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Kerr metric in the deSitter background

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Abstract. In addition to the Kerr metric with cosmological constant Λ several other metrics are presented giving a Kerr-like solution of Einstein's equations in the background of deSitter universe. A new metric of what may be termed as rotating deSitter space-time—a space-time devoid of matter but containing null fluid with twisting null rays, has been presented. This metric reduces to the standard deSitter metric when the twist in the rays vanishes. Kerr metric in this background is the immediate generalization of Schwarzschild's exterior metric with cosmological constant.

Keywords. Kerr metric; deSitter universe.

1. Introduction

In an earlier paper (Vaidya 1977, referred to hereafter as I) we have considered Kerr metric in cosmological background, the background metric being the Robertson-Walker metric. In the present paper we single out the background universe as empty deSitter space-time for several reasons. One reason is that though deSitter metric can be expressed as a particular case of the general Robertson-Walker metric, the deSitter space-time has features which are geometrically distinct from Robertson-Walker models. Again the simple deSitter space-time representing an expanding, curved and yet empty open universe is the immediate generalization of Minkowski flat space-time and has very similar properties as a background for physical phenomena.

Schwarzschild's exterior metric in deSitter background is the well-known Schwarzschild's solution with cosmological constant Λ

$$\mathrm{d}s^2 = \left(1-\frac{2m}{r}-\frac{r^2}{R^2}\right)\mathrm{d}t^2 - \left(1-\frac{2m}{r}-\frac{r^2}{R^2}\right)^{-1}\mathrm{d}r^2 - r^2\,\mathrm{d}\Omega^2 \tag{1}$$

$$\mathrm{d}\Omega^2 = \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2 \quad \text{and} \quad \Lambda = \frac{3}{R^2}.$$

On parallel lines we may expect that Kerr metric in deSitter background will be the Kerr solution with cosmological constant which has been derived by several authors earlier (Carter 1968; Demianski 1973; Frolov 1974). However we shall see that one can have other (non-equivalent) forms for Kerr metric in deSitter background.

In the next section we shall derive the known form of Kerr-metric with cosmological constant. In the third section we shall write down a simple form of metric for anti deSitter space-time (Λ negative) and get a Kerr-like solution in this background.

In the last section we first derive the metric for what can be termed as rotating deSitter space-time—a space-time devoid of matter but containing null fluid with twisting null rays. This metric reduces to the standard deSitter metric when the twist in the rays vanishes. The Kerr metric in this background is the immediate generalization of Schwarschild's exterior metric (1) with cosmological constant.

2. Kerr metric with Λ

We begin with the standard deSitter metric

$$ds^2 = \left(1 - \frac{\rho^2}{R^2}\right) dt^2 - \left(1 - \frac{\rho^2}{R^2}\right)^{-1} d\rho^2 - \rho^2 \, d\Omega^2$$

$$\Lambda = 3/R^2, \; d\Omega^2 = d\theta^2 + \sin^2\theta \, d\varphi^2,$$

and carry out the transformation

$$\rho^2 = r'^2/[1 + (r'^2/R^2)].$$

It then transforms to

$$ds^2 = \left(1 + \frac{r'^2}{R^2}\right)^{-1} \left[dt^2 - \left(1 + \frac{r'^2}{R^2}\right)^{-1} dr'^2 - r'^2 \, d\Omega^2 \right]. \tag{2}$$

(2) shows that the deSitter metric can be put in a form conformal to Einstein-Universe—metric with negative curvature $(-1/R^2)$. We rewrite (2) in terms of the usual Cartesian coordinates (t, x, y, z) as

$$ds^2 = \frac{R^2}{R^2 + (x^2 + y^2 + z^2)} \left[dt^2 - dx^2 - dy^2 - dz^2 + \frac{(x dx + y dy + z dz)^2}{R^2 + (x^2 + y^2 + z^2)} \right], \tag{3}$$

and then follow the steps initiated in **I** to carry out the transformations to spheroidal polar coordinates (t, r, α, β) by the substitutions $t = t$

$$x = R \sinh \frac{r}{R} \sin\alpha \cos\beta - a \cosh \frac{r}{R} \sin\alpha \sin\beta,$$

