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A generalized Kerr-Schild metric

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Abstract. A metric $g_{ik} = \eta_{ik} + H\xi_i\xi_k + 2J\xi_{(i}p_{k)}$ is investigated. When $J = 0$ this reduces to the well-known Kerr metric. Conditions on the vector p_i are obtained under which a geodetic, shear-free null congruence ξ_i in the Minkowskian space-time (with metric η_{ik}) will continue to remain geodetic and shear-free in the Riemannian space-time of g_{ik}. A general solution of Einstein's equation $R_{ik} = \sigma\xi_i\xi_k$ is obtained when $p_i\xi^i = 0$ and ξ_i is twist-free.

Keywords. Kerr-Schild metric; relativity, general.

1. Introduction

A metric of the form

$$g_{ik} = \eta_{ik} + H\xi_i\xi_k \tag{1.1}$$

is known in the literature as the Kerr-Schild metric. Here η_{ik} is the metric of Minkowskian space in coordinates which are cartesian but not necessarily rectangular, *i.e.*, η_{ik} are constants and ξ_i is null. Many exact solutions of Einstein's equations or Einstein-Maxwell equations can be expressed in terms of this metric. In 1924 Eddington expressed Schwarzchild's exterior metric in the form

$$ds^2 = du^2 + 2du dr - r^2 (d\theta^2 + \sin^2 \theta d\phi^2) - (2m \mid r)\, du^2, \quad m = \text{constant} \tag{1.2}$$

which essentially is a metric of the form (1.1). Vaidya (1951, 1953) rederived (1.2) with $m = m(u)$ as the metric for a radiating star. Again Whitehead (1922) based his new theory of gravitation with Minkowskian space-time in the background and in his theory the gravitational field of a particle was described by a metric of the form (1.1).

Kerr and Schild (1965) systematically studied this metric and derived a very general class of vacuum solutions of Einstein's equations. In addition to Schwarzschild solution and Kerr-solution (Kerr 1963) several other solutions of Einstein's equations can be couched in terms of this metric. Debney *et al* (1969), Kinnersley (1969), Bonnor and Vaidya (1970, 1972), Vaidya (1972, 1973, 1974), Vaidya and Patel (1973) have worked out non-vacuum solutions of Einstein's equations with this metric.

A special feature of the Kerr-Schild metric (1.1) is that if the congruence ξ_i is (i) null, (ii) geodetic, and (iii) shear-free in the Minkowskian space-time with metric η_{ik}, it continues to satisfy the same three properties in the Riemannian space-time with the metric (1.1). This is the reason why this metric is extensively used in working out Einstein fields or Einstein-Maxwell fields permitting the existence of geodetic, shear-free, null congruences.

But studies of vacuum solutions of Einstein's equations by Robinson and Trautman (1962), Robinson and Robinson (1969) and others also show that there are other interesting metrics for which the above three properties continue to hold good when one passed from a Minkowskian space-time to a related Riemannian space-time. It is our aim to systematically study one such metric by generalizing the Kerr-Schild metric (1.1).

We shall study the metric

$$g_{ik} = \eta_{ik} + H\xi_i\xi_k + 2J\xi_{(i}p_{k)} \tag{1.3}$$

with p_k an arbitrary vector field which may be taken as space-like without any loss of generality. Throughout the paper the round brackets and the square brackets including the indices will denote symmetrization and anti-symmetrization respectively.

2. The metric

Take a Minkowskian space-time $\mathcal{M}$ with signature $(-, -, -, +)$. Assume that it is pervaded by a null, geodetic and shear-free congruence ξ_i so that

$$\xi_i\xi^i = 0, \quad \xi^i{}_{,k}\xi^k = 0, \quad (\xi^i{}_{,k} + \xi_{k,n}\,\eta^{ni})\,\xi^k{}_{,i} - (\xi^i{}_{,i})^2 = 0 \tag{2.1}$$

η_{ik} being the metric for M, and a comma denoting ordinary derivative.

