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HYPERBOLIC UNIT GROUPS

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ABSTRACT. In this paper we study the groups $\mathcal{G}$ whose integral group rings have hyperbolic unit groups $\mathcal{U}(\mathbb{Z}\mathcal{G})$. We classify completely the torsion subgroups of $\mathcal{U}(\mathbb{Z}\mathcal{G})$ and the polycyclic-by-finite subgroups of the group $\mathcal{G}$. Finally, we classify the groups for which the boundary of $\mathcal{U}(\mathbb{Z}\mathcal{G})$ has dimension zero.

1. Introduction

The study of hyperbolic groups has been an active topic of research in recent years. It started with the work of M. Gromov [4] and has developed very rapidly since then.

Let $\mathcal{G}$ be a group and $\Gamma := \mathcal{U}_1(\mathbb{Z}\mathcal{G})$ the group of normalized units of the integral group ring $\mathbb{Z}\mathcal{G}$. It is well known that if $\mathcal{G}$ is finite, then Γ is finitely presented (see, for instance, [8]). Since almost every finitely presented group is hyperbolic [10], it is natural to investigate when Γ is hyperbolic. Motivated by these considerations, we are led to pose the following:

Problem 1. Classify the groups $\mathcal{G}$ for which Γ is hyperbolic.

This paper is a contribution to the above problem. After giving in section 2 the basic facts about hyperbolic groups needed in this work, we characterize, in section 3, the torsion subgroups of Γ and the polycyclic-by-finite subgroups of $\mathcal{G}$, thus answering the problem, in particular, for torsion groups. Contrary to the theorem that almost all finitely presented groups are hyperbolic, we find that the unit group of a finite group is hyperbolic only in a very small number of cases, which we enumerate explicitly. It turns out that if $\mathcal{G}$ is finite and Γ is hyperbolic, then $\mathcal{G}$ has a normal free complement (i.e., there exists a normal free subgroup F in Γ such that $\Gamma = F\mathcal{G},\ F \cap \mathcal{G} = 1$), and furthermore every torsion-free complement of $\mathcal{G}$ is free.

Finally, we completely characterize the groups $\mathcal{G}$ such that Γ is hyperbolic with its hyperbolic boundary having dimension zero, or equivalently that Γ is virtually free.

Jespers ([6], [7]) has classified those finite groups $\mathcal{G}$ that have a normal free complement in Γ. This property implies that Γ is quasi-isometric to a free group of finite rank and hence is hyperbolic. In view of this work, our Theorems 2 and

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3 can be considered to be basically due to Jespers; however, our proofs are quite different.

2. Preliminaries

Let (X, d) be a metric space. For $x,\ y,\ z \in X$, the Gromov product of $y,\ z$ with respect to x is defined to be

$$(y.z)_x = \frac{1}{2}\{d(y, x) + d(z, x) - d(y, z)\}.$$

The metric space is said to be δ-hyperbolic ($\delta \geq 0$) if

$$(x.y)_w \geq \min\{(x.z)_w,\ (y.z)_w\} - \delta$$

for all $w,\ x,\ y,\ z \in X$. Let G be a finitely generated group and S a finite set of generators for G. The Cayley graph $\mathcal{G}(G, S)$ of G with respect to S is the metric graph whose vertices are in one-to-one correspondence with the elements of G and which has an edge (labeled s) of length 1 joining g to gs for each $g \in G$ and $s \in S$. The group G is said to be hyperbolic (in the sense of Gromov) if its Cayley graph $\mathcal{G}(G, S)$ is a δ-hyperbolic metric space for some $\delta \geq 0$. This definition does not depend on the choice of the generating set S.

For the reader's convenience, we collect some of the facts about hyperbolic groups that we need. Let $\mathbb{Z}^2$ denote the free Abelian group of rank two.

Theorem 1. *Let Γ be a hyperbolic group. Then*

(a) *$\mathbb{Z}^2$ does not embed as a subgroup of Γ.*
(b) *If $g \in \Gamma$ has infinite order, then $[\,C_\Gamma(g) : \langle g\rangle\,]$ is finite, where $C_\Gamma(g)$ is the centralizer of g in Γ.*
(c) *Torsion subgroups of Γ are finite of bounded order.*
(d) *Γ is virtually free if and only if its boundary has dimension zero.*
(e) *If Γ is quasi-isometric to a free group, then Γ is virtually free. If, moreover, Γ is torsion-free, then it is free.*

For the theory of hyperbolic groups, the reader may refer to [1], [3] or [4] and, for standard results and notation in group rings, to [11, 12, 13, 14].

