THE GRAVITATIONAL FIELD OF A RADIATING STAR

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1. Introduction

No non-static solutions, with physical significance, of Einstein's field equations are known outside the field of Cosmology. The field of a radiating mass presents a problem for which general relativity has, so far, not been able to provide a solution. Schwarzschild's external solution deals with the gravitational field of a cold dark body whose mass is constant. The application of this solution to describe the sun's gravitational field should only be regarded as approximate. Various attempts have been made to generalize Schwarzschild's solution in order to make it applicable to non-static masses, (Narlikar, 1936; Narlikar and Moghe, 1936).

While discussing this outstanding unsolved problem of general relativity, Professor Narlikar (1939) remarks:

> " If the principle of energy is to hold good, that is, if the combined energy of the matter and field is to be conserved, the system must be an isolated system surrounded by flat space-time. A spherical radiating mass would probably be surrounded by a finite and non-static envelope of radiation with radial symmetry. This would be surrounded by a radial field of gravitational energy becoming weaker and weaker as it runs away from the central body until at last the field is flat at infinity. It has yet to be seen whether and how this view of the distribution of energy is substantiated by the field equations of relativity."

We represent below the solution of the field equations which substantiates the views expressed above. We begin with the derivation of the energy tensor for the radiation envelope surrounding a star.

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* The treatment as given here is essentially different from that of Professor H. Mineur as it appears in *Ann. de l'Ecole Normale Superieure*, Ser. 3, 5, 1, 1933. Our attention was kindly drawn to it by Professor Mineur some years ago. — *V. V. N.* 23–4–1950.

2. Energy Tensor for a Directed Flow of Radiation

By the term "directed flow of radiation" we mean a distribution of electro-magnetic energy such that a local observer at any point of the region of space under consideration finds one and only one direction in which the radiant energy is flowing at the point. Using natural co-ordinates at the point of interest, we may take the components of the energy tensor as being given in terms of electric and magnetic field strengths E and H by the typical examples given by Tolman (1934).

$$T_0^{11} = -\tfrac{1}{2}(E_x^2 - E_y^2 - E_z^2 + H_x^2 - H_y^2 - H_z^2), \tag{2·1}$$

$$T_0^{12} = -(E_xE_y + H_xH_y), \tag{2·2}$$

$$T_0^{14} = (E_yH_z - E_zH_y), \tag{2·3}$$

$$T_0^{44} = \tfrac{1}{2}(E_x^2 + E_y^2 + E_z^2 + H_x^2 + H_y^2 + H_z^2). \tag{2·4}$$

The suffix 0 to a component of a tensor indicates that the component is evaluated in natural co-ordinates at the point of interest. Considering, for simplicity, that the axes of our natural co-ordinates are oriented in such a way that the flow of radiation at the point of interest is in the x-direction and further that the radiation is polarised with the electric vector parallel to y-direction, we shall find

$$E_x = E_z = H_x = H_y = 0; \quad E_y = H_z \tag{2·5}$$

and so the only surviving components of the tensor $T_0^{\mu\nu}$ would be

$$T_0^{11} = T_0^{44} = T_0^{14} = \tfrac{1}{2}(E_y^2 + H_z^2) = \rho, \tag{2·6}$$

ρ being the density of the radiant energy at the point.

Having obtained the components of $T^{\mu\nu}$ for one system of co-ordinates, we can find them in any other system by the rules of tensor transformation. For a general co-ordinate system with the line-element

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu, \tag{2·7}$$

the components of $T^{\mu\nu}$ will be given by

$$T^{\mu\nu} = \frac{\partial x^\mu}{\partial x^\alpha_0}\frac{\partial x^\nu}{\partial x^\beta_0} T_0^{\alpha\beta}. \tag{2·8}$$

