ring $\mathbb{Z}[R_2]$. Thus, going modulo $[R_1,\, R_2]$ and invoking Lemma 2.1 with $G = R_1/[R_1,\, R_2]$, $N = (R_1 \cap R_2)/[R_1,\, R_2]$, we must have $f \in [R_1,\, R_2]$. Consequently θ induces a monomorphism

$$f_{R_1,R_2} : \frac{R_1\cap R_2}{[R_1,\, R_2]} \hookrightarrow \frac{\mathbf{r}_1\cap\mathbf{r}_2}{\mathbf{r}_1\mathbf{r}_2+\mathbf{r}_2\mathbf{r}_1}.$$

Let $\alpha \in \mathbf{r}_1$. Then $\alpha = \sum_i (r_i - 1)\beta_i$ with $r_i \in R_1$ and $\beta_i \in \mathbb{Z}[R_2]$. Now $r_i = t_{i_j}s_{i_J}$ with $t_{i_j} \in T$ and $s_{i_j} \in R_1 \cap R_2$. Therefore,

$$\alpha \equiv (w-1) + \sum_k m_k(t_k - 1) \mod \mathbf{r}_1\mathbf{r}_2 + \mathbf{r}_2\mathbf{r}_1$$

with $m_k \in \mathbb{Z}$ and $w \in R_1 \cap R_2$. It follows that if $\alpha \in \mathbf{r}_1 \cap \mathbf{r}_2$, then $m_k = 0$ for all k, and we thus conclude that f_{R_1,R_2} is an epimorphism and hence an isomorphism. □

Lemma 2.2. *Let $X = X_1 \sqcup \dots \sqcup X_n$ $(n \geq 2)$ be a disjoint union of sets. Let $p_i : F(X) \to F(X_1 \sqcup \dots \hat{X}_i \dots \sqcup X_n)$, $i = 1, \dots, n$, be the natural projections induced by*

$$p_i(x) = \begin{cases} x, & for\ x \in X \backslash X_i \\ 1, & for\ x \in X_i \end{cases}$$

and $R_i = \ker(p_i)$. Then

(i) *$R_1 \cap \dots \cap R_n = [R_1, \dots, R_n]_S$ in $F(X)$;*

(ii) *$\mathbf{r}_1 \cap \dots \cap \mathbf{r}_k = (\mathbf{r}_1 \dots \mathbf{r}_n)_S$ in $\mathbb{Z}[F(X)]$.*

Proof. The statement (i) follows from ([14], Corollary 3.5) (see [1]).

For the proof of (ii) observe first that, for each $i, j \in \{1, 2, \dots, n\}$, $i \neq j$, we have

$$R_i = \langle X_i \rangle^{F(X \backslash X_j)}[R_i,\, R_j].$$

Since $\mathbb{Z}[F(X)] = \mathbb{Z}[F(X \setminus X_j)] + \mathbf{r}_j$ and $[R_i,\, R_j] - 1 \subseteq \mathbf{r}_i\mathbf{r}_j + \mathbf{r}_j\mathbf{r}_i$, it follows that

$$\mathbf{r}_i = (\langle X_i \rangle^{F(X \backslash X_j)} - 1)\mathbb{Z}[F(X \setminus X_j)] + \mathbf{r}_i\mathbf{r}_j + \mathbf{r}_j\mathbf{r}_i, \tag{2.1}$$

and consequently, we have

$$\mathbf{r}_i \cap \mathbf{r}_j = \mathbf{r}_i\mathbf{r}_j + \mathbf{r}_j\mathbf{r}_i,\ i \neq j.$$

Suppose that, for some k, $2 \leq k < n$, we have shown that

$$\mathbf{r}_{i1} \cap \dots \cap \mathbf{r}_{ik} = (\mathbf{r}_{i1} \dots \mathbf{r}_{ik})_S$$

for all subsets of k elements from $\{1, \dots, n\}$, and let j be an integer, $1 \leq j \leq n$, $j \notin \{i1, \dots, ik\}$. From (2.1), we have

$$\mathbf{r}_{il} = (\langle X_{il} \rangle^{F(X \backslash X_j)} - 1)\mathbb{Z}[F(X \setminus X_j)] + \mathbf{r}_{il}\mathbf{r}_j + \mathbf{r}_j\mathbf{r}_{il},\ l = i1, \dots, ik.$$