3. Groups with $\mathcal{U}_1(\mathbb{Z}\mathcal{G})$ hyperbolic

Let $\mathcal{G}$ be an arbitrary group, and let $G \subseteq \Gamma := \mathcal{U}_1(\mathbb{Z}\mathcal{G})$ be a subgroup of normalized units. In this section, unless otherwise stated, we shall always assume that Γ is a hyperbolic group.

We first make some easy observations:

1^0. In case G is finite, then it constitutes a $\mathbb{Z}$-linearly independent subset of $\mathbb{Z}\mathcal{G}$ and consequently $\mathbb{Z}[G]$, the subring of $\mathbb{Z}\mathcal{G}$ generated by G, is isomorphic to the group ring $\mathbb{Z}G$. This fact shall be used freely. It is a simple consequence of a well-known theorem of Berman: If u is an element of finite order in the unit group $\mathcal{U}_1(\mathbb{Z}\mathcal{G})$, then the coefficient in u of the identity element of $\mathcal{G}$ is zero unless u equals 1.

2^0. Let $g \in \Gamma$ be a torsion element and $n = o(g)$ its order. Since the torsion-free rank of $\mathcal{U}_1(\mathbb{Z}\langle g\rangle)$ is $\sum_{d|n,d>2}(\frac{\phi(d)}{2} - 1)$ (see [13], Theorem 3.1, p. 54) where ϕ is the Euler phi-function, it follows, from Theorem 1(a), that n divides 5, 8, or 12.

3^0. Let G be a finite subgroup of $\varGamma$. Since $\mathbb{Z}^2$ does not embed in $\varGamma$, $\mathbb{Q}G$ has at most one Wedderburn component that is not a division ring, and that component must be $M_2(\mathbb{Q})$. (This follows by looking at the upper triangular matrices in a Wedderburn component $M_n(D)$, which gives rise to units of the form $GL_n(\mathcal{O})$ for an order $\mathcal{O}$ in D.) Hence if G is non-Abelian and none of its non-Abelian quotient embeds into a division ring, then $\Delta(G, G')$, the kernel of the natural projection $\mathbb{Q}G \longrightarrow \mathbb{Q}(G/G')$, where G' is the derived group of G, is isomorphic to $M_2(\mathbb{Q})$, and so its dimension over $\mathbb{Q}$ is 4.

4^0. In view of $\mathbf{2}^0$ and $\mathbf{3}^0$, it is easy to see that the groups $C_5 \times C_3$, $C_5 \times C_4$, $C_5 \times C_5$, D_5, $C_5 \rtimes C_4$, $C_5 \rtimes C_8$, $C_8 \times C_2$, $K_8 \times C_3$, $(C_2 \times C_2) \rtimes C_4$ and Q_{16} cannot appear as subgroups of $\varGamma$, where Q_{16} is the generalized quaternion group of order 16, K_8 denotes the quaternion group of order 8, D_n the dihedral group of order $2n$ and C_n the cyclic group of order n.

5^0. We will have occasion to use the general result that if $\varGamma := \mathcal{U}_1(\mathbb{Z}\mathcal{G})$ is finitely generated, so is $\mathcal{G}$; in particular if $\varGamma$ is hyperbolic, $\mathcal{G}$ is finitely generated. We supply a proof of this assertion. Suppose that $\varGamma$ is generated by a finite set, say $u_1, \cdots, u_n$. Let G_0 be the subgroup of $\mathcal{G}$ generated by the $\operatorname{supp}(u_i)$, $\operatorname{supp}(u_i^{-1}), i = 1, \cdots, n$. This is a finitely generated group. All the u_i's belong to the unit group of $\mathbb{Z}G_0$, and hence $\varGamma$ is contained in the unit group of $\mathbb{Z}G_0$. But $\mathcal{G}$ is a subset of $\varGamma$, and so $\mathcal{G}$ is contained in $\mathbb{Z}G_0$. This implies that $\mathcal{G} = G_0$, and hence $\mathcal{G}$ is finitely generated, as G_0 is.