On using (2·6) this yields

$$T^{\mu\nu} = \left[\frac{\partial x^\mu}{\partial x_0^1}\frac{\partial x^\nu}{\partial x_0^1} + \frac{\partial x^\mu}{\partial x_0^4}\frac{\partial x^\nu}{\partial x_0^4} + \frac{\partial x^\mu}{\partial x_0^1}\frac{\partial x^\nu}{\partial x_0^4} + \frac{\partial x^\mu}{\partial x_0^4}\frac{\partial x^\nu}{\partial x_0^1}\right]\rho. \tag{2·9}$$

As the radiant energy travels along null-geodesics

$$dx_0^1 = dx_0^4 = d\tau \text{ (say)}. \tag{2·10}$$

By (2·10) along the radiation flow we find

$$g_{\mu\nu}dx^\mu dx^\nu = 0. \tag{2·11}$$

Next we use (2·10) in

$$\frac{dx^\mu}{d\tau} = \frac{\partial x^\mu}{\partial x_0^{\alpha}} \frac{dx_0^{\alpha}}{d\tau}$$

and find

$$\frac{dx^\mu}{d\tau} = \frac{\partial x^\mu}{\partial x_0^1} + \frac{\partial x^\mu}{\partial x_0^4}. \tag{2·12}$$

With the help of (2·12), (2·9) finally reduces to

$$T^{\mu\nu} = \rho \frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}, \tag{2·13}$$

with

$$g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0. \tag{2·14}$$

Thus for our case of the outside field of a non-static mass the energy tensor is to be taken of the form

$$T^{\mu\nu} = \rho v^\mu v^\nu, \tag{2·15}$$

with

$$v_\mu v^\mu = 0; \quad (v^\mu)_\nu v^\nu = 0. \tag{2·16}$$

3. The Field Equations

A star of mass M and radius r_0 is supposed to start radiating at time t_0. As the star continues to radiate the zone of radiation increases in thickness, its outer surface at a later instant t_1 being $r = r_1$. For $r_0 \leqslant r \leqslant r_1$, $t_0 \leqslant t \leqslant t_1$ let the line-element be assumed to be of the form

$$ds^2 = -e^\lambda dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) + e^\nu dt^2,$$
$$\lambda = \lambda(r, t), \quad \nu = \nu(r, t). \tag{3·1}$$

For the nature of radiation we have found the energy tensor $T^{\mu\nu}$ of the form

$$T^{\mu\nu} = \rho v^\mu v^\nu, \tag{3·2}$$

ρ is the density of radiation and the lines of flow are null-geodesics:

$$v_\mu v^\mu = 0; \quad (v^\mu)_\nu v^\nu = 0. \tag{3·3}$$

Since $(T^{\mu\nu})_\nu = 0$, we have the analogue of the equation of continuity

$$(\rho v^\mu)_\mu = 0. \tag{3·4}$$

As the flow is to be radial, $v^2 = 0$, $v^3 = 0$ and

$$T_1^1 = \rho v_1 v^1, \quad T_4^4 = \rho v_4 v^4, \quad T_1^4 = \rho v_1 v^4, \quad T_2^2 = T_3^3 = 0. \tag{3·5}$$

Also $v_\mu v^\mu = 0$ simplifies to

$$- e^{\lambda} (v^1)^2 + e^{\nu} (v^4)^2 = 0. \tag{3·6}$$

With the usual expression for the components of $T^{\mu\nu}$ in terms of $g_{\mu\nu}$ and their derivatives, (3·5) gives the following three field equations:

(i) $$T_1^4 e^{(\nu-\lambda)/2} + T_4^4 = 0, \tag{3·7}$$

or $$e^{-\lambda}\left(\frac{\lambda'}{r} - \frac{1}{r^2}\right) + \frac{1}{r^2} + \frac{\dot{\lambda}}{r} e^{-(\lambda+\nu)/2} = 0; \tag{3·8}$$

(ii) $$T_1^1 + T_4^4 = 0, \tag{3·9}$$

or $$e^{-\lambda}\left(\frac{\lambda' - \nu'}{r} - \frac{2}{r^2}\right) + \frac{2}{r^2} = 0; \tag{3·10}$$

(iii) $$T_2^2 = 0, \tag{3·11}$$

or $$- e^{-\lambda}\left(\frac{\nu''}{2} + \frac{\nu'^2}{4} - \frac{\lambda'\nu'}{4} + \frac{\nu' - \lambda'}{2r}\right) + e^{-\nu}\left(\frac{\ddot{\lambda}}{2} + \frac{\dot{\lambda}^2}{4} - \frac{\dot{\lambda}\dot{\nu}}{4}\right) = 0. \tag{3·12}$$

Here and in what follows an overhead dash or dot indicates a differentiation with regard to r or t.

