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The Kerr metric in cosmological background

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Abstract. A metric satisfying Einstein's equations is given which in the vicinity of the source reduces to the well-known Kerr metric and which at large distances reduces to the Robertson-Walker metric of a homogeneous cosmological model. The radius of the event horizon of the Kerr black hole in the cosmological background is found out.

Keywords. Kerr metric; Kerr black holes; general relativity; expanding universe.

1. Introduction

The metric given by Kerr (1963) is recognised as the metric describing space-time exterior to a finite rotating body. Many attempts have been made to understand the source of this metric but the investigations are still inconclusive. On the other hand, the exterior metric itself has been very widely used in astrophysics in connection with the blackhole theories.

The standard Kerr metric is described in flat background. Because of the potential use of Kerr black holes in the story of the cosmological evolution of the universe, it would be interesting to find out what the Kerr metric looks like in the background of a homogeneous model of the universe rather than in the standard Minkowskian background. In the following, a metric is presented which in the vicinity of the source reduces to Kerr metric and which in the absence of the source reduces to Robertson-Walker metric.

In a series of papers (Vaidya 1972, 1973, 1974), we have developed a specific coordinate system to describe the general Kerr-Schild (1965) metric and in particular the Kerr metric. This coordinate system is quite useful in the present work because it uses the Minkowskian time t as the time-like coordinate for the Kerr metric.

Take the Minkowskian background metric in the form

$$ds^2 = dt^2 - dx^2 - dy^2 - dz^2. \tag{1}$$

In the physical 3-space given by $t = \text{const.}$, one can use rotating ellipsoidal polar coordinates (r, α, β) in place of (x, y, z) given by

$$x = \rho \cos \beta - w \sin \beta, \quad y = \rho \sin \beta + w \cos \beta, \tag{2}$$

$$\rho = r \sin \alpha, \quad z = r \cos \alpha, \quad w = a \sin \alpha,$$

so that $(x^2 + y^2)/(r^2 + a^2) + z^2/r^2 = 1$,

i.e., surfaces $r =$ constant are ellipsoids of revolution. Then (1) will transform to

$$ds^2 = dt^2 - (dr - a\sin^2\alpha d\beta)^2 - (r^2 + a^2\cos^2\alpha)(d\alpha^2 + \sin^2\alpha\, d\beta^2). \tag{3}$$

Using the retarded null coordinate $u = t - r$ in place of the space-like coordinate r, we write (3) in the form

$$\begin{aligned} ds^2 &= 2(du + a\sin^2\alpha d\beta)\, dt - (r^2 + a^2\cos^2\alpha)(d\alpha^2 + \sin^2\alpha\, d\beta^2) \\ &\quad - (du + a\sin^2\alpha d\beta)^2 \\ r &= t - u. \end{aligned} \tag{4}$$

The Minkowskian metric (4) forms the background of the Kerr metric written in the form (Vaidya 1974)

$$\begin{aligned} ds^2 &= 2(du + a\sin^2\alpha d\beta)\, dt - (r^2 + a^2\cos^2\alpha)(d\alpha^2 + \sin^2\alpha\, d\beta^2) \\ &\quad - [1 + 2mr/(r^2 + a^2\cos^2\alpha)](du + a\sin^2\alpha d\beta)^2. \end{aligned} \tag{5}$$

Thus the time coordinate t of the Kerr metric (5) is the Galilian time t of (1).

In the next section we begin with the static metric of Einstein's universe in place of (1), leave t unaltered, but transform (x, y, z) to rotating ellipsoidal polar coordinates and thus get the background cosmological metric. In the following section a Kerr-like solution of the field equations of Einstein will be given in this cosmological background and in the last section the ellipsoid of revolution which describes the event horizon in the cosmological background is presented.