Recall that a non-Abelian group G is called Hamiltonian if all its subgroups are normal. We will abuse terminology to denote a non-Abelian, non-Hamiltonian group simply as a non-Hamiltonian group; thus in our usage, a non-Hamiltonian group is always non-Abelian. If G is a finite Hamiltonian group, then $G \simeq K_8 \times A \times E$, where K_8 is the quaternion group of order 8, A an Abelian group of odd order and E an elementary Abelian 2-group (see [5], Theorem 7.12, p. 308). As in [6], we begin with the following result.

Lemma 1. *Let $H \subseteq G \subseteq \varGamma$ be groups with G finite and non-Abelian. Then one of the following holds:*

(a) *H is Abelian.*
(b) *H is a (non-Abelian) Hamiltonian 2-group.*
(c) *$\mathbb{Q}H$ contains a unique matrix Wedderburn component that is isomorphic to $M_2(\mathbb{Q})$ and $H = G$.*

Proof. Note first that if $M_n(D)$ is a Wedderburn component of $\mathbb{Q}H$, then we must have $n \leq 2$.

Suppose H is non-Abelian. In case $\mathbb{Q}H$ is a direct sum of division rings, then $H \simeq K_8 \times E \times A$, with E an elementary Abelian 2-group and A an Abelian group of odd order ([13], Theorem 1.17, p. 172). In case $A \neq 1$, then, because of the restriction on the orders of the elements of finite order, A is an elementary Abelian 3-group. Hence $H_0 \simeq K_8 \times C_3$ is a subgroup of G, which is not possible by $\mathbf{4}^0$, and so H is a Hamiltonian 2-group.

Next suppose that $\mathbb{Q}H$ is not a direct sum of division rings. Then, by $\mathbf{3}^0$, $\mathbb{Q}H$ has a unique matrix Wedderburn component $\mathcal{A}$ and $\mathcal{A} \simeq M_2(\mathbb{Q})$. Let $e \in \mathbb{Q}H$ be the primitive central idempotent such that $\mathbb{Q}He = \mathcal{A}$. Let $\{f_i \,|\, 1 \leq 1 \leq n\}$ be the set of primitive central idempotents of $\mathbb{Q}G$, and so $e = \sum_{ef_i \neq 0} ef_i$. Since $\mathcal{A}$ is simple, $ef_i \neq 0$ implies that $\mathbb{Q}Gf_i$ contains a copy of $\mathcal{A}$ and hence, since the only matrix Wedderburn component in $\mathbb{Q}G$ is $M_2(\mathbb{Q})$, $\mathbb{Q}Gf_i \simeq M_2(\mathbb{Q})$. Also, there is exactly one index i, such that ef_i is non-zero; for, otherwise, there will be more than one Wedderburn component in $\mathbb{Q}G$ which is $M_2(\mathbb{Q})$, which is not allowed by $\mathbf{3}^0$. So $e = ef_i$ and $\mathbb{Q}He = \mathbb{Q}Ge$ for a central primitive idempotent e in H. We conclude from this that $H = G$. For this, let $g \in G$; thus $ge \in \mathbb{Q}He$. Since the trace of a primitive central idempotent in a group ring, i.e., the coefficient of 1, is non-zero, g belongs to the support of ge, which is contained in H. Thus G is contained in H, and hence $G = H$. □

Corollary 1. *Let G be a finite non-Hamiltonian subgroup of Γ. Then every Wedderburn component of $\mathbb{Q}G$ is either $\mathbb{Q}$, or an imaginary quadratic extension of $\mathbb{Q}$, or $M_2(\mathbb{Q})$ or a totally definite quaternion algebra over $\mathbb{Q}$. Furthermore, G has a subgroup of index 2.*

Proof. By the previous lemma there exists a unique Wedderburn component of the form $M_2(\mathbb{Q})$. Let $\mathcal{A}$ be any other Wedderburn component. Then $\mathcal{A}$ is a division ring. Let K be a maximal subfield of $\mathcal{A}$. Then the unit group of the ring of integers of K must be finite, and hence K is at most a quadratic extension of $\mathbb{Q}$. So if $\mathcal{A}$ is non-commutative, then its centre must equal $\mathbb{Q}$, and hence $dim_{\mathbb{Q}}(\mathcal{A}) = 4$, and thus $\mathcal{A}$ is a totally definite quaternion algebra over $\mathbb{Q}$.