Consequently

$$(\mathbf{r}_{i1} \dots \mathbf{r}_{ik})_S \subseteq \mathbb{Z}[F(X \setminus X_j)] + (\mathbf{r}_{i1} \dots \mathbf{r}_{ik}\mathbf{r}_j)_S.$$

An application of the natural projection $\mathbb{Z}[F(X)] \to \mathbb{Z}[F(X \setminus X_j)]$ induced by the map which is identity on $X \setminus X_j$ and trivial on X_j then shows that

$$\mathbf{r}_j \cap \mathbf{r}_{i1} \cap \dots \cap \mathbf{r}_{ik} \subseteq (\mathbf{r}_{i1} \dots \mathbf{r}_{ik}\mathbf{r}_j)_S.$$

The reverse inclusion being trivial, it follows that the intersection of $k+1$ distinct ideals out of $\mathbf{r}_1, \dots, \mathbf{r}_n$ equals the corresponding symmetric sum of their products, and thus, by induction, assertion (ii) is proved. □

3. Simplicial constructions

3.1. **Milnor's construction.** Recall that, for a given pointed simplicial set K, the Milnor $F(K)$-construction [11] is the simplicial group with $F(K)_n = F(K_n \setminus *)$, where $F(-)$ is the free group functor. Consider the simplicial circle $S^1 = \Delta[1]/\partial\Delta[1]$:

$$S^1_0 = \{*\},\ S^1_1 = \{*,\, \sigma\},\ S^1_2 = \{*,\, s_0\sigma,\, s_1\sigma\},\ \dots,\ S^1_n = \{*,\, x_0,\, \dots,\, x_{n-1}\}, \tag{3.1}$$

where $x_i = s_{n-1} \dots \hat{s}_i \dots s_0\sigma$. For the Milnor construction $F(S^1)$, $F(S^1)_n$ is a free group of rank n, for $n \geq 1$:

$$F(S^1): \quad \dots \;\substack{\longrightarrow\\ \cdots\\ \longleftarrow\!\!\!\longrightarrow\\ \cdots\\ \longleftarrow}\; F_3 \;\substack{\longrightarrow\\ \longrightarrow\\ \longleftarrow\!\!\!\longrightarrow\\ \longleftarrow\!\!\!\longrightarrow\\ \longleftarrow}\; F_2 \;\substack{\longrightarrow\\ \longleftarrow\!\!\!\longrightarrow\\ \longleftarrow}\; \mathbb{Z}$$

with face and degeneracy homomorphisms:

$$\partial_i : F_n \to F_{n-1},\; i = 0, \dots, n,\; n = 2, 3, \dots$$
$$s_i : F_n \to F_{n+1},\; i = 0, \dots, n,\; n = 1, 2, \dots .$$

There is a homotopy equivalence [11]:

$$|F(S^1)| \simeq \Omega S^2.$$

Hence, for $n \geq 2$, the nth homotopy group of S^2 can be described as an intersection of kernels in degree $n-1$ modulo simplicial boundaries. Following [14], denote the elements from a basis of F_{n+1} as follows:

$$y_n = s_{n-1} \dots s_0\sigma,$$
$$y_i = s_n \dots \hat{s}_i \dots s_0\sigma(s_n \dots \hat{s}_{i+1} \dots s_0\sigma)^{-1},\; 0 \leq i < n.$$

Then, it follows from standard simplicial identities that, in the free group F_{n+1}, one has

$$\ker(\partial_0) = y_0 \dots y_n,$$
$$\ker(\partial_i) = \langle y_{i-1}\rangle^{F_{n+1}},\; 0 < i \leq n+1$$