$$y = R \sinh \frac{r}{R} \sin\alpha \sin\beta + a \cosh \frac{r}{R} \sin\alpha \cos\beta,$$

$$z = R \sinh \frac{r}{R} \cos\alpha.$$

The deSitter metric then takes the form

$$ds^2 = \left[\cosh^2 \frac{r}{R} \left(1 + \frac{a^2}{R^2} \sin^2\alpha\right) \right]^{-1} ds_0^2, \tag{4}$$

$$ds_0^2 = 2\,(dt - dr + a \sin^2\alpha \, d\beta)\, dt - (dt - dr + a \sin^2\alpha \, d\beta)^2$$
$$- M^2 \left[\left(1 + \frac{a^2}{R^2} \sin^2\alpha\right)^{-1} d\alpha^2 + \sin^2\alpha \, d\beta^2 \right] \tag{4a}$$

$$M^2 = (R^2 + a^2) \sinh^2 \frac{r}{R} + a^2 \cos^2\alpha. \tag{5}$$

Following the scheme of I, one can write down the Kerr-metric in the deSitter background as

$$ds^2 = \left[\cosh^2 \frac{r}{R}\left(1 + \frac{a^2}{R^2}\sin^2\alpha\right)\right]^{-1} ds_1^2, \tag{6}$$

$$ds_1^2 = ds_0^2 - 2m\mu\,(dt - dr + a\sin^2\alpha\, d\beta)^2, \tag{6a}$$

$$\mu = R\sinh\frac{r}{R}\cosh^3\frac{r}{R}, \quad m = \text{constant}.$$

If we transform our spheroidal polar coordinates to the conventional Boyer-Lindquist type coordinates, (6) is transformed to Kerr metric with cosmological constant as given by Demianski (1973). The explicit transformation equations are given in appendix (A). Alternatively using a result recently obtained by Taub (1981) and the form of Kerr metric in the background of Einstein's universe as given in I one can verify that (6) is Kerr metric in the background of deSitter universe. The verification is given in appendix (B).

3. Anti-deSitter background

When the cosmological constant Λ is negative so that one can write $\Lambda = -3/R^2$ the space-time represented by deSitter metric is known in literature as anti-deSitter space-time. A surprising result is that the following axially symmetric metric, conformal to the usual deSitter metric, represents anti-deSitter space-time

$$ds^2 = \frac{R^2}{r^2\cos^2\alpha}\left[\left(1 - \frac{r^2}{b^2}\right)dt^2 - \left(1 - \frac{r^2}{b^2}\right)^{-1} dr^2 - r^2 d\alpha^2 - r^2\sin^2\alpha\, d\beta^2\right]. \tag{7}$$

It satisfies $R_{ik} = \Lambda g_{ik}$ with $\Lambda = -3/R^2$, b^2 being an undetermined constant. Now it is known (Hawking and Ellis 1973) that in a certain definite space-time region anti-deSitter space-time metric is conformal to Einstein universe metric. Thus one can transform (7) to the form

$$ds^2 = \frac{R^2}{z^2}\left[dt^2 - dx^2 - dy^2 - dz^2 - \frac{(xdx + ydy + zdz)^2}{b^2 - (x^2 + y^2 + z^2)}\right]. \tag{8}$$

One can now use the method of transforming to spheroidal polar coordinates initiated in I and used in §2 above to get the Kerr metric in the background (7). However we shall not write down the resulting metric here. Instead, we note a simple particular case of (7) or (8) obtained by choosing the undetermined constant $b \to \infty$. The background anti-deSitter metric (7) or (8) then simplifies to the following plane symmetric metric

$$ds^2 = \frac{R^2}{z^2}[dt^2 - dx^2 - dy^2 - dz^2]. \tag{9}$$

The Kerr metric in this background is given by

$$ds^2 = \frac{R^2}{r^2\cos^2\alpha}\left[2(dt - dr + a\sin^2\alpha\, d\beta)\, dt - \left(1 + \frac{2mr^3}{r^2 + a^2\cos^2\alpha}\right) \times (dt - dr + a\sin^2\alpha\, d\beta)^2 - (r^2 + a^2\cos^2\alpha)\,(d\alpha^2 + \sin^2\alpha\, d\beta^2)\right] \tag{10}$$

It can again be verified that for the metric (10) $R_{ik} = \Lambda g_{ik}$, $\Lambda = -3/R^2$, m = constant (10) gives a very simple metric for the Kerr-like gravitational field satisfying $R_{ik} = \Lambda g_{ik}$ with Λ a non-zero negative constant. That Λ has to be non-zero is a condition for the existence of the background metric of anti-deSitter space-time given by (7) or (9).