Considering the complex group algebra $\mathbb{C}G$ and using what we proved above about the Wedderburn components of $\mathbb{Q}G$, we see that $\mathbb{C}G$ is the direct sums of copies of $\mathbb{C}$ and two-by-two matrices over $\mathbb{C}$. Hence, by Corollary 12.9 of [2], G contains a subgroup of index 2. □

We note that if G is a finite non-Hamiltonian subgroup of Γ (and therefore has a Wedderburn component that is not a division algebra), then $\mathbb{Z}G$ does not contain a central unit of infinite order; for, if α is such an element, then for a θ in $\mathbb{Q}G$, a non-zero nilpotent element with $\theta^2 = 0$, $\langle \alpha, 1+\theta \rangle \cong \mathbb{Z}^2$, a contradiction to Theorem 1(a). Hence central units of $\mathbb{Z}G$ are trivial. The following result extends this observation to arbitrary subgroups of Γ.

Lemma 2. *Let G be any subgroup of Γ such that $\mathcal{U}_1(\mathbb{Z}[G])$ contains a central unit of infinite order. If G is finite, then it is Abelian; if G is infinite and contained in $\mathcal{G}$, then $G = \mathcal{U}_1(\mathbb{Z}G)$ and G is centre-by-finite.*

Proof. If G is finite, then by Lemma 1 and the above observation G must be Abelian.

Suppose that G is an infinite subgroup of $\mathcal{G}$ and α is a central unit in $\mathbb{Z}G$ of infinite order. Then, since Γ is hyperbolic, by Theorem 1(b) we have that $[\mathcal{U}_1(\mathbb{Z}G) : \langle \alpha \rangle] < \infty$. It follows that G is an elementary hyperbolic group with a central element g_o, say, of infinite order. If $0 \neq \theta$ is a nilpotent element in $\mathbb{Z}G$, then $\langle 1+\theta, g_0 \rangle \simeq \mathbb{Z}^2$, since the Kaplansky trace of a nilpotent element is zero. Hence $\mathbb{Z}G$ has no non-zero nilpotent elements. For an element x of G of order d, and an arbitrary element g of G, the element $(1-x)g(1+x+\cdots+x^{d-1})$ is nilpotent, hence is zero. From this we see that the element g normalizes the cyclic subgroup generated by x for every torsion element x in G. Thus the torsion elements $T(G)$ of G form a subgroup of G, which is finite by Theorem 1(c). Furthermore, since all elements of $\mathcal{U}_1(\mathbb{Z}T(G))$ will have to be of finite order, $\mathcal{U}_1(\mathbb{Z}T(G)) = T(G)$. Since $G/T(G)$ is ordered it follows from ([14], Proposition 45.5, p. 277) that $\mathcal{U}_1(\mathbb{Z}G) = (\mathcal{U}_1(\mathbb{Z}T(G)))G = G$. □

Lemma 3. *Let G be any group, and let $x, y \in G$ be such that $\langle x\rangle \cap \langle y\rangle = 1$, $o(x) < \infty$, $o(y) \geq 5$ and $x^y \notin \langle x\rangle$. Then $\mathbb{Z}^2$ embeds into $\mathcal{U}_1(\mathbb{Z}G)$.*

Proof. Define $\hat{x} := 1 + x + \cdots + x^{n-1}$, $n = o(x)$, and $\theta_k = (1-x)y^k\hat{x}, 1 \leq k \leq o(y)$. Now observe that, under the given hypothesis, it is possible to choose k such that $\langle 1+\theta_1,\, 1+\theta_k\rangle$ is a subgroup of $\mathcal{U}_1(\mathbb{Z}G)$ and isomorphic to $\mathbb{Z}^2$. □

Lemma 4. *If Γ has an element of order 5, then $\mathcal{G} \simeq C_5$.*

Proof. By general properties of elements of prime order in a group ring (see [14], Theorem 45.11, p. 278), if Γ has an element of order 5, so does $\mathcal{G}$. Let $x \in \mathcal{G}$ be such an element.