Lemma 2.2 applied to the case $X = \{y_0, \dots, y_n\}$, implies that

$$\ker(\partial_1) \cap \dots \cap \ker(\partial_{n+1}) = [\ker(\partial_1), \dots, \ker(\partial_{n+1})]_S$$

Therefore, there is a natural presentation of the $(n+1)$st homotopy group of S^2 given first by Wu [14]:

$$\pi_{n+1}(S^2) \simeq \frac{\ker(\partial_0) \cap \dots \cap \ker(\partial_n)}{[\ker(\partial_0), \dots, \ker(\partial_n)]_S},\; n \geq 1.$$

For similar results obtained without simplicial constructions see [7].

Theorem 3.1. *Let $n \geq 3$, F_n a free group with a basis $\{x_1, \dots, x_n\}$. Let $R_i = \langle x_i\rangle^{F_n}$, $i = 1, \dots, n$, $R_{n+1} = \langle x_1 \dots x_n\rangle^{F_n}$. Then*

(*i*) *there is an isomorphism $Q(\mathbf{r}_1, \dots, \mathbf{r}_{n+1}) \simeq \mathbb{Z}$;*

(*ii*) *$D(F; (\mathbf{r}_1 \dots \mathbf{r}_{n+1})_S) = R_1 \cap \dots \cap R_{n+1}$.*

Furthermore, $R_1 \cap \dots \cap R_{n+1} \neq [R_1, \dots, R_{n+1}]_S$ for $n \neq 0 \mod 8$.

Proof. First apply the functor $\mathbb{Z}[-]$ to the Milnor construction $F(S^1)$:

$$\mathbb{Z}[F(S^1)]: \quad \dots \;\substack{\longrightarrow\\ \cdots\\ \longleftarrow\!\!\!\longrightarrow\\ \cdots\\ \longleftarrow}\; \mathbb{Z}[F_3] \;\substack{\longrightarrow\\ \longrightarrow\\ \longleftarrow\!\!\!\longrightarrow\\ \longleftarrow\!\!\!\longrightarrow\\ \longleftarrow}\; \mathbb{Z}[F_2] \;\substack{\longrightarrow\\ \longleftarrow\!\!\!\longrightarrow\\ \longleftarrow}\; \mathbb{Z}[\mathbb{Z}]$$

By definition of homology, we have

$$\pi_n\mathbb{Z}[F(S^1)] = H_n(\Omega S^2).$$

From the classical suspension splitting theorem of loop suspensions [8], we have

$$\Sigma\Omega S^2 \simeq \bigvee_{k=2}^{\infty} S^k$$

and so $H_n(\Omega S^2) = \mathbb{Z}$ for each $n \geq 1$. Thus $\pi_n\mathbb{Z}[F(S^1)] = \mathbb{Z},\ n \geq 1$. The kernels of homomorphisms

$$\bar{\partial}_i : \mathbb{Z}[F_{n+1}] \to \mathbb{Z}[F_n],\ i = 0,\ \dots,\ n+1$$

are ideals

$$(\ker(\partial_i) - 1)\mathbb{Z}[F_{n+1}],\ i = 0,\ \dots,\ n+1.$$

Making the enumeration in the free group F_n: $x_i = y_{i+1},\ i = 0,\ \dots,\ n-1$, lemma 2.2 (ii) implies that

$$H_n(\Omega S^2) \simeq Q(\mathbf{r}_1,\ \dots,\ \mathbf{r}_{n+1}) \simeq \mathbb{Z}$$

and the statement (i) is proved.

