Pramāṇa – J. Phys., Vol. 32, No. 6, June 1989, pp. 731–739. © Printed in India.

Radiating Kerr particle in Einstein universe

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MS received 6 October 1988; revised 8 March 1989

Abstract. A generalized Kerr-NUT type metric is considered in connection with Einstein field equations corresponding to perfect fluid plus a pure radiation field. A general scheme for obtaining the exact solutions of these field equations is developed. Two physically meaningful particular cases are investigated in detail. One gives the field of a radiating Kerr particle embedded in the Einstein universe. The other solution may probably represent a deSitter-like universe pervaded by a pure radiation field.

Keywords. Radiating Kerr metric; Einstein universe; de-Sitter universe.

PACS No. 04·20

1. Introduction

It is well-known that the exterior gravitational field of a rotating black hole is described by the Kerr metric (Kerr 1963). The Kerr solution is most interesting and intriguing. The cause of its complexity as well as beauty is the inherent rotation. There is considerable interest in generalizing Kerr metric to describe the non-static field of a radiating rotating star. Vaidya (1974) has given a very general form of Kerr-Schild metric satisfying the Einstein's field equations $R_{ik} = \sigma\xi_i\xi_k$, $\xi^i\xi_i = 0$. The radiating Kerr metric, discussed by Vaidya and Patel (1973), is a particular case of Vaidya's general solution. Vaidya *et al* (1976) have also discussed radiating Kerr and radiating NUT (Newman *et al* 1963) solutions.

In the present paper we shall present an exact solution of Einstein's equations which describes the field of a radiating Kerr particle embedded in Einstein static universe. We shall also discuss another solution which describes pure radiation field in a deSitter-like universe.

2. The geometry and the field equations

The radiating Kerr metric discussed by Vaidya and Patel (1973) can be expressed in the form (Vaidya 1974)

$$\begin{aligned} ds^2 = &-(N^2+q^2)(d\alpha^2+\sin^2\alpha\, d\beta^2) + 2(\xi_i dx^i)\, dt \\ &- 2(\xi_i dx^i)[NW_u\, d\alpha - qW_u \sin\alpha\, d\beta] \\ &-(\xi_i dx^i)^2[1 - W_u^2 - 2Mq|(N^2+q^2)], \end{aligned} \tag{1}$$

where

$$MW^3 \operatorname{cosec}^3 \alpha = \text{constant},\ N = VW_u + W_\alpha,$$

$$q = V_\alpha + u - t, \quad V = (a/2)\sin\alpha\cos\alpha,$$

$$W^2 = \left[k^2 + a\left(u - \frac{a}{4}\cos^2\alpha\right)\right]\sin^2\alpha, \tag{2}$$

$$\xi_i\,\mathrm{d}x^i = \mathrm{d}u - V\mathrm{d}\alpha + W\sin\alpha\mathrm{d}\beta.$$

Here and in what follows a suffix denotes partial derivatives e.g. $V_\alpha = \partial V/\partial\alpha$, $W_u = \partial W/\partial u$ etc., a and k are arbitrary constants. When $a = 0$, the radiation disappears and the metric (1) becomes the Kerr metric.

Applying the transformations

$$\bar{u} = u - (a/4)\cos^2\alpha, \quad T = t + (a/4)\cos^2\alpha, \tag{3}$$

the metric (1) reduces to

$$\begin{aligned}\mathrm{d}s^2 &= 2(\mathrm{d}\bar{u} + W\sin\alpha\,\mathrm{d}\beta)(\mathrm{d}T + qW_{\bar{u}}\sin\alpha\,\mathrm{d}\beta)\\ &\quad - (N^2 + q^2)(\mathrm{d}\alpha^2 + \sin^2\alpha\,\mathrm{d}\beta^2)\\ &\quad - [1 - W_{\bar{u}}^2 - 2Mq/(N^2 + q^2)](\mathrm{d}\bar{u} + W\sin\alpha\,\mathrm{d}\beta)^2.\end{aligned} \tag{4}$$

Note that

$$q = \bar{u} - T + (a/2)\sin^2\alpha, \quad W^2 = (k^2 + a\bar{u})\sin^2\alpha. \tag{5}$$