Suppose first that $\mathcal{G}$ is non-Abelian. If x were central, then Γ would have a central element α of infinite order, with support in $\langle x\rangle$, and hence, by Lemma 2, $\mathcal{G}$ would have an element g_0 of infinite order. Since $\langle \alpha,\ g_o\rangle \simeq \mathbb{Z}^2$, we have a contradiction. On the other hand, if $\langle x\rangle$ were normal but not central, then there would exist an element $g_0 \in \mathcal{G}$ with $o(g_0) \in \{2, 4, 8, \infty\}$ such that $\langle x\rangle \rtimes \langle g_o\rangle$ is a subgroup of G. In view of $\mathbf{4}^0$, $o(g_0) = \infty$ and with α as above, we have $\langle \alpha,\ g_0^4\rangle \simeq \mathbb{Z}^2$, again a contradiction. Hence there must exist another element $y \in \mathcal{G}$ of order 5 that does not commute with x; but then x and y satisfy the conditions of Lemma 3, which is a contradiction.

Next suppose that $\mathcal{G}$ is Abelian. Then, as seen above, $\mathcal{G}$ must be torsion and hence finite. From rank considerations of Γ, it is clear that $\mathcal{G} \simeq C_5$. □

Lemma 5. *If $\mathcal{G}$ is a non-torsion group, then $T(\mathcal{G})$ is a finite Hamiltonian group and $\mathcal{U}_1(\mathbb{Z}T(\mathcal{G})) = T(\mathcal{G})$. Moreover, the primitive central idempotents of $\mathbb{Q}T(\mathcal{G})$ are central in $\mathbb{Q}\mathcal{G}$.*

Proof. Let $x, y \in \mathcal{G}$, with $o(x) < \infty = o(y)$. Then, by Lemma 3, we must have that $x^y \in \langle x\rangle$. Since the orders of the torsion elements of $\mathcal{G}$ divide 8 or 12, it follows that y^4 must centralize $T(\mathcal{G})$, the set of torsion elements of $\mathcal{G}$. Let $y_0 = y^4$, and let $z \in \mathcal{G}$ be any other torsion element. If z does not normalize $\langle x\rangle$, then $(1-x)z\hat{x}$ and $y_0(1-x)z\hat{x}$ are $\mathbb{Q}$-linearly independent commuting nilpotent elements, and so $\mathbb{Z}^2$ embeds into Γ, a contradiction. It follows that $T(\mathcal{G})$ is a subgroup, is locally finite, and hence, since Γ is hyperbolic, it is finite. If $\mathcal{U}_1(\mathbb{Z}T(\mathcal{G}))$ is not trivial, then, since $y_0 \in \mathcal{G}$ has infinite order, we can embed $\mathbb{Z}^2$ into Γ. Finally, let $e \in \mathbb{Q}T(\mathcal{G})$ be a central primitive idempotent that is not fixed by $g \in \mathcal{G}$. Then $o(g) = \infty$ and so $\langle 1 + eg,\, 1 + y_0eg\rangle \simeq \mathbb{Z}^2$, a contradiction. □

Lemma 6. *Let G be a finite non-Abelian subgroup of Γ. Then $exp(G)$ divides 12 and $G = \langle H, x\rangle$, where H is a subgroup of index 2 and x is a 2-element. Furthermore, (i) if 3 divides $|G|$, then G is isomorphic either to S_3 or to Q_{12}; (ii) if G is a 2-group having a non-central element of order 2, then $G \simeq D_4$.*

Proof. In view of Corollary 1, observe that G has a subgroup H of index 2 that is either Abelian or a Hamiltonian 2-group and its order is not divisible by 5. So we may choose a 2-element $x \in G$ such that $x^2 \in H$ and $G = \langle H, x\rangle$.