One can show that (10) satisfies $R_{ik} = \Lambda g_{ik}$ by the method given in appendix (B).

4. deSitter space-time with twisting null rays

In I it was shown that the Minkowski metric

$$ds^2 = dt^2 - dx^2 - dy^2 - dz^2$$

can be transformed in ellipsoidal coordinates to the form

$$ds^2 = 2(du + a\sin^2\alpha d\beta)dt - (du + a\sin^2\alpha\, d\beta)^2 - (r^2 + a^2\cos^2\alpha)d\Omega^2 \quad (11)$$

$$d\Omega^2 = d\alpha^2 + \sin^2\alpha\, d\beta^2, \quad r = t - u,$$

and that (11) forms the background metric for the Kerr field. Our aim is to get an immediate generalization of (11) which will take us from the Minkowski background to deSitter background. Let us therefore begin with the Schwarzschild's exterior metric in deSitter background *i.e.* the metric (1) above and introduce the retarded time u in place of the coordinate t. The equation defining u is

$$\left(1 - \frac{2m}{r} - \frac{r^2}{R^2}\right)\frac{\partial u}{\partial r} + \frac{\partial u}{\partial t} = 0,$$

a solution of which is

$$u = t - \int \left(1 - \frac{2m}{r} - \frac{r^2}{R^2}\right)^{-1} dr.$$

Using u as a time co-ordinate in place of t one transforms (1) to

$$ds^2 = 2du\,dr + \left(1 - \frac{2m}{r} - \frac{r^2}{R^2}\right)du^2 - r^2\, d\Omega^2. \quad (12)$$

But in our scheme as used in § 2 onwards we do not use u in place of t, but we use u in place of r. And one can do this because u is a null coordinate. We return to a time-like coordinate T by the substitution $u = T - r$ and use (T, u, α, β) as coordinates *i.e.* replace r by $T - u$. Metric (12) then takes the form

$$ds^2 = 2du\,T - \left(1 + \frac{2m}{r} + \frac{r^2}{R^2}\right)du^2 - r^2\, d\Omega^2. \quad (13)$$

This is the Schwarzschild's exterior metric in deSitter background.

The form (11) of the Minkowski metric (which is the background metric of Kerr solution) suggests an immediate generalization of (13) to the following

$$ds^2 = 2\,(du + g\sin\alpha\, d\beta)\, dt - \left(1 + \frac{2mr}{r^2 + y^2} + \frac{r^2 + y^2}{R^2}\right)(du + g\sin\alpha\, d\beta)^2$$
$$- H(r^2 + y^2)\, d\Omega^2, \quad (14)$$

$$g = g(\alpha), y = y(\alpha), H = H(\alpha), \Lambda = \frac{3}{R^2} \text{ and } d\Omega^2 = d\alpha^2 + \sin^2\alpha\, d\beta^2.$$

It may be noted that if $\Lambda = 0$, with $g = a \sin\alpha$, $y = -a\cos\alpha$ and $H = 1$ (14) gives us the Kerr metric. To determine these functions when $\Lambda \neq 0$ we further note that the metric (14) is in the form which we have termed a Kerr-NUT metric (Vaidya *et al* 1976) *viz*

$$ds^2 = 2(du + g\sin\alpha\, d\beta)\, dx - 2L(du + g\sin\alpha\, d\beta)^2 - M^2\, d\Omega^2 \tag{15}$$

and so we can use the tetrad formalism developed in that paper to work out the physics of metric (14). Using the tetrad

$$\theta^1 = du + g\sin\alpha\, d\beta,\ \theta^2 = M d\alpha,\ \theta^3 = M\sin\alpha\, d\beta,\ \theta^4 = dx - L\theta^1$$

(so that (15) takes the form $ds^2 = 2\theta^1\theta^4 - (\theta^2)^2 - (\theta^3)^2$) the tetrad components $R_{(ab)}$ of the Ricci tensor are also recorded there.