2. The Robertson-Walker metric in rotating coordinates

Consider the metric of the Einstein universe in the form

$$ds^2 = dt^2 - dx^2 - dy^2 - dz^2 - \frac{(xdx + ydy + zdz)^2}{R^2 - (x^2 + y^2 + z^2)}. \tag{6}$$

Here again we shall leave the cosmic time t unaltered and carry out the following transformation from (x, y, z) to spheroidal polar coordinates (r, α, β):

$$\begin{aligned} &x = \rho\cos\beta - w\sin\beta,\ \ y = \rho\sin\beta + w\cos\beta \\ &\rho = R\sin(r/R)\sin\alpha,\ \ z = R\sin(r/R)\cos\alpha,\ \ w = a\cos(r/R)\sin\alpha. \end{aligned} \tag{7}$$

It will be seen that transformations (7) reduce to transformations (2) when $R \to \infty$. Under (7), the Einstein's metric (6) transforms to

$$\begin{aligned} ds^2 &= dt^2 - dr^2 + 2a\sin^2\alpha d\beta dr - M^2[1 - (a^2/R^2)\sin^2\alpha]^{-1} d\alpha^2 \\ &\quad - [M^2 + a^2\sin^2\alpha]\sin^2\alpha d\beta^2 \end{aligned} \tag{8}$$

$$M^2 = (R^2 - a^2)\sin^2(r/R) + a^2\cos^2\alpha. \tag{9}$$

Again using the null coordinate $u = t - r$ in place of r, we express (8) in the form

$$ds^2 = 2(du + a\sin^2\alpha d\beta)\,dt - M^2[(1 - a^2\sin^2\alpha / R^2)^{-1}\,d\alpha^2 + \sin^2\alpha d\beta^2]$$
$$- (du + a\sin^2\alpha d\beta)^2, \quad (10)$$
$$= ds_E^2 \text{ (say)}.$$

(10) is the metric of Einstein's universe.

Turning to Robertson-Walker metric, one writes it in the form

$$ds^2 = d\tau^2 - e^{g(\tau)}\left[dx^2 + dy^2 + dz^2 + \frac{(xdx + ydy + zdz)^2}{R^2 - (x^2 + y^2 + z^2)}\right] \quad (11)$$

One can replace the cosmic time τ by t such that

$\int e^{-\frac{1}{2}g(\tau)}\,d\tau = t$ and define the function $F(t)$ by $g(\tau) = 2F(t)$.

Then the Robertson-Walker metric becomes

$$ds^2 = e^{2F(t)}\left[dt^2 - dx^2 - dy^2 - dz^2 - \frac{(xdx + ydy + zdz)^2}{R^2 - (x^2 + y^2 + z^2)}\right].$$

But we have already transformed the quadratic form within the brackets on the right hand side by changing only the coordinates x, y, z in the physical space, leaving the time t unaltered; so that when the same transformation is now carried out in (11), the conformal factor $e^{2F(t)}$ will remain unaffected and the quadratic form within the brackets will transform to ds_E^2 of (10). Hence one can immediately write down the Robertson-Walker metric for expanding model of the universe in the new coordinates in the form

$$ds^2 = e^{2F(t)}\,ds_E^2 \quad (12)$$

where ds_E^2 stands for the metric (10) and $F(t)$ is an undetermined function of t.

In the next section we present Kerr-like solutions of Einstein's field equations in the homogeneous background given by (10) and (12).

3. Kerr-like solutions

We first take the Einstein's universe given by (10) as the background universe. The Kerr-metric in the cosmological background of Einstein's universe is given by

$$ds^2 = 2\,(du + a\sin^2\alpha d\beta)\,dt - (1 + 2\,m\mu)(du + a\sin^2\alpha d\beta)^2$$
$$- M^2[(1 - a^2\sin^2\alpha/R^2)^{-1}\,d\alpha^2 + \sin^2\alpha\, d\beta^2] \quad (13)$$

with M given by (9) $r = t - u$ $m =$ constant and

$$\mu = R\,\sin\,(r/R)\,\cos\,(r/R)/M^2. \quad (14)$$

If m is put equal to zero, the metric (13) reduces to the metric of the Einstein's universe (10). When $R \to \infty$, it becomes the Kerr metric (5). The material content of the field described by this metric can be found from Einstein's equations

$$R_{ik} - \tfrac{1}{2}g_{ik}\,R = -\,8\,\pi\,[(p + \rho)\,v_i\,v_k - pg_{ik}] - \Lambda g_{ik}.$$