If the total energy is to be conserved, the line-element obtained by solving these equations must reduce to the static form

$$ds^2 = -\left(1 - \frac{2M}{r}\right)^{-1} dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\phi^2) + \left(1 - \frac{2M}{r}\right) dt^2 \tag{3·13}$$

at $r = r_0$, $t = t_0$ and for $r \geqslant r_1$ at $t = t_1$.

4. The Solution of the Field Equations

On putting $$e^{-\lambda} = 1 - \frac{2m}{r}, \quad m = m(r, t) \tag{4·1}$$

in the field equation (3·8) we find that it is equivalent to

$$e^{-\lambda/2} \frac{\partial m}{\partial r} + e^{-\nu/2} \frac{\partial m}{\partial t} = 0. \tag{4·2}$$

Using the operator

$$\frac{d}{d\tau} \equiv v^1 \frac{\partial}{\partial r} + v^4 \frac{\partial}{\partial t}, \tag{4·3}$$

we may express this as

$$\frac{dm}{d\tau} = 0. \tag{4·4}$$

From (4·2) we can express $e^{\nu/2}$ in terms of m:

$$e^{\nu/2}=-\frac{\dot{m}}{m'}\left(1-\frac{2m}{r}\right)^{-1/2}. \tag{4·5}$$

Now we can take the second field equation (3·10). On substituting the values of λ and ν from (4·1) and (4·5), we find that

$$\left(\frac{\dot{m}'}{\dot{m}}-\frac{m''}{m'}\right)\left(1-\frac{2m}{r}\right)=\frac{2m}{r^2}. \tag{4·6}$$

The first integral of the above equation is

$$m'\left(1-\frac{2m}{r}\right)=f(m), \tag{4·7}$$

$f(m)$ being an arbitrary function. (4·7) is the differential equation to be solved for m.

We now take the third field equation (3·12). We shall show that when λ and ν are given by (4·1), (4·5) together with the last differential equation (4·7), the equation (3·12) is automatically satisfied. The following is an identity holding between the components of the tensor T_μ^ν.

$$\frac{\partial}{\partial r}(T_1^1)+\frac{\partial}{\partial t}(T_1^4)-\frac{\nu'}{2}(T_4^4-T_1^1)+\frac{2}{r}(T_1^1-T_2^2)+T_1^4\left(\frac{\dot{\lambda}+\dot{\nu}}{2}\right)=0. \tag{4·8}$$

With the help of this identity and the two equations (3·7) and (3·9) the equation (3·11) can be transformed into

$$\frac{d}{d\tau}(r^2e^{-\lambda}\,T_4^4)=0. \tag{4·9}$$

Thus the third field equation is satisfied, *i.e.*, $T_2^2=0$ provided (4·9) is satisfied, *i.e.*, provided

$$\frac{d}{d\tau}\left\{m'\left(1-\frac{2m}{r}\right)\right\}=0, \tag{4·10}$$

i.e., provided $\frac{dm}{d\tau}=0$ when we use (4·7). And the last relation is already proved as (4·4) above.