For proving (ii), observe now that there is a natural diagram

$$\begin{tikzcd} \frac{R_1\cap\cdots\cap R_{n+1}}{[R_1,\dots,R_{n+1}]_S} \arrow[r, "f_{F;R_1,\dots,R_{n+1}}"] \arrow[d, equal] & \frac{\mathbf{r}_1\cap\cdots\cap\mathbf{r}_{n+1}}{(\mathbf{r}_1\dots\mathbf{r}_{n+1})_S} \arrow[d, equal] \\ \pi_n(\Omega S^2) \arrow[r] & H_n(\Omega S^2) \end{tikzcd}$$

The homotopy groups $\pi_n(\Omega S^2) = \pi_{n+1}(S^2)$ are finite for $n \geq 3$, hence the homomorphism $f_{F;R_1,\dots,R_{n+1}}$ is the zero map and therefore,

$$R_1 \cap \cdots \cap R_{n+1} \subseteq D(F; (\mathbf{r}_1 \dots \mathbf{r}_{n+1})_S).$$

The reverse inclusion follows trivially, hence the statement (ii) follows.

Finally, the remark that $R_1 \cap \cdots \cap R_{n+1} \neq [R_1, \dots, R_{n+1}]_S$ for $n \neq 0 \mod 8$ is just a reformulation of the result of Curtis [4] that $\pi_n(S^2) \neq 0,\ n \neq 1 \mod 8$. □

Remark 3.1. For $n = 2$, we have the following situation:

Let $F = F(x_1, x_2),\ R_1 = \langle x_1\rangle^F,\ R_2 = \langle x_2\rangle^F,\ R_3 = \langle x_1x_2\rangle^F$. Then the following diagram consists of isomorphisms

$$\begin{tikzcd} \mathbb{Z} \arrow[d, equal] & \mathbb{Z} \arrow[d, equal] \\ \pi_2(\Omega S^2) \arrow[r, "\simeq"] \arrow[d, equal] & H_2(\Omega S^2) \arrow[d, equal] \\ \frac{R_1\cap R_2\cap R_3}{[R_1, R_2, R_3]_S} \arrow[r, "\simeq"] & \frac{\mathbf{r}_1\cap\mathbf{r}_2\cap\mathbf{r}_3}{(\mathbf{r}_1\mathbf{r}_2\mathbf{r}_3)_S} \end{tikzcd}$$

and therefore $D(F;\ (\mathbf{r}_1\mathbf{r}_2\mathbf{r}_3)_S) = [R_1,\ R_2,\ R_3]_S$.

3.2. **Carlsson's Construction.** For any pointed simplicial set K and any group G, the Carlsson construction [3, 13] is the simplicial group $F^G(K)$ in which $F^G(K)_n$ is the self-free product of the group G indexed by non-identity elements in K_n with the face and degeneracy homomorphisms canonically induced from that of K. There is a homotopy equivalence [3, 13] $|F^G(K)| \simeq \Omega(BG\wedge|K|)$, where BG is the classifying space of the group G. We consider the case where $K = S^1$ is the simplicial circle and G is an arbitrary group. Note that the suspension of any path-connected space is (weak) homotopy equivalent to ΣBG for certain group G by Kan-Thurston's theorem [9]. Thus the construction $F^G(S^1)$ gives the loop space model for the suspensions.

The specific information on the simplicial structure of $F^G(S^1)$ is as follows:

The elements in the simplicial circle S^1 are listed in (3.1). Thus

$$F^G(S^1)_{n+1} = G_{x_0} * G_{x_1} * \cdots * G_{x_n},$$

where G_{x_i} is a copy of G labeled by $x_i = s_n \dots \hat{s}_i \dots s_0\sigma \in S^1_{n+1}$. The face $\partial_j\colon S^1_{n+1} \to S^1_n$ is given by the formula:

$$\partial_j x_i = \begin{cases} x_i & \text{for} \quad i < j \\ x_{i-1} & \text{for} \quad i \geq j, \end{cases}$$

where $x_{-1} = x_n = *$ in S^1_n. Write $g(x_i)$ for the element $g \in G$ located in the copy G_{x_i} of G. The group homomorphism

$$\partial_j\colon F^G(S^1)_{n+1} = G_{x_0} * G_{x_1} * \cdots * G_{x_n} \longrightarrow F^G(S^1)_n = G_{x_0} * G_{x_1} * \cdots * G_{x_{n-1}}$$

is given by the formulae:

$$\begin{aligned} \partial_0(g(x_i)) &= \begin{cases} 1 & \text{for} \quad i = 0 \\ g(x_{i-1}) & \text{for} \quad 0 < i \leq n, \end{cases} \\ \partial_{n+1}(g(x_i)) &= \begin{cases} g(x_i) & \text{for} \quad 0 \leq i \leq n-1 \\ 1 & \text{for} \quad i = n, \end{cases} \end{aligned} \tag{3.2}$$

and for $0 < j < n+1$,

$$\partial_j(g(x_i)) = \begin{cases} g(x_i) & \text{for} \quad i < j \\ g(x_{i-1}) & \text{for} \quad i \geq j. \end{cases} \tag{3.3}$$

In the free product $G^{*n+1} = G_{x_0} * G_{x_1} * \cdots * G_{x_n}$, let $R^G_{n+1,0} = \langle g(x_0) \mid g \in G\rangle^{G^{*n+1}}$, $R_{n+1,n+1} = \langle g(x_n) \mid g \in G\rangle^{G^{*n+1}}$ and $R^G_{n+1,j} = \langle g(x_{j-1})^{-1} g(x_j) \mid g \in G\rangle^{G^{*n+1}}$ for $0 < j < n+1$. Let $\mathbf{r}^G_{n+1,j} = (R^G_{n+1,j} - 1)\mathbb{Z}[G^{*n+1}]$.

Theorem 3.2. *Let G be any group. Then there is an isomorphism of groups*

$$Q(\mathbf{r}^G_{n+1,0}, \mathbf{r}^G_{n+1,1}, \dots, \mathbf{r}^G_{n+1,n+1}) \cong H_{n+1}(\Omega\Sigma BG; \mathbb{Z}) \cong \bigoplus_{k=1}^{\infty} H_{n+1}((BG)^{\wedge k}; \mathbb{Z}),$$

where $X^{\wedge k}$ is the k-fold self smash product of X.

Proof. Let $T = \{t_\alpha \mid \alpha \in J\}$ be a set of generators for G and let F be the free group generated by T. Consider Carlsson's construction $F^F(S^1)$. The group $F^F(S^1)_{n+1}$ is the free group generated by $\{t_\alpha(x_j) \mid \alpha \in J,\ 0 \leq j \leq n\}$. Let $y^{(\alpha)}_j = t_\alpha(x_j)t_\alpha(x_{j+1})^{-1}$ for $-1 \leq j \leq n$ with $t_\alpha(x_{-1}) = t_\alpha(x_{n+1}) = 1$. Then

$$\{y^{(\alpha)}_j \mid \alpha \in J,\ 0 \leq j \leq n\}$$

is also a basis for $F^F(S^1)_{n+1}$. From formulae (3.2) and (3.3),

$$\partial_j(y_k^{(\alpha)}) = \begin{cases} y_{k-1}^{(\alpha)} & \text{for} \quad j \le k \\ 1 & \text{for} \quad j = k+1 \\ y_k^{\alpha} & \text{for} \quad j > k+1. \end{cases}$$

Thus

$$\begin{aligned} \ker(\partial_j \colon F^F(S^1)_{n+1} \to F^F(S^1)_n) &= \langle y_{j-1}^{(\alpha)} \mid \alpha \in J \rangle^{F^F(S^1)_{n+1}} \\ &= \langle t_\alpha(x_{j-1}) t_\alpha(x_j)^{-1} \mid \alpha \in J \rangle^{F^F(S^1)_{n+1}} \\ &= R^F_{n+1,\,j}. \end{aligned}$$

for $0 \le j \le n+1$. The canonical epimorphism $\phi\colon F \to G$ induces a simplicial epimorphism

$$\tilde{\phi}\colon F^F(S^1) \twoheadrightarrow F^G(S^1),$$

which induces the epimorphism

$$\tilde{\phi}|\colon \ker(\partial_j \colon F^F(S^1)_{n+1} \to F^F(S^1)_n) \twoheadrightarrow \ker(\partial_j \colon F^G(S^1)_{n+1} \to F^G(S^1)_n).$$