It is easy to check that

$$\partial W/\partial\bar{u} = -\partial(qW_{\bar{u}})/\partial T.$$

Thus we have seen that the radiating Kerr metric can be expressed in the form

$$\begin{aligned}\mathrm{d}s^2 &= 2(\mathrm{d}u + g\sin\alpha\,\mathrm{d}\beta)(\mathrm{d}t + H\sin\alpha\,\mathrm{d}\beta)\\ &\quad - 2L(\mathrm{d}u + g\sin\alpha\,\mathrm{d}\beta)^2 - M^2(\mathrm{d}\alpha^2 + \sin^2\alpha\,\mathrm{d}\beta^2),\end{aligned} \tag{6}$$

with

$$g = g(\alpha, u), \quad H = H(\alpha, u, t), \quad L = L(\alpha, u, t), \quad M = M(\alpha, u, t).$$

It satisfies the relation $\partial g/\partial u = -\partial H/\partial t$.

For the general metric (6), the tetrad components of the curvature tensor R_{ijkl} are given in the Appendix. From these components the other geometrical quantities e.g. the Ricci tensor R_{ik} and the curvature scalar R can be calculated.

We shall prove the following general theorem.

Theorem. *For the metric* (6) *if* (*i*) $g_u + H_t = 0$ *and* (*ii*) $g = R(u)A(\alpha)$, *then one can always transform metric* (6) *to the form where* $g_u = 0$, $H_t = 0$, (*here* R *is a function of* u *only and* A *is a function of* α *only*).

Proof: $g = R(u)A(\alpha)$ and $g_u + H_t = 0$ imply

$$H = -AR_u t + B(\alpha, u), \tag{7}$$

where B is a function of α and u. Using this value of H and putting $\bar{u} = \int (\mathrm{d}u/R)$ and $\bar{t} = Rt$ one can easily verify that the metric (6) transforms to the form

$$\mathrm{d}s^2 = 2(\mathrm{d}\bar{u} + A \sin\alpha\, \mathrm{d}\beta)(\mathrm{d}\bar{t} + BR \sin\alpha\, \mathrm{d}\beta)$$
$$- M^2(\mathrm{d}\alpha^2 + \sin^2\alpha\, \mathrm{d}\beta^2) - 2\bar{L}(\mathrm{d}\bar{u} + A \sin\alpha\, \mathrm{d}\beta)^2$$

where

$$2\bar{L} = (2LR + tR_u)R.$$

This proves the theorem. Thus for the metric (6), we can take

$$g_u = 0, \quad H_t = 0. \tag{8}$$

Therefore, now onwards, we assume that the functions g and H satisfy the conditions (8). Introducing the tetrad

$$\theta^1 = \mathrm{d}u + g \sin\alpha\, \mathrm{d}\beta, \quad \theta^2 = M\, \mathrm{d}\alpha,$$
$$\theta^3 = M \sin\alpha\, \mathrm{d}\beta, \quad \theta^4 = \mathrm{d}t + H \sin\alpha\, \mathrm{d}\beta - L\theta^1 \tag{9}$$

the metric (6) becomes

$$\mathrm{d}s^2 = 2\theta^1\theta^4 - (\theta^2)^2 - (\theta^3)^2 = g_{(ab)}\theta^a\theta^b. \tag{10}$$

Here and in what follows the bracketed indices indicate tetrad components with respect to the tetrad (9). When $H = 0$, the metric (6) reduces to our metric discussed earlier (Vaidya *et al* 1976).

Using the Cartan's equations of structure

$$d\theta^a + w^a_b \Lambda \theta^b = 0,$$
$$dw^a_b + w^a_c \Lambda w^c_b = \tfrac{1}{2} R^a_{bcd}\theta^c \Lambda \theta^d$$

and the results of the appendix, one can find the tetrad components R^a_{bcd} of the curvature tensor for the metric (6) with $g_u = H_t = 0$. From R^a_{bcd} the tetrad components $R_{(ab)} = R_{abc}$ of the Ricci tensor can be obtained easily. One gets