Using the method of tetrads and the exterior calculus, one can see that the material content of the metric (13) can be described as a perfect fluid with the following pressure and density

$$8\pi p = -\frac{1}{R^2}(1 - 2m\mu) + \Lambda$$

$$8\pi\rho = \frac{3}{R^2}(1 - 2m\mu) - \Lambda.$$

The detailed calculations are lengthy but straightforward and so are not given here.

One can now get the Kerr metric in the background of the expanding universe given by (12). The metric will be conformal to the metric (13), with a little change in the multiplying factor of m. This metric turns out to be

$$ds^2 = e^{2F(t)}\,[2\,(du + a\sin^2\alpha d\beta)\,dt - (1 + 2m\mu e^{-2F})\;(du + a\sin^2\alpha d\beta)^2$$
$$- M^2[(1 - a^2\sin^2\alpha/R^2)^{-1}\;d\alpha^2 + \sin^2\alpha d\beta^2]] \tag{15}$$

μ being again defined by (14).

The metric (15) bears the same relation to the metric (12) of the expanding universe as the metric (13) bears to the metric (10) of Einstein's static universe. However, there is a qualitative difference. The resultant effect of the isotropic expansion of the cosmic fluid and the presence of the rotating source is that the cosmic fluid in the vicinity of the source exhibits anisotropy in the pressure which diminishes as we go away from the source.

At a point in the 3-space $t = \text{constant}$, if we choose 3 mutually orthogonal infinitesimal vectors θ^1, θ^2, θ^3, defined by

$$\theta^1 = e^F\,(du + a\sin^2\alpha d\beta)$$

$$\theta^2 = e^F\,M\,(1 - a^2\sin^2\alpha/R^2)^{-\frac{1}{2}}\,d\alpha$$

$$\theta^3 = e^F\,M\sin\alpha d\beta$$

then the pressures in the direction θ^2 and θ^3 are equal (say, p) while the pressure in the direction θ^1 is $q \neq p$. The mathematical expressions for the pressures p and q and the density ρ of the cosmic fluid at a point, as disturbed by the presence of the Kerr source are given in the appendix.

4. The event horizon

The coordinate system which we have been using, though very convenient for expressing the Kerr metric in the background of a homogeneous universe is not convenient to locate the event horizon. This is because in our system the null coordinate u (which is essentially a retarded time) replaces the space-like radial coordinate r. In order, therefore, to discuss the event horizon we shall have to replace u in the Kerr metric (5), in the Einstein-Kerr metric (13) and in the Robertson-Walker-Kerr metric (15) by r through $u = t - r$.

Let us take the Kerr metric (5). We first replace the coordinate u by r (*i.e.*, put $du = dt - dr$). Then since the event horizon is spheroid given by a constant value of r, in the transformed metric we put $dr = 0$ and thus get the metric on the spheroid $r = \text{constant}$ in the form

$$d\sigma^2 = -M^2 d\alpha^2 - \frac{B}{M^2}\sin^2\alpha\left(d\beta - \frac{2amr}{B}\,dt\right)^2 + \frac{\Delta M^2}{B}\,dt^2 \tag{16}$$

where

$$M^2 = r^2 + a^2\cos^2\alpha,$$

$$\Delta = r^2 - 2mr + a^2$$

$$B = (r^2 + a^2)^2 - \Delta a^2 \sin^2\alpha.$$

It will be seen that (16) agrees with the expression for $d\sigma^2$ obtained from the form of Kerr metric given by Boyer and Lindquist (1967), when we put $r =$ constant in it and so we get the event horizon of the Kerr metric as the spheroid with r parameter given by

$$\Delta = 0, \text{ i.e., } r^2 - 2mr + a^2 = 0, \text{ i.e., } r = m + (m^2 - a^2)^{\frac{1}{2}}. \tag{17}$$