Thus

$$\begin{aligned} \ker(\partial_j \colon F^G(S^1)_{n+1} \to F^G(S^1)_n) &= \tilde{\phi}(R^F_{n+1,\,j}) \\ &= R^G_{n+1,\,j} \end{aligned}$$

for $0 \le j \le n+1$. Note that the faces

$$\partial_1, \ldots, \partial_n \colon F^F(S^1)_{n+1} \longrightarrow F^F(S^1)_n$$

are natural projections under the basis $\{y_j^{(\alpha)} \mid \alpha \in J,\ 0 \le j \le n\}$. By Lemma 2.2, the Moore chains of the simplicial group $\mathbb{Z}[F^F(S^1)]$

$$N_{n+2}(\mathbb{Z}[F^F(S^1)]) = \mathbf{r}^F_{n+2,1} \cap \mathbf{r}^F_{n+2,\,2} \cap \cdots \cap \mathbf{r}^F_{n+2,\,n+2} = (\mathbf{r}^F_{n+2,\,1}\mathbf{r}^F_{n+2,\,2} \cdots \mathbf{r}^F_{n+2,\,n+2})_S.$$

Thus the Moore boundary

$$\begin{aligned} \mathcal{B}_{n+1}(\mathbb{Z}[F^F(S^1)]) &= \partial_0(N_{n+2}(\mathbb{Z}[F^F(S^1)])) \\ &= \partial_0((\mathbf{r}^F_{n+2,\,1}\mathbf{r}^F_{n+2,\,2} \cdots \mathbf{r}^F_{n+2,\,n+2})_S) \\ &= (\mathbf{r}^F_{n+1,\,0}\mathbf{r}^F_{n+1,\,1} \cdots \mathbf{r}^F_{n+1,\,n+1})_S. \end{aligned}$$

Now the simplicial epimorphism $\tilde{\phi}\colon F^F(S^1) \twoheadrightarrow F^G(S^1)$ extends canonically to a simplicial epimorphism

$$\mathbb{Z}[\tilde{\phi}]\colon \mathbb{Z}[F^F(S^1)] \twoheadrightarrow \mathbb{Z}[F^G(S^1)],$$

which induces an epimorphism on the Moore boundaries

$$\mathbb{Z}(\tilde{\phi})|\colon \mathcal{B}_{n+1}(\mathbb{Z}[F^F(S^1)]) \twoheadrightarrow \mathcal{B}_{n+1}(\mathbb{Z}[F^G(S^1)]).$$

Thus

$$\begin{aligned} \mathcal{B}_{n+1}(\mathbb{Z}(F^G(S^1))) &= \mathbb{Z}[\tilde{\phi}]((\mathbf{r}^F_{n+1,\,0}\mathbf{r}^F_{n+1,\,1} \cdots \mathbf{r}^F_{n+1,\,n+1})_S) \\ &= (\mathbf{r}^G_{n+1,\,0}\mathbf{r}^G_{n+1,\,1} \cdots \mathbf{r}^G_{n+1,\,n+1})_S. \end{aligned}$$

Note that the Moore cycles

$$\begin{aligned} \mathcal{Z}_{n+1}(F^G(S^1)) &= \bigcap_{j=0}^{n} \ker(\partial_j \colon \mathbb{Z}[F^G(S^1)]_{n+1} \to \mathbb{Z}[F^G(S^1)]_n) \\ &= \mathbf{r}^G_{n+1,\,0} \cap \mathbf{r}^G_{n+1,\,1} \cap \cdots \cap \mathbf{r}^G_{n+1,\,n+1}. \end{aligned}$$