We use the geometrical framework developed earlier (Vaidya 1974) to obtain a real tetrad system in $\mathcal{M}$ appropriate to the congruence ξ_i. Let $(A_i, B_i, C_i, \lambda_i)$ be the four mutually orthogonal Galilean uniform vector fields which can always be found to pervade a Minkowskian space-time. λ_i is a time-like unit vector while A_i, B_i, C_i are space-like unit vectors. The four uniform vectors give rise to a rectangular cartesian frame with coordinates (x, y, z, t) for a field point P such that

$$x_{,i} = A_i, \quad y_{,i} = B_i, \quad Z_{,i} = C_i, \quad t_{,i} = \lambda_i.$$

Let us denote by $\mathcal{S}$ the flat 3-space at right angles to λ_i at the point P. Then if l_i is the projection of ξ_i on $\mathcal{S}$ at P, we shall take $\xi_i = \lambda_i - l_i$. In 3-flat $\mathcal{S}$ let l_i have the spherical angles α and β with respect to the triad A_i, B_i, C_i. We can now define an orthogonal triad $l_i, \bar{l}_i, \bar{m}_i$ as follows:

$$l_i = \cos\alpha\, C_i + \sin\alpha\, m_i$$

$$\bar{l}_i = -\sin\alpha\, C_i + \cos\alpha\, m_i$$

$$\bar{m}_i = -\sin\beta A_i + \cos\beta B_i$$

where

$$m_i = \cos\beta A_i + \sin\beta B_i$$

Consider now a Riemannian space $\mathcal{R}$ with the metric

$$g_{ik} = \eta_{ik} + H\xi_i\xi_k + 2J\xi_{(i}p_{k)} \tag{2.2}$$

H and J being scalar functions of the coordinates. η_{ik}, ξ_i and p_i are defined in $\mathcal{M}$, ξ_i and η_{ik} having been already defined above and p_i can be taken as a unit space-like vector lying in the 3-flat $\mathcal{S}$, without impairing in any way the generality of the metric (2.2).

$$p_i p^i = -1, \qquad p_i \lambda^i = 0.$$

It can be verified that the determinant g of the metric (2.2) is $-(1 + Jp)^2$ where $p = p_i \xi^i = \eta^{ik} \xi_i p_k$ and that the inverse metrix g^{ik} is given by

$$g^{ik} = \eta^{ik} - (H + J^2)(1 + Jp)^{-2} \xi^i \xi^k - 2J(1 + Jp)^{-1} \xi^{(i} p^{k)} \tag{2.3}$$

At this stage we shall specify the convention about raising and lowering suffixes. We shall take ξ^i and p^i of $\mathcal{M}$, by definition as contravariant vectors in $\mathfrak{R}$ and denote by ξ_i and p_i the vectors $\eta_{ik}\xi^k$ and $\eta_{ik}p^k$ even in $\mathfrak{R}$. As a matter of fact throughout the paper we shall use the convention that raising and lowering of suffixes is done by η^{ik} and η_{ik} whether we work in $\mathcal{M}$ or $\mathfrak{R}$ and that while working in $\mathfrak{R}$ the dependence on g_{ik} or g^{ik} will be explicitly written down.

It can now be verified that if covariant differentiation in $\mathfrak{R}$ is denoted by a semicolon,

$$\xi^i_{\ ;k} \xi^k = \xi^i_{\ ,k} \xi^k + \left\{ {i \atop k\ l} \right\} \xi^k \xi^l = \xi^i_{\ ,k} \xi^k + (1 + Jp)^{-1} (Jp)_{,k} \xi^k \xi^i$$

and

$$\xi^i_{\ ;i} = \xi^i_{\ ,i} + (1 + Jp)^{-1} (Jp)_{,k} \xi^k$$

so that if we take $(Jp)_{,k} \xi^k = 0$,

i.e.,

$$\xi^k (\partial | \partial x^k) [Jp_a \xi^a] = 0. \tag{2.4}$$

We shall find that if ξ^i is geodetic in $\mathcal{M}$, it is also geodetic in $\mathfrak{R}$ and that expansion θ of ξ^i is the same whether we calculate it in $\mathcal{M}$ or in $\mathfrak{R}$. We shall, in the remainder of this paper, assume (2.4) to hold good. It can now be verified that when (2.4) holds good,

$$(\xi^i_{\ ;k} + g_{mk} \xi^m_{\ ;n} g^{ni}) \xi^k_{\ ;i} - (\xi^i_{\ ;i})^2 = (\xi^i_{\ ,k} + \eta_{mk} \xi^m_{\ ,n} \eta^{ni}) \xi^k_{\ ,i} - (\xi^i_{\ ,i})^2$$

so that when we assume (2.4) to hold good, a geodetic, shear-free, null congruence in $\mathcal{M}$ will continue to have the same properties in $\mathfrak{R}$.