Suppose that 3 divides $|G|$; then H must be Abelian. If $Syl_3(G)$, the Sylow 3-subgroup of G, were central, then, by Lemma 1, $G = Syl_3(G) \times Syl_2(G)$ with $Syl_2(G)$ a Hamiltonian 2-group, and so $G \simeq C_3 \times K_8$, a contradiction to $\mathbf{4}^0$. Hence there exists $a \in H$ of order 3 such that $a^x \neq a$. Then, since clearly $(a^{-1}a^x)^x = (a^{-1}a^x)^{-1}$, we have that $G \simeq C_3 \rtimes \langle x\rangle$. We only need to rule out the case when

$o(x) = 8$. In this case $\mathbb{Q}(G/G') \simeq \mathbb{Q}C_8$, and so $\mathcal{U}_1(\mathbb{Z}G)$ has a central element of infinite order. Therefore, by Lemma 2, G cannot be a subgroup of Γ.

Suppose next that G is a 2-group and $y \in G$ is a non-central element of order 2. Suppose first that $y \in H$. Then, by Lemma 1, $G = \langle x, y\rangle$. Since $x^2 \in H$, it follows that $[y, x^2] = 1$ and so $G' = \langle yy^x\rangle$. Since such a group does not possess a non-Abelian quotient that embeds in a non-commutative division ring, it follows by $\mathbf{3}^0$ that $dim_{\mathbb{Q}}(\Delta(G, G')) = 4$ and so $|G| = 8$; it thus follows that $G \simeq D_4$. Suppose that $y \notin H$. Then $G = \langle H, y\rangle$. Choose $a \in H$ such that $[a, y] \neq 1$. Then $G = \langle a, y\rangle$. Note that $z := a^{-1}a^y$ is inverted by y. If it is not fixed, then $G = \langle z, y\rangle = \langle z\rangle \rtimes \langle y\rangle$. On the other hand, if z is fixed, then $G' = \langle z\rangle$ and the same argument as given above shows that $|G| = 8$. Thus in any case it follows, by $\mathbf{3}^0$, that $G \simeq D_4$.

It remains to show that G has no element of order 8. Suppose $y \in G$ is an element of order 8. Then, in view of the previous paragraph, elements of order two are central. Let z be such an element. If $z \notin \langle y\rangle$, then $C_8 \times C_2$ would embed in G, which is not the case. Hence $z \in \langle y\rangle$, and thus $z = y^4$. Hence G has a unique element of order 2. Since G is not Abelian, it follows that G is isomorphic to Q_{16}, which is ruled out by $\mathbf{4}^0$. □

We are now ready to present our main results.

Theorem 2. *Let G be a finite non-Hamiltonian group. Then the following are equivalent:*

(1) *Exactly one Wedderburn component of $\mathbb{Q}G$ is $M_2(\mathbb{Q})$, and any other component is either $\mathbb{Q}$, or an imaginary quadratic extension of $\mathbb{Q}$ or a totally definite quaternion algebra over $\mathbb{Q}$.*
(2) *G has a normal free complement in $\mathcal{U}_1(\mathbb{Z}G)$.*
(3) *$\mathcal{U}_1(\mathbb{Z}G)$ is virtually free.*
(4) *$\mathcal{U}_1(\mathbb{Z}G)$ is hyperbolic.*

Moreover, if one of the above conditions holds, then every finitely generated torsion-free subgroup of $\mathcal{U}_1(\mathbb{Z}G)$ is free. In particular, any normal torsion-free complement of G in $\mathcal{U}_1(\mathbb{Z}G)$ is free.

Proof. $(1) \Rightarrow (2)$: From (1) it easily follows that $c.d.(G)$, the set of complex character degrees of G, is $\{1, 2\}$; hence, by Corollary 12.9 of [2], G is metabelian. Furthermore, (1) also implies that $\mathbb{Q}(G/G')$ is a direct sum of copies of $\mathbb{Q}$ and imaginary quadratic fields. Hence the exponent of G/G' divides 4 or 6, and so $\mathcal{U}_1(\mathbb{Z}(G/G'))$ is trivial. Therefore $F := \mathcal{U}_1(\mathbb{Z}G) \cap (1 + \Delta(G)\Delta(G'))$, which is known to be torsion-free (see [14]; for a more general result see [9]), is a complement of G in $\mathcal{U}_1(\mathbb{Z}G)$. Since $SL(2, \mathbb{Z})$ contains a free group of rank 2 as a subgroup of finite index, it easily follows that F is quasi-isometric to a free group and so, by Theorem 1(e), F is a free group.