We use the same procedure to find the spheroid representing the event horizon for the Kerr black hole immersed in the Einstein universe. We shall find that it is given by $r =$ constant satisfying the equation

$$R^2 \tan^2 (r/R) - 2m R \tan(r/R) + a^2 = 0$$

$$\text{i.e., } R\tan(r/R) = m + (m^2 - a^2)^{\frac{1}{2}}. \tag{18}$$

Since R is large compared to r, a comparison of (17) and (18) shows that the effect of the curvature of the universe on the radius of the event horizon (*i.e.*, on the r-parameter of the spheroid) is to reduce this radius by a fraction $\frac{1}{3}(r^2/R^2)$.

Again taking a Kerr black hole immersed in the expanding universe (15), we shall find that the radius of the event horizon is given by eq. (18) with m replaced by me^{-2F}, *i.e.*, by

$$R^2 \tan^2(r/R) - 2me^{-2F} R\tan(r/R) + a^2 = 0. \tag{19}$$

This event horizon exists if $a^2/m^2 < e^{-4F}$. If m_0 is the gravitational mass and J is the total angular momentum of the black hole then our parameters a and m are given by $m = Gm_0/c^2$, $a = J/cm_0$ and therefore the condition of existence of an event horizon becomes

$$CJ/Gm_0^2 < e^{-2F(t)} = e^{-g(\tau)}. \tag{20}$$

Now at the present epoch $e^{g(\tau)}$ is almost unity but for an expanding universe model, in the early stages of evolution $e^{g(\tau)} \sim \tau^{\frac{1}{4}}$ when τ is small. Thus $e^{g(\tau)}$ is much larger than unity at the earlier stages and so conditions at the early stages of evolution are much more favourable for the e inequality (20) to be satisfied. Again for large values of τ, *i.e.*, in far future, $e^{g(\tau)}$ will be much less than 1 and so conditions in the later stages of evolution are much less favourable for the restriction (20) to be satisfied. We may therefore draw a general conclusion that during the earlier stages of the evolution of an expanding universe the conditions were far more favourable for the existence of Kerr black holes than are at the present epoch and they will be much less favourable at a distant future epoch.

Appendix

We use the following notations

$$e^{-F} = T.$$

$$X = mR\sin(r/R)\cos(r/R)$$

$$T_t = \frac{dT}{dt}, \; X_r = \frac{dX}{dr}.$$

Define

$R_{(11)}, \; R_{(22)}, \; R_{(14)}, \; R_{(44)}$ by

$$R_{(11)} = -\frac{T^2}{2}\,[(1 - 2T^2 m\mu)^2(1/R^2) + (1 + 2T^2 m\mu)^2(T_{tt}/T)$$
$$- 2T^2 m\,\mu\,(T_t/T)(X_r/X)]$$

$$R_{(22)} = T^2[(1 - 2T^2 m\mu)(-2/R^2) + (1 + 2T^2 m\mu)(T_{tt}/T)$$
$$- (3 + 2T^2 m\mu)(T_t/T)^2 - T^2 m\mu\,(T_t/T)(X_r/X)]$$

$$R_{(14)} = T^2[(1 - 2T^2 m\mu)(1/R^2) - 2(1 + T^2 m\mu)(T_{tt}/T) + 3(T_t/T)^2]$$

$$R_{(44)} = -2T^2[(1/R^2) + (T_{tt}/T)].$$

Then

$$8\pi\rho = \frac{-1}{x} R_{(11)} - R_{(22)} - \Lambda$$

$$8\pi p = -R_{(14)} + \Lambda,$$

$$8\pi q = -\frac{1}{x} R_{(11)} + R_{(22)} + \Lambda,$$

where x is the positive root of

$$R_{(44)} x^2 - R_{(11)} = 0.$$

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