For the metric (2.2), it is not easy to write down Christoffel symbols directly and the simplification introduced by (2.4) is not of much help in the matter. But, the following results can be easily established, in view of (2.4)

$$\left\{ {i \atop k\ l} \right\} \xi^k \xi^l = 0, \qquad \left\{ {i \atop i\ l} \right\} \xi^l = 0, \qquad \left\{ {i \atop k\ l} \right\} \xi_i \xi^l = 0.$$

It is also possible to obtain explicit expressions for $\left\{ {i \atop k\ l} \right\} \xi_i$ and $\left\{ {i \atop k\ l} \right\} \xi^l$ at this stage. These expressions are recorded in Appendix 1. These explicit expressions are enough to show that

$$R_{ik} \xi^i \xi^k = \tfrac{1}{2} Jp (2 + Jp) \Omega^2 \tag{2.5}$$

where $\Omega^2 = (\xi_{i,k} - \xi_{k,i}) \xi^i_{\ ,n} \eta^{nk}$, so that Ω measures the twist of the congruence in $\mathcal{M}$. We see from (2.5) that $R_{ik} \xi^i \xi^k$ will vanish if anyone or more of the following hold good.

(i) $\Omega = 0$, (ii) $J = 0$, (iii) $p = 0$, (*iv*) $Jp = -2$.

We shall not consider the case $J = 0$ because in that case our metric reduces to the Kerr-Schild metric. The two cases (iii) and (iv) are not essentially different because both of them lead to the value -1 of the determinant of the metric, in one case $(-g)^{\frac{1}{2}}$ being $+1$ and in the other $(-g)^{\frac{1}{2}}$ being -1. We are thus left with the cases (i) and (iii). We shall distinguish the two cases by (*a*) $p=0$ (*b*) $p\neq 0$. Of course when $p\neq 0$ we must take $\Omega=0$.

3. A particular case

It is comparatively easier to work with the metric (2.2) where $p = p_i\xi^i = 0$. For, in that case the determinant g becomes -1 and so raising or lowering of indices of ξ^i by η_{ik} or g_{ik} produces the same result. Again as p_i is a unit vector in the 3-flat ϑ, and l_i is a unit vector in ϑ along the projection of ξ_i on ϑ, the relation $\xi^i p_i = 0$ implies $l^i p_i = 0$. One can therefore get an idea of the gravitational situation described by the metric (2.2) by choosing either $p_i = \bar{l}_i$ or $p_i = \bar{m}_i$. We shall choose $p_i = \bar{m}_i$. (A more general choice would be $p_i = \cos\psi \bar{l}_i + \sin\psi \bar{m}_i$ with $\psi = \psi(x, y, z, t)$. This will be treated in a subsequent paper).

In order to work out R_{ik} for this metric knowledge about relations between derivatives of ξ_i, l_i, $\bar{l}_i$, $\bar{m}_i$, α, β, θ, Ω will be necessary. We shall not reproduce all these known results which have been derived in detail ealrier (Vaidya 1972, 1973, 1974). However, for the sake of ready reference the results which are useful for the present investigation are recorded in Appendix 3.

It can now be verified that $R_{ik}\xi^i\bar{l}^k = 0$ would lead to two alternatives; either $J_{,i}\xi^i = 0$ or $\Omega = 0$ and if we further want $R_{ik}\xi^i \bar{m}^k$, to vanish, the only alternative available is $\Omega = 0$ along with an equation for J. Thus with $p_i = \bar{m}_i$, the necessary and sufficient condition for ξ_i to be an eigenvector of R_{ik} is $\Omega = 0$. The equation for J can be solved to give $J = M\theta^2 + N\theta^{-1}$, $M_{,i}\xi^i = 0$, $N_{,i}\xi^i = 0$. With $\Omega = 0$, we further impose a symmetry restriction. We shall be considering the field to be symmetric round the axis $\alpha = 0$, so that various functions are independent of the azimuthal coordinate β. In that case $\Omega = 0$ would imply that the geometrical parameter W of Appendix 2 will vanish. We can now solve the field equation $g^{ik}R_{ik} = \eta^{ik} R_{ik} = 0$ to get