It can be verified that if one chooses

$$g\, d\alpha = dy,\ H = f/(-y),\ \text{with } 2f = (\partial g/\partial\alpha) + g\cot\alpha, \tag{16}$$

one finds that for the metric (14)

$$R_{(12)} = R_{(13)} = R_{(23)} = R_{(24)} = R_{(34)} = R_{(44)} = 0,$$

$$R_{(14)} = \Lambda, R_{(22)} = -\Lambda + [2 + \tfrac{4}{3}\Lambda y^2 - yG]\,(r^2 + y^2)^{-1} = R_{33},$$

$$R_{(11)} = -\frac{2y}{3}\Lambda\,[(g^2/f) + 2y]\,(r^2 + y^2)^{-1}, \Lambda = \frac{3}{R^2}.$$

In the above G is defined by

$$2fG = g^2\left[\frac{1}{y^2} + \frac{\partial}{\partial y}\left(\frac{1}{f}\frac{\partial f}{\partial y}\right)\right] + 2\frac{\partial f}{\partial y} - 2.$$

It is clear that if we choose

$$8\pi\sigma = \frac{2y}{3}\Lambda[(g^2/f) + 2y]\,(r^2 + y^2)^{-1} \text{ and}$$

$$Gy - 2 - \frac{4}{3}\Lambda y^2 = 0, \tag{17}$$

we shall have $R_{ik} = \Lambda g_{ik} - 8\pi\sigma\xi_i\xi_k$,

where ξ_i is a null vector defined by

$$\xi_i\, dx^i = du + g\sin\alpha\, d\beta.$$

Equations (16) and (17) are the three equations which determine the three unknown functions of α in the metric (14).

If $\Lambda = 3/R^2 = 0$, (16) and (17) are satisfied by $f = -y = +a\cos\alpha$, $g = a\sin\alpha$, a being constant. The metric (14) then becomes the Kerr metric.

The other simple case is $\Lambda \neq 0$, $m = 0$. Then

$$ds^2 = 2(du + g\sin\alpha\, d\beta)\, dt - \left(1 + \frac{r^2 + y^2}{R^2}\right)(du + g\sin\alpha\, d\beta)^2 - (f/-y)\,(r^2 + y^2)\, d\Omega^2 \tag{18}$$

with $$dy = g d\alpha,\ 2f = \frac{dg}{d\alpha} + g \cot\alpha \text{ and } f = f(y)$$

satisfying $$Gy - 2 - (4y^2/R^2) = 0. \tag{19}$$

One can interpret this simple solution as a rotating deSitter space-time because (i) it represents a universe devoid of matter (ii) its metric (18) reduces to the usual deSitter metric when the rotation parameter $g = 0$. It has an additional feature that it is pervaded by a unidirectional flow of null radiation which arises solely due to rotation. As a matter of fact if one solves (19) correctly upto the first power of $1/R^2$ (*i.e.* upto first power of Λ), one can show that

$$8\pi\sigma = \frac{2a^2(3\cos^2\alpha - 1)}{R^2(r^2 + a^2\cos^2\alpha)},$$

so that

$$\int_0^{\pi/2}\int_0^{2\pi} M^2 \sigma \sin\alpha \, d\alpha \, d\beta = 0,\ M^2 = (f/ - y)(r^2 + y^2)$$

i.e. the net outflow of null radiation across the 2-space with metric $M^2(d\alpha^2 + \sin^2\alpha\, d\beta^2)$ is zero. Thus there is no net loss of energy due to this flowing radiation. The expanding nature of 3-space and the rotation introduced in it, together, so to say, lead to a churning of gravitational energy in deSitter space-time which flows out from a cone of semiangle $\cos^{-1}(1/\sqrt{3})$ and with the axis coinciding with the axis of rotation and returns through the rest of the surface.