$$R_{(23)} = 0, \quad R_{(22)} = R_{(33)} \tag{11}$$

and

$$R_{(24)} = (1/M)[(M_t/M)_\alpha - g(f/M^2)_u - H(f/M^2)_t],$$
$$R_{(34)} = -(1/M)[g(M_t/M)_u + H(M_t/M)_t + (f/M^2)_\alpha],$$
$$R_{(12)} = LR_{(24)} + [(L_t + M_u/M)_\alpha + g\{(2fL - h)/M^2\}_u$$
$$+ H\{(2fL - h)/M^2\}_t - fH_u/M^2],$$

$$R_{(13)} = LR_{(34)} + (1/M)[\{(2fL - h)/M^2\}_\alpha - g(L_t + M_u/M)_u - H(L_t + M_u/M)_t - H_u M_t/M],$$

$$R_{(44)} = (2/M)[M_{tt} - f^2/M^3],$$

$$R_{(14)} = L_{tt} + (2/M)[M_{ut} + (LM_t)_t + f(fL - h)/M^3],$$

$$R_{(22)} = (1/M^2)[(M\alpha/M)_\alpha + (M\alpha/M)\cot\alpha - 1 - \{L(M^2)_t\}_t - (M^2)_{ut} + 4f(fL-h)/M^2 + g^2(M_u/M)_u + H^2(M_t/M)_t + 2gH(M_t/M)_u + gH_u M_t/M],$$

$$R_{(11)} = L^2 R_{(44)} + 2h(2fL - h)/M^4 + (1/M^2)[L_{\alpha\alpha} + L_\alpha \cot\alpha + H^2 L_{tt} + g^2 L_{uu} + 2gHL_{ut} + 2MM_{uu} + 4LMM_{ut} - 2L_t MM_u + 2L_u MM_t + gL_t H_u - gH_{uu}]. \tag{12}$$

where f and h are defined by the relations

$$2f = g_\alpha + g\cot\alpha, \quad 2h = H_\alpha + H\cot\alpha. \tag{13}$$

We assume that the space-time is filled with a mixture of perfect fluid and a pure flowing radiation field. The Einstein field equations for such a distribution are

$$R_{ik} - \tfrac{1}{2}g_{ik}R + \Lambda g_{ik} = -8\pi[(p+\rho)v_i v_k - pg_{ik} + \sigma w_i w_k]. \tag{14}$$

where

$$v^i v_i = 1, \quad w^i w_i = 0, \quad v^i w_i = 1. \tag{15}$$

Here p, ρ, σ and Λ are respectively the pressure, the material density, the radiation density and the cosmological constant. The last condition in (15) is the normalization condition for the null vector w_i. It is not hard to see that the field equations (14) can be expressed in the tetrad form as

$$R_{(ab)} = -8\pi[(p+\rho)v_{(a)}v_{(b)} - \tfrac{1}{2}(\rho - p)g_{(ab)}] + \Lambda g_{(ab)} - 8\pi\sigma w_{(a)}w_{(b)}, \tag{16}$$

where $v_{(a)}$ and $w_{(b)}$ are the tetrad components of the flow vector v_i and the null vector w_i respectively. We take the tetrad components $v_{(a)}$ and $w_{(a)}$ as

$$v_{(a)} = (1/2z, 0, 0, z), \quad w_{(a)} = (1/z, 0, 0, 0), \tag{17}$$

where z is a function of co-ordinates to be determined from the field equations. It is easy to verify that v_i and w_i given by (17), satisfy the conditions (15). The results (10), (16) and (17) imply the following relations.

$$R_{(24)} = 0, \quad R_{(34)} = 0, \tag{18}$$

$$R_{(12)} = 0, \quad R_{(13)} = 0, \tag{19}$$

$$z^2 = R_{(44)}/2[R_{(14)} + R_{(22)}], \tag{20}$$

$$8\pi p = \Lambda - R_{(14)}, \tag{21}$$

$$8\pi(p+\rho) = -2[R_{(14)} + R_{(22)}], \tag{22}$$

$$8\pi\sigma = -z^2 R_{(11)} - 2\pi(p+\rho) \tag{23}$$

where $R_{(ab)}$ are given by (12).

From the differential eqs (18) and (19) we have to find the metric potentials g, H, M and L. Equations (20)–(23) will then give the four physical parameters p, ρ, z^2 and σ. We shall integrate these equations for two physically significant cases in the next two sections.