$$H = -\tfrac{1}{4}M^2\theta^4 + K\theta^2 + 2L\theta - N^2\theta^{-2} \qquad K_{,i}\, \xi^i = 0,\ L_{,i}\xi^i = 0 \tag{3.1}$$

It can now be verified that with these H and J, $R_{ik}\, \xi^k = -\tfrac{1}{4}K\theta^4\xi_i$ so that if we wish to satisfy the field equation $R_{ik} = \sigma\xi_i\xi_k$ then we must have $R_{ik}\xi^k = 0$, *i.e.*, $K = 0$ and $R_{ik}\bar{l}^k = 0$, $R_{ik}\bar{m}^k = 0$. The case $K\neq 0$ corresponds to the electromagnetic field which we shall discuss in section 5. So up to this stage we have

$$g_{ik} = \eta_{ik} + H\xi_i\xi_k + 2J\xi_{(i}\bar{m}_{k)} \tag{3.2}$$

with

$$H = -\tfrac{1}{4}M^2\theta^4 + 2L\theta - N^2\theta^{-2} \tag{3.3}$$

$$J = M\theta^2 + N\theta^{-1} \tag{3.4}$$

and

$$R_{ik}\xi^k = 0, \qquad g^{ik}R_{ik} = 0.$$

In the restricted case $\Omega = 0$, with the symmetry assumption that all functions are independent of the azimuthal angle β, the retarded time u and the geometrical parameters V and W of Appendix 2 are related as follows:

$$W = 0, \qquad V\operatorname{cosec}\alpha = f(x), \qquad x = u - V\cot\alpha$$

f being an undetermined function of argument. Now a function $M(x, y, z, t)$ satisfying $M_{,i}\, \xi^i = 0$, can be regarded as a function of u and α only, so that in

(3.3) and (3.4) we may take $M = M(u, \alpha)$ and $N = N(u, \alpha)$, $L = L(u, \alpha)$ as undetermined functions of u and α.

The form of $R_{ik}\bar{l}^k$ is given in Appendix 4. On setting $R_{ik}\;\bar{l}^k = 0$, we get three equations

$$M = 0 \tag{3.5}$$

$$VN_u + NV_u + N\cot\alpha - N_\alpha = 0 \tag{3.6}$$

$$VL_u + 3LV_u - L_\alpha = 0 \tag{3.7}$$

$N_u = \partial N/\partial u$, etc. It can now be verified that with equations (3.5), (3·6) and (3.7), $R_{ik}\bar{m}^k$ identically vanishes.

Equations (3.6) and (3.7) can be solved to get

$$N = n(x)\sin\alpha\,(f'\cos\alpha + 1)^{-1} \tag{3·8}$$

$$L = m(x)\,(f'\cos\alpha + 1)^{-3} \tag{3.9}$$

n and m are undetermined functions of x and f' denotes df/dx. Finally, we get $R_{ik}\lambda^i\lambda^k = -\,\theta^2 L_u$ so that $R_{ik} = \sigma\xi_i\xi_k$, $\sigma = -\,\theta^2 L_u$.

4. Explicit line element

In order to write down the metric in an explicit form we use (x, α, β, r) as coordinates. x is already defined above as $x = u - V\cot\alpha$, α and β are spherical angles of l_i and r is defined as

$$r = -\,x + t = -\,u + V\cot\alpha + t.$$

The metric (2.2) with H and J defined by (3.3), (3.4), (3.8) and (3.9) can now be written in the full form

$$\begin{aligned} ds^2 = &-r^2(d\alpha^2 + \sin^2\alpha d\beta^2) + 2(f'\cos\alpha + 1)\;dxdr \\ &+ [1 - f'^2 + 4m(x)\,r^{-1}\,(f'\cos\;\alpha{+}1)^{-1} - \tfrac{1}{4}r^2\;\sin^2\alpha\,n^2\,(x)]\;dx^2 \\ &- 2rf'\sin\alpha\,dxd\alpha + r^2\sin^2\alpha\,n(x)\,dxd\beta \end{aligned} \tag{4.1}$$