It follows that

$$\begin{array}{rcl} Q(\mathbf{r}^G_{n+1,0},\, \mathbf{r}^G_{n+1,1},\, \dots\, ,\, \mathbf{r}^G_{n+1,n+1}) & = & \mathcal{Z}_{n+1}(F^G(S^1))/\mathcal{B}_{n+1}(F^G(S^1)) \\ & = & \pi_{n+1}(\mathbb{Z}(F^G(S^1))) \\ & \cong & H_{n+1}(F^G(S^1);\mathbb{Z}) \\ & \cong & H_{n+1}(\Omega\Sigma BG;\mathbb{Z}) \\ & \cong & \bigoplus_{k=1}^{\infty} H_{n+1}((BG)^{\wedge k};\, \mathbb{Z}), \end{array}$$

where the last isomorphism follows from the classical suspension splitting theorem of loop suspensions [8], hence the assertion. ☐

Corollary 3.1. *Let G be a group. Then*

$$Q(\mathbf{r}^G_{n+1,0},\, \mathbf{r}^G_{n+1,1},\, \dots\, ,\, \mathbf{r}^G_{n+1,n+1}) = 0$$

for all $n \geq 0$ if and only if the reduced homology $\tilde{H}_*(G;\, \mathbb{Z}) = 0$.

Corollary 3.2. *Let G be a group. Then the groups*

$$Q(\mathbf{r}^G_{n+1,0},\, \mathbf{r}^G_{n+1,1},\, \dots\, ,\, \mathbf{r}^G_{n+1,n+1})$$

is torsion-free for all $n \geq 0$ if and only if the integral homology $\tilde{H}_*(G;\, \mathbb{Z})$ is torsion-free.

Example 3.1. The group $F = F^{\mathbb{Z}/2}(S^1)_2 = \mathbb{Z}/2*\mathbb{Z}/2$ is generated by x_0, x_1 with defining relations $x_0^2 = x_1^2 = 1$. In this case,

$$\begin{aligned} \mathbf{r}^{\mathbb{Z}/2}_{2,0} &= (\langle x_0\rangle^F - 1)\mathbb{Z}[\mathbb{Z}/2 * \mathbb{Z}/2], \\ \mathbf{r}^{\mathbb{Z}/2}_{2,1} &= (\langle x_0 x_1\rangle^F - 1)\mathbb{Z}[\mathbb{Z}/2 * \mathbb{Z}/2], \\ \mathbf{r}^{\mathbb{Z}/2}_{2,2} &= (\langle x_1\rangle^F - 1)\mathbb{Z}[\mathbb{Z}/2 * \mathbb{Z}/2] \end{aligned}$$

with

$$\begin{array}{rcl} Q(\mathbf{r}^{\mathbb{Z}/2}_{2,0}, \mathbf{r}^{\mathbb{Z}/2}_{2,1}, \mathbf{r}^{\mathbb{Z}/2}_{2,2}) & \cong & H_2(\mathbb{R}\mathrm{P}^\infty;\, \mathbb{Z}) \oplus H_2(\mathbb{R}\mathrm{P}^\infty \wedge \mathbb{R}\mathrm{P}^\infty;\, \mathbb{Z}) \\ & = & \mathbb{Z}/2. \quad \square \end{array}$$

Remark 3.2. Observe that, for every group G, there is the following natural diagram (see [2]):

$$\begin{tikzcd} H_3(G) \arrow[r] \arrow[d] & \Gamma_2(G_{ab}) \arrow[r] \arrow[d, equal] & \pi_2(\Omega\Sigma BG) \arrow[r, two heads] \arrow[d] & H_2(G) \arrow[d] \\ H_3(G_{ab}) \arrow[r] & \Gamma_2(G_{ab}) \arrow[r] & G_{ab} \otimes G_{ab} \arrow[r, two heads] & H_2(G_{ab}), \end{tikzcd}$$

where Γ_2 is the universal Whitehead quadratic functor. For a group G with $\ker\{H_2(G) \to H_2(G_{ab})\} = 0$ and torsion-free G_{ab}, the natural map $\pi_2(\Omega\Sigma BG) \to G_{ab}\otimes G_{ab} (\subseteq H_2(\Omega\Sigma BG))$ is a monomorphism. In particular, this covers the case mentioned in Remark 3.1.

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