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We now assume that the metric function M is independent of t i.e. $M_t = 0$. Equation (18), then imply that

$$f = RM^2, \tag{24}$$

where R is a constant. Using (24), the differential eq. (19) can be explicitly written as

$$L_{t\alpha} + 2gRL_u - g(h/M^2)_u + 2HRL_t - RH_u = 0 \tag{25}$$

and

$$2RL_\alpha - (h/M^2)_\alpha - gL_{tu} - HL_{tt} = 0. \tag{26}$$

We further assume that $H = 0$. Then it is easy to see that the general solution of (25) and (26) is

$$L = S(t) + \phi(u, y)\sin 2Rt + \psi(u, y)\cos 2Rt$$

where S is an arbitrary function of t and the functions ϕ and ψ satisfy the relations

$$\phi_y = -\psi_u, \quad \phi_u = \psi_y, \tag{27}$$

the variable y being defined by the differential relation $g\,\mathrm{d}\alpha = \mathrm{d}y$. As we are interested in the field of a radiating Kerr particle in Einstein static universe, we take $S = 1/2$ and consequently the function L is given by

$$L = \tfrac{1}{2} + \phi(u, y)\sin 2Rt + \psi(u, y)\cos 2Rt. \tag{28}$$

As suggested from the field of a Kerr particle in Einstein Universe (Patel and Vaidya 1983) we take

$$\phi = -\exp(ky)\cos ku, \; \psi = \exp(ky)\sin ku, \tag{29}$$

where k is an arbitrary constant. It is not difficult to see that ϕ and ψ given by (29) satisfy the relations (27). Therefore $2L$ can be expressed as

$$2L = 2\exp\{(a + 2R)y\}\sin(2R\bar{r} + au) + 1, \tag{30}$$

where $K = 2R + a$ and $\bar{r} = u - t$. For the Einstein Universe (Vaidya, 1978)

$$g\sin\alpha = 2 - \exp(-2yR) \tag{31}$$

Now the parameters z^2, p, ρ and σ can be determined from (20), (21), (22) and (23) respectively. They are given by

$$z^2 = 1/2(1-L), \tag{32}$$

$$8\pi p = \Lambda + 2R^2(L-1), \tag{33}$$

$$8\pi\rho = -\Lambda + 6R^2(1-L), \tag{34}$$

$$16\pi\sigma = aR(2L-1)/(L-1), \tag{35}$$

where L is given by (30).

When $a=0$, the radiation density σ vanishes and our solution reduces to the solution discussed by Patel and Vaidya (1983) which describes the gravitational field of a Kerr particle embedded in the Einstein universe.

Hence we are tempted to say that the solution discussed in the present section represents the field of a radiating Kerr particle embedded in Einstein universe.

In the next section we shall take up another simple particular case.

4. Pure radiation field in deSitter-like universe

Let us asume that $g=0$ and $M_t=0$. With these assumptions the differential eqs (18) are identically satisfied. In this section we shall use the field equations

$$R_{(ab)} = \Lambda g_{(ab)} - 8\pi\sigma\xi_{(a)}\xi_{(b)}, \quad \xi_{(a)}\xi^{(a)} = 0. \tag{36}$$

We can take $\xi_{(a)} = (1,0,0,0)$. The field eqs (36) imply (19) and

$$-(1/H)(h/M^2)_\alpha = \Lambda, \quad 2h = H_\alpha + H\cot\alpha \tag{37}$$

$$(1/M^2)[(M_\alpha/M)_\alpha + (M_\alpha/M)\cot\alpha - 1] = -\Lambda \tag{38}$$

The solution of eqs (19) in this case is

$$L = (C - M_u/M)t + (1/2)\Lambda t^2 + D(u,\alpha), \tag{39}$$

where C is an arbitrary function of u and D is an arbitrary function of u and α. The radiation density σ in this case is given by

$$\begin{aligned} 8\pi\sigma = {} & 2h^2/M^4 - \Lambda H^2/M^2 - 2(M_{uu}/M + M_u^2/M^2) \\ & + 2CM_u/M - 2\Lambda t M_u/M - (1/M^2)[D_{\alpha\alpha} + D_\alpha \cot\alpha]. \end{aligned} \tag{40}$$