Special cases: (i) when $f' = 0$, $V_u = 0$ and we get $V = k\sin\alpha$ where k is the constant of integration which by a Lorentz transformation can be made to vanish. So $V = 0$ and the metric (4.1) reduces to

$$\begin{aligned} ds^2 = &-r^2(d\alpha^2 + \sin^2\alpha\,d\beta^2) + 2dudr\,+ \\ &+ [1 + 4m(u)\,r^{-1} - \tfrac{1}{4}r^2\sin^2\alpha\,n^2(u)]\,du^2 - r^2\sin^2\alpha\,n(u)\,dud\beta. \end{aligned}$$

This metric is the transform of the solution obtained by Vaidya and Pandya (1966) and also of the well-known Robinson-Trautman metric (Robinson and Trautman 1962). Further, when $n = 0$ it reduces to the radiating star metric (Vaidya 1953).

(ii) If $f' =$ constant, it can be verified that the metric (4.1) is the transform of the metric of Vaidya and Pandya (1966) when the origin is given a uniform velocity f' along z-axis.

The general solution $f' \neq$ constant derived here represents the gravitational field when the source of the Robinson-Trautman field is given an arbitrary acceleration.

5. The electromagnetic case

It can now be verified that the metric (4.1) also represents the null solution of Einstein-Maxwell equations.

$$R_{ik} = -8\pi E_{ik}$$

where E_{ik} is the electromagnetic energy tensor corresponding to the four potential $\phi_i = q(x)(f' \cos\alpha + 1)^{-1} \xi_i$, $q(x)$ being an undetermined function of x. This of course follows from Robinson's theorem (1961) because in this case $\xi_i = (f' \cos\alpha + 1)\, x_{,i}$ so that all the conditions of that theorem are satisfied. The corresponding electromagnetic field is null.

The case of a non-null electromagnetic field can also be incorporated in the present scheme by taking K defined in section 3 above as not equal to zero. When $K \neq 0$ it is possible to determine the metric form (1.3) so as to satisfy the field equations,

$$R_{ik} = -8\pi E_{ik} + \sigma \xi_i \xi_k$$

E_{ik} being the electromagnetic energy tensor. Details of this case will be discussed elsewhere.

Appendix 1

$$\{{}_k{}^i{}_l\}\, \xi_i = (1 + Jp)^{-1} [Jp\xi_{(k,l)} + pJ,{}_{(k}\xi_{l)} + J\xi^a (p_{[a,k]}\, \xi_l + p_{[a,l]}\, \xi_k)$$
$$- (J_{,a}\xi^a)\, \xi_{(k}p_{l)} - \tfrac{1}{2}(H_{,a}\xi^a)\, \xi_k\xi_l]$$

$$\{{}_k{}^i{}_l\}\, \xi^l = (1 + Jp)^{-1} [\tfrac{1}{2}\{ H_{,a}\, \xi^a + J(J_{,a}\xi^a) + J^2 p_{a,b}\xi^a p^b + Jp(J_{,a}p^a)\}\, \xi_k$$
$$+ Jp_{(k,l)}\, \xi^l + \tfrac{1}{2}(J_{,a}\xi^a)\, p_k + \tfrac{1}{2}pJ_{,k} - J^2pp^l\, \xi_{[l,k]}]\, \xi^i + \tfrac{1}{2}(J_{,a}\xi^a)\, \xi_k p_u$$
$$+ pJ\eta^{nl}\, \xi_{[l,k]} + J\eta^{nl}\, p_{[l,a]}\, \xi^a\, \xi_k - \tfrac{1}{2}p\eta^{na}\, J_{,a}\xi_k$$