With this interpretation (18) becomes the metric of deSitter space-time with rotating null rays and metric (14)

$$ds^2 = 2(du + g\sin\alpha\, d\beta)\, dt - \left(1 + \frac{2mr}{r^2 + y^2} + \frac{r^2 + y^2}{R^2}\right)(du + g\sin\alpha\, d\beta)^2$$
$$- (f/ - y)(r^2 + y^2)(d\alpha^2 + \sin^2\alpha\, d\beta^2)$$

with $dy = g d\alpha$ and $f = f(y)$ given by (19) becomes Kerr metric in the background of this rotating deSitter universe. And this last metric is easily seen to be a simple generalization of Schwarzschild's exterior metric in deSitter background.

Appendix A

Let $(\bar{r}, \theta, \varphi, \bar{t})$ be the Boyer-Lindquist type coordinates, then the required transformation equations are

$$\bar{r} = R \tanh\frac{r}{R},\ \bar{t} = t + \int \frac{2m\mu}{\Delta} dr$$

$$\cos\theta = \cos\alpha\left(1 + \frac{a^2\sin^2\alpha}{R^2}\right)^{-1/2}$$

$$(a^2 + R^2)\varphi = R^2\beta - at - a\int \frac{R^2 + 2m\mu}{\Delta} dr,$$

where $2m\mu$ is defined by (6a) and

$$\Delta = (R^2 + a^2)\sinh^2\frac{r}{R} + a^2 - 2mR\sin\frac{r}{R}\cosh^3\frac{r}{R}.$$

Appendix B

Taub (1981) has shown that if a space-time $\hat{V}$ has metric given by

$$\hat{g}_{\mu\nu} = g_{\mu\nu} + 2H\, l_\mu l_\nu, \tag{B.1}$$

where $g_{\mu\nu}$ is the metric for an arbitrary space V, H is a scalar field over V and l_μ is a null, geodetic and shear-free vector field in V, then l_μ is also null geodetic and shear-free with respect to $\hat{g}_{\mu\nu}$. He has further shown that if the space-time $\overline{V}$ with the metric tensor $\overline{g}_{\mu\nu}$ is conformal to space-time $\hat{V}$ with metric tensor $\hat{g}_{\mu\nu}$ that is, if

$$\overline{g}_{\mu\nu} = \exp(2\sigma)\hat{g}_{\mu\nu}, \tag{B.2}$$

and if l_μ is null, geodetic and shear-free in $\hat{V}$, it is also one in $\overline{V}$.

Using (B.1) and (B.2) one can write

$$\begin{aligned}\overline{g}_{\mu\nu} &= \exp(2\sigma)(g_{\mu\nu} + 2Hl_\mu l_\nu)\\ &= \exp(2\sigma)g_{\mu\nu} + 2H_0 l_\mu l_\nu\end{aligned} \tag{B.3}$$

with $H_0 = H\exp(2\sigma)$. Taub has also worked out expressions for the Ricci tensor $\overline{R}^\nu_\mu$ for $\overline{V}$ in terms of R^ν_μ of V and the scalars H, σ and their derivatives.

One can see that our metric (6) has the metric tensor of the form given by (B.3) with

$$\exp(2\sigma) = \left[\cosh^2\frac{r}{R}\left(1 + \frac{a^2}{R^2}\sin^2\alpha\right)\right]^{-1},$$

$g_{\mu\nu}$ is the metric tensor obtained from

$$ds_0^2 = g_{\mu\nu}\,dx^\mu\,dx^\nu \ \text{ of } (4a) \quad x^1 = r, x^2 = \alpha, x^3 = \beta, x^4 = t,$$

$$H_0 = -m\mu = -mR\sinh\frac{r}{R}\cosh^3\frac{r}{R} \ \text{ and}$$

$$l_\mu = \left(1 + \frac{a^2}{R^2}\sin^2\alpha\right)\xi_\mu, \ \xi_\mu\,dx^\mu = dt - dr + a\sin^2\alpha\,d\beta.$$

It may be noted that ξ_μ is a geodetic, shear-free null congruence and since $\alpha_\mu\xi^\mu = 0$, l_μ is also a similar congruence in V.

With these substitutions and following Taub's calculations one can verify that

$$\overline{R}_{\mu\nu} = \Lambda\overline{g}_{\mu\nu}.$$

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