$(2) \Rightarrow (3)$: This implication is trivial, since G is finite.

$(3) \Rightarrow (4)$: This is a consequence of the fact that a free group is hyperbolic and hyperbolicity is stable under quasi-isometry.

$(4) \Rightarrow (1)$: This follows from Corollary 1.

Finally, if H is a finitely generated torsion-free subgroup of $\mathcal{U}_1(\mathbb{Z}G)$ and F is a free subgroup of finite index in $\mathcal{U}_1(\mathbb{Z}G)$, then $H \cap F$ is a finitely generated free subgroup of finite index in H. Hence, once again by Theorem 1(e), we have that H is free. □

The following result characterizes the torsion groups that can occur as subgroups of hyperbolic unit groups.

Theorem 3. *If a torsion group G embeds into a hyperbolic unit group, then G must be finite and isomorphic to one of the following groups:*
(1) *C_5, C_8, C_{12}, an Abelian group of exponent dividing 4 or 6;*
(2) *a Hamiltonian 2-group;*
(3) *$S_3, D_4, Q_{12}, C_4 \rtimes C_4$.*

Conversely, all of the groups listed above have hyperbolic unit groups.

Proof. Since a torsion subgroup of a hyperbolic group is finite, G must be so.

Suppose first that G is Abelian. Write

$$\mathbb{Q}G = \bigoplus a_d \mathbb{Q}(\xi_d),$$

where $a_d = \frac{n_d}{\phi(d)}$, $n_d =$ the number of elements of order d, and ϕ is the Euler phi-function. Then $\mathcal{U}_1(\mathbb{Z}G)$ is hyperbolic if and only if its torsion-free rank is at most one, i.e.,

$$\sum_d a_d(\frac{\phi(d)}{2} - 1) \leq 1.$$

Hence either $\mathcal{U}_1(\mathbb{Z}G)$ is finite, and so G has exponent dividing 4 or 6, or there exists a unique integer d such that $\phi(d) = 4$ and $a_d = 1$. It then follows that $G \in \{C_5, C_8, C_{12}\}$.

Clearly, if G is one of these groups, then the unit group of $\mathcal{U}_1(\mathbb{Z}G)$ is either trivial or has torsion-free rank equal to one and so is hyperbolic.

Suppose G is non-Hamiltonian. In view of Lemma 6, we can suppose that G is a 2-group of order at least 16 in which all elements of order 2 are central.

Since G is non-Hamiltonian, we may choose $a, x \in G$ such that $a^x \notin \langle a \rangle$ and so, by Lemma 1, $G = \langle a, x \rangle$ and $[a, x] \neq a^2$. If $x^a \in \langle x \rangle$, then $G \simeq C_4 \rtimes C_4$. So we also suppose that $x^a \notin \langle x \rangle$. Let $\overline{G} = G/\langle a^2 \rangle$; then the Wedderburn components of $\mathbb{Q}\overline{G}$ are among those of $\mathbb{Q}G$. Hence, by Theorem 2, $\overline{G}$ embeds into a hyperbolic unit group. Since the image of a in $\overline{G}$ is not central, it follows, by Lemma 6, that $\overline{G} \simeq D_4$. So G has order 16 and we may choose a non-central element $y \in G$ whose image has order 4 in $\overline{G}$. We still have that $G = \langle y, x \rangle$ and either $[y, x] = y^2$ or $[y, x] = y^2x^2$. In particular, $[y, x] = [x, y]$. If $[y, x] = y^2$, then $G \simeq C_4 \rtimes C_4$. On the other hand, if $[y, x] = y^2x^2$, then xy would be a non-central element of order 2 and so, by Lemma 6, G would have order 8, which is not the case.

If G is Hamiltonian, then, by Lemma 6, G is a 2-group.