Appendix 2

The retarded time u and the geometrical parameters V and W are given by

$$u = -x \sin\alpha \cos\beta - y \sin\alpha \sin\beta - z\cos\alpha + t$$

$$V = x\cos\alpha\cos\beta + y\cos\alpha\sin\beta - z\sin\alpha$$

$$W = x\sin\beta - y\cos\beta$$

Appendix 3

The following results of null geometry of $\mathcal{M}$ in terms of the tetrad $(\lambda_i, l_i, \bar{l}_i, \bar{m}_i)$ given by Vaidya (1974) are reproduced here for ready reference. An occasional change of sign in some results is due to the fact that in Vaidya (1974) ξ_i is taken as $\lambda_i + l_i$, whereas in the present investigation we are taking the "outward moving radiation" $\xi_i = \lambda_i - l_i$

$$l_{i,k} = \bar{l}_i\alpha_{,k} + \bar{m}_i \sin\alpha\, \beta_{,k} = -\xi_{i,k}$$

$$m_{i,k} = \bar{m}_i\, \beta_{,k}$$

$$\bar{l}_{i,k} = -l_i\alpha_{,k} + \bar{m}_i \cos\alpha\, \beta_{,k}$$

$$\bar{m}_{i,k} = -m_i\beta_{,k}$$

$$\theta_{,i}\xi^i + \tfrac{1}{2}(\theta^2 - \Omega^2) = 0, \qquad \Omega_{,i}\xi^i + \Omega\theta = 0,$$
$$(\theta^2 + \Omega^2)_{,i}\,\xi^i + (\theta^2 + \Omega^2)\,\theta = 0,$$
$$a_{,i} = (\tfrac{1}{2}\theta)\,(\bar{l}_i - V_u\xi_i) - (\tfrac{1}{2}\Omega)\,(\bar{m}_i + W_u\xi_i)$$
$$\sin a\beta_{,i} = (\tfrac{1}{2}\Omega)\,(\bar{l}_i - V_u\xi_i) + (\tfrac{1}{2}\theta)\,(\bar{m}_i + W_u\xi_i)$$
$$\theta_{,i}\bar{l}^i - \Omega_{,i}\bar{m}^i = \Omega\,(\Omega V_u - \theta W_u)$$
$$\theta_{,i}\bar{m}^i + \Omega_{,i}\bar{l}^i = -\,\Omega\,(\theta V_u + \Omega W_u)$$
$$\eta^{ik}\,a_{,ik} + \tfrac{1}{4}(\theta^2 + \Omega^2)\cot a = 0, \qquad \eta^{ik}\,(\sin a\beta_{,i})_{,k} = 0$$
$$\eta^{ab}\,\xi^i_{\ ,ab} + \tfrac{1}{2}(\theta^2 + \Omega^2)\,l^i = 0$$

Appendix 4

$$\begin{aligned} R_{ik}\bar{l}^k = & -\tfrac{1}{2}\,[\theta^2(H + J^2) + \{\theta\,(H + J^2)\}_{,k}\xi^k]\,(\bar{l}_i - \tfrac{1}{2}V_u\xi_i) \\ & - [X^*\theta + X^*_{,a}\,\xi^a + \tfrac{1}{2}\,\theta\,(H_{,k}\xi^k)\,V_u - \tfrac{1}{8}\,\theta^2(H + J^2)\,V_u \\ & - \tfrac{1}{4}\,\theta\,(H_{,k}\bar{l}^k) + \tfrac{1}{4}\,J\theta V_u\,(J_{,k}\xi^k + \tfrac{1}{2}\,J\theta) - \tfrac{1}{2}\,J\theta Y^*]\,\xi_i \\ & - [Z^*\theta + Z^*_{,a}\xi^a - \tfrac{1}{4}(\tfrac{1}{2}\,J\theta - J_{,k}\xi^k)\,\theta V_u \\ & + \tfrac{1}{4}(\tfrac{1}{2}\,J\theta + J_{,k}\xi^k)\,\theta V_u]\,\bar{m}_i \end{aligned}$$

where

$$X^* = \tfrac{1}{2}H_{,a}\bar{l}^a + JY^*$$
$$Y^* = -\,\tfrac{1}{2}\,(\tfrac{1}{2}\,J\theta\cot a - J_{,a}\bar{l}^a)$$
$$Z^* = \tfrac{1}{2}\,(\tfrac{1}{2}\,J\theta\cot a + J_{,a}\bar{l}^a).$$

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