Eqs (37) and (38) can be integrated to have the functions H and M. Their general solution can be expressed in the form

$$\begin{aligned} M^2\sin^2\alpha &= n^2 V r^n/(1+\Lambda V r^n)^2 \\ H\sin\alpha &= U(1-\Lambda V r^n)/n(1+\Lambda V r^n) \\ &\quad - 2\Lambda V U r^n \log r/(1+\Lambda V r^n)^2, \end{aligned} \tag{41}$$

where n, V and U are arbitrary functions of u only and the variable r is defined by

$$r = \tan(\alpha/2). \tag{42}$$

When $\Lambda = 0$, the radiation density σ becomes

$$\begin{aligned} 8\pi\sigma = 2C[&n_u/n + V_u/2V + n_u \log r] \\ &- 2[n_{uu}/n + n_u^2/n^2 + V_{uu}/2V + 2V_u n_u/Vn] \\ &+ (1/2)n_{uu}\log r + (n_u/n)V_u \log r + (2/n)n_u^2 \log r \\ &+ (n_u^2/2)(\log r)^2] - (1/M^2)(D_{\alpha\alpha} + D_\alpha \cot\alpha). \end{aligned}$$

This radiation density σ can be made zero by choosing the functions C and D as

$$C = [\log(MM_u)]_u, \; D_{\alpha\alpha} + D_\alpha \cot\alpha = 0.$$

But these conditions are equivalent to

$$n = \text{constant}, \quad C = V_{uu}/V_u, \quad D = (1/2) + m(u)\log r \tag{43}$$

where m is an arbitrary function of u only. Therefore the following metric describes an empty space time.

$$\begin{aligned} ds^2 = 2\,du[dt + (U/n)\,d\beta] - n^2 V r^{n-2}(dr^2 + r^2\,d\beta^2) \\ - [1 + 2m(u)\log r + t\{2(V_{uu}/V_u) - (V_u/V)\}]\,du^2, \end{aligned} \tag{44}$$

where U and V are arbitrary functions of u and n is a constant. The metric (44) is cylindrically symmetric. We have verified that the curvature tensor for (44) is non-zero.

A particular case of the solution (41) is noteworthy. Let us assume that $n = 2$ and $\Lambda V = 1$. In this case we get $M^2 = (1/\Lambda)$, $\Lambda \neq 0$,

$$H \sin\alpha = U(1 - r^2)/2(1 + r^2) - 2Ur^2 \log r/(1 + r^2)^2. \tag{45}$$

Let us choose $C(u) = 0$. The function h and the radiation density σ in this case become

$$\begin{aligned} 2h &= -U(1 + \cos\alpha \log r), \\ 8\pi\sigma &= (1/M^2)[2h^2/M^2 - \Lambda H^2 - (D_\alpha \sin\alpha)_\alpha/\sin\alpha], \end{aligned} \tag{46}$$

where $r = \tan(\alpha/2)$. Using (45), we can integrate the eq. $\sigma = 0$ for the function D. It is given by

$$D = A \log\tan(\alpha/2) + B - (\Lambda U^2/4)\{\cos\alpha \log r + (1/2)\cos^2\alpha(\log r)^2\}, \tag{47}$$

where A and B are arbitrary functions of u. The term $A \log\tan(\alpha/2)$ corresponds to a mass particle in deSitter-like universe. Choosing the functions A and B as

$$A = 0, \quad B = (1/2) - (\Lambda U^2/8), \tag{48}$$

we obtain the functions D and L as

$$D = (1/2)(1 - \Lambda h^2), \quad 2L = 1 + \Lambda(t^2 - h^2) \tag{49}$$

where h is given by (46). Thus the deSitter-like metric $[R_{(ab)} = \Lambda g_{(ab)}]$ can be expressed in the final form as

$$\begin{aligned} ds^2 &= 2\,du[dt + U\{(1-r^2)/2(1+r^2) - 2r^2 \log r/(1+r^2)^2\}\,d\beta] \\ &\quad - (4/\Lambda)[(1+r^2)^2 dr^2 + r^2(1+r^2)^{-2} d\beta^2] \\ &\quad - [1 + \Lambda(t^2 - h^2)]\,du^2, \end{aligned} \tag{50}$$

where h is given by

$$2h = -U[1 + (1-r^2)\log r/(1+r^2)].$$

5. Concluding remarks

In the above analysis, we have investigated two particular cases of the metric (6) with the conditions (8). They are (1) $H = 0$, $M_t = 0$ and (ii) $g = 0$, $M_t = 0$. The first case gives us the field of a radiating spinning source embedded in Einstein universe. The second case describes the propagation of a flowing radiation field in deSitter-like universe. If $\Lambda = 0$ and the pure radiation field is switched off, we get a non-flat empty space-time. The deSitter-like metric is derived by showing of $R_{ik} = \Lambda g_{ik}$ as a particular case of the second case.