For the converse, since the unit group of a Hamiltonian 2-group is trivial, it only remains to show that the groups in (3) have hyperbolic unit groups. To do so, we prove that for all these groups Theorem 2 (1) is satisfied. Indeed, for all of them we have that G/G' has exponent dividing 4, and so $\mathbb{Q}(G/G')$ is a direct sum of copies of $\mathbb{Q}$ and $\mathbb{Q}(\sqrt{-1})$. Also, for all of them, $dim_{\mathbb{Q}}(\Delta(G, G') \leq 8$. Each of the groups Q_{12} and $C_4 \rtimes C_4$ embeds into a division ring (see [15], Theorem 2.1.5, p. 47). Because of the limitation on the dimension, these division rings are four dimensional over $\mathbb{Q}$ and hence are totally definite quaternion algebras, and the proof is complete by Theorem 2. □

We next characterize the infinite polycyclic-by-finite groups G that embed in a group $\mathcal{G}$ whose unit group is hyperbolic.

Theorem 4. *An infinite polycyclic-by-finite group G embeds into a group $\mathcal{G}$ whose unit group $\mathcal{U}_1(\mathbb{Z}\mathcal{G})$ is hyperbolic if and only if*
(1) *$T(G)$, the set of elements of finite order in G, is a subgroup of G;*
(2) *$G \simeq T(G) \rtimes \mathbb{Z}$;*
(3) *$\mathcal{U}_1(\mathbb{Z}T(G)) = T(G)$.*

Proof. Let G be infinite and polycyclic-by-finite; then G has an element of infinite order and so, by Lemma 5, $T(G)$ is a finite subgroup and $\mathcal{U}_1(\mathbb{Z}(T(G))) = T(G)$. Since G is polycyclic-by-finite, it contains a normal free Abelian subgroup A, say. Since $\mathcal{U}_1(\mathbb{Z}\mathcal{G})$ is hyperbolic, it follows that $A = \langle x \rangle$ is cyclic. Applying Theorem 1 (b), it is easy to see that $[\, G : \langle x \rangle \,] < \infty$ and so G has Hirsch length one and thus, $G \simeq T(G) \rtimes \mathbb{Z}$. Since the hypothesis of ([14], Proposition 45.5, p. 277) is satisfied, we conclude that $\mathcal{U}_1(\mathbb{Z}G) = (\mathcal{U}_1(\mathbb{Z}(T(G)))G = (T(G))G = G$. The converse being trivial, the proof is complete. □

Finally, we give a complete characterization of the groups $\mathcal{G}$ whose unit group Γ is finitely generated virtually free, or equivalently the boundary $\partial(\Gamma)$ has dimension zero [4]. For convenience we adopt the following:

Definition 1. A group $\mathcal{G}$ is called a $*$-group if one of the following conditions holds.

(1) $\mathcal{G}$ is a finite Abelian group of exponent dividing 4 or 6.
(2) $\mathcal{G}$ is a finite Hamiltonian 2-group.
(3) $\mathcal{G} \in \{C_5, C_8, C_{12}, S_3, D_4, Q_{12}, C_4 \rtimes C_4\}$.
(4) $\mathcal{G} = H \rtimes F$, where H is of type (1) or (2) above and F is a finitely generated free group.

Theorem 5. *The unit group $\Gamma = \mathcal{U}_1(\mathbb{Z}G)$ of a group G is finitely generated virtually free if and only if G is a $*$-group. Furthermore, in case G is infinite, $\Gamma = G$.*

Proof. Suppose that Γ is finitely generated virtually free, i.e., there exist a finitely generated free group F contained in Γ of finite index. Then F is finitely generated and Γ is hyperbolic. Hence if G is a finite group, then by Theorem 3, G is of type (1), (2) or (3). So suppose that G is infinite. By $\mathbf{5^0}$, it follows that $G \cap F$ is a finitely generated free subgroup of G of finite index in G, and thus G is virtually free and so is hyperbolic. Since G must necessarily be non-torsion, therefore, by Lemma 5, we have that $T(G)$ is finite and $\mathcal{U}_1(\mathbb{Z}T(G)) = T(G)$. The quotient group $H := G/(T(G))$ is quasi-isometric to the free group $G \cap F$, both being quasi-isometric to G; therefore, by Theorem 1 (e), H is virtually free. Since H is torsion-free, it follows that H is free. Hence $G \simeq T(G) \rtimes H$. Since H is ordered and $\mathcal{U}_1(\mathbb{Z}T(G)) = T(G)$, it follows from Proposition 45.5 of [14] that $\Gamma = G$. So in any case G is a $*$-group.

The converse follows by the previous results and ([14], Prop. 45.5). □

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