The general case $H \neq 0$, $g \neq 0$ of the metric (6) alongwith (8) in connection with the field equations (14) is currently under investigation. The results of this investigation will be discussed later.

Appendix

Let us introduce the tetrad (9) for the metric (6) with $g_u + H_t = 0$. The tetrad components R^a_{bcd} of the curvature tensor can be obtained from the equation of structure $\Omega^a_b = \frac{1}{2} R^a_{bcd} \theta^c \Lambda \theta^d$ where $\Omega^a_b = dw^a_b + w^a_c \Lambda w^c_b$. The non-zero independent components of R^a_{bcd} are listed below for ready reference:

$$R^1_{141} = L_{tt}, \quad R^1_{112} = (1/M)[L_{t\alpha} + L_\alpha(M_t/M) + (fN/M)],$$

$$R^1_{212} = (1/M)[LM_{tt} + M_{tu} + L_t M_t + f(fL - h)/M^3] = R^1_{313}$$

$$R^1_{113} = (1/M)[g_u L_t + g L_{tu} + H L_{tt} + g_{uu} + L_\alpha(f/M^2) + NM_t],$$

$$R^1_{123} = (1/M^2)[2fL_t + 2f_u + 2(M_t/M)(h - fL) - 2f\Delta],$$

$$R^1_{242} = R^1_{343} = (1/M)[M_{tt} - f^2/M^3],$$

$$R^1_{213} = (1/M^2)[fL_t + f_u - f\Delta + (M_t/M)(h - fL)],$$

$$R^1_{223} = (1/M)[(f/M^2)_\alpha + g(M_t/M)_u + H(M_t/M)_t + g_u(M_t/M)],$$

$$R^1_{312} = (1/M^2)[\Delta f - f_u - fL_t - (M_t/M)(h - fL)],$$

$$R^1_{323} = (1/M)[(M_t/M)_\alpha + (f/M^2)g_w - g(f/M^2)_u - H(f/M^2)_t],$$

$$R^2_{112} = (1/M)[M_{uu} + 2LM_{tu} + L^2M_{tt} + (L_\alpha/M)_\alpha + L_uM_t - L_tM_u - LM_tL_t - NEM],$$

$$R^2_{113} = (1/M)[2\Delta(h - fL)/M - 2g_u(L_\alpha/M) - g(L_\alpha/M)_u - H(L_\alpha/M)_t - L_t(h - fL)/M - NFM + M\{(h - fL)/M^2\}_u + LM\{(h - fL)/M^2\}_t],$$

$$R^2_{123} = (1/M)[\Delta g_u/M + g\Delta_u + H\Delta_t - 2fL_\alpha/M^2 + \{(h - fL)/M^2\}_\alpha],$$

$$R^3_{113} = [\Delta^2 + \Delta_u + L\Delta_t - 2g_u(N/M) - N_u(g/M) - N_t(H/M) - \Delta L_t + FL_\alpha/M^2 - (h - fL)^2/M^4],$$

$$R^3_{123} = [\Delta\alpha/M^2 - 2N(f/M^2) - g_u(h - fL)/M^3 - (g/M)\{(h - fL)/M^2\}_u - (H/M)\{(h - fL)/M^2\}_t],$$

$$R^2_{323} = [6f(h - fL)/M^4 - F^2 - E^2 - F_\alpha/M - gE_u/M - HE_t/M + 2\Delta(M_t/M)],$$

where $2f = g_\alpha + g\cot\alpha$, $2h = H_\alpha + H\cot\alpha$

$$\Delta = (M_u/M) + L(M_t/M), \quad FM^2 = M_\alpha + M\cot\alpha$$

$$EM^2 = gM_u + HM_t \text{ and}$$

$$MN = H_u - (2Lg_u + HL_t + gL_u).$$

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