Hence we have solved all the field equations and the final line-element describing the radiation envelope of a star is

$$ds^2=-\left(1-\frac{2m}{r}\right)^{-1}dr^2-r^2(d\theta^2+\sin^2\theta d\phi^2)+\frac{\dot{m}^2}{f^2}\left(1-\frac{2m}{r}\right)dt^2, \tag{4·11}$$

with $$m'\left(1-\frac{2m}{r}\right)=f(m),\ m=m(r,t)$$

for $$r_0 \leqslant r \leqslant r_1,\ t_0 \leqslant t \leqslant t_1.$$

The surviving components of energy tensor are

$$-T_1{}^1=T_4{}^4=\frac{m'}{4\pi r^2},\ T_1{}^4=\frac{m'^2}{4\pi \dot{m} r^2},\ T_4{}^1=-\frac{\dot{m}}{4\pi r^2}. \tag{4·12}$$

(Vaidya, 1943).

5. The Operator $\frac{d}{d\tau}$

The relation (4·4)

$$\frac{dm}{d\tau}=0 \tag{5·1}$$

is a type of relation peculiar to the field we are investigating. In this section we obtain some more relations of this type. On eliminating v^4 from $v_\mu v^\mu=0$ and $(v^\mu)_\nu v^\nu=0$ we find

$$\frac{\partial v^1}{\partial r}+\frac{\partial v^1}{\partial t}e^{(\lambda-\nu)/2}+v^1\left(\frac{\lambda'+\nu'}{2}+\dot{\lambda}\, e^{(\lambda-\nu)/2}\right)=0. \tag{5·2}$$

But the last term on the left hand side can be shown to vanish by using the field equations (3·8) and (3·10). Hence (5·2) becomes

$$\frac{dv^1}{d\tau}=0. \tag{5·3}$$

Another such relation can be obtained by starting with the equation of continuity (3·4)

$$(\rho\, v^\mu)_\mu=0$$

which when written out in full gives

$$\frac{\partial}{\partial x^\mu}(r^2\sin\theta\rho v^\mu\, e^{(\lambda+\nu)/2})=0. \tag{5·4}$$

When v^4 is eliminated again we find

$$\frac{\partial}{\partial r}(r^2\rho v^1)+e^{(\lambda-\nu)/2}\frac{\partial}{\partial t}(r^2\rho v^1)+(r^2\rho v^1)\left(\frac{\lambda'+\nu'}{2}+\dot{\lambda}e^{(\lambda-\nu)/2}\right)=0. \tag{5·5}$$

Like (5·2) this also reduces further to give

$$\frac{d}{d\tau}(r^2\rho v^1)=0. \tag{5·6}$$

(5·3) and (5·6) together can be used to obtain

$$\frac{d}{d\tau}(r^2\rho v^1v^1)=0 \quad \text{or} \quad \frac{d}{d\tau}(r^2T^{11})=0 \tag{5·7}$$

and $$\frac{d}{d\tau}(r^2\rho)=0 \tag{5·8}$$

the former of which would again imply $T_2^2=0$ as seen in the last section.

From the definition of the operator

$$\frac{d}{d\tau}\equiv v^1\frac{\partial}{\partial r}+v^4\frac{\partial}{\partial t},$$

it is clear that it differentiates following the lines of flow. Hence the relations (5·1), (5·3) and (5·8) show that m, v^1 and $r^2\rho$ are conserved along the lines of flow. Here we shall try to understand the phrase "conserved along a line of flow". At any time t, a spherical wave-front of radius $r=$ the radius of the star starts moving onwards. At every point of this wave-front the functions m, v^1, $r^2\rho$ have certain values at the start. The functions will retain these values at any point of this wave-front throughout the motion of the wave-front. The boundary $r=r_1$ at $t=t_1$ is a wave-front. At $t=t_0$, this wave-front started moving with radius $r=r_0$. At that time the value of m on the wave-front was M the total mass of the star. Our conservation result now asserts that on this first wave-front, the value of m will always be M. Thus at $t=t_1$ on the boundary of the radiation zone $r=r_1$ we find $m=\text{M}$.

We may add some simple mathematical properties of the conserved results. If

$$\frac{d\psi}{d\tau}=0,$$

in our co-ordinate system, it means

$$g^{\mu\nu}\frac{\partial\psi}{\partial x^\mu}\frac{\partial\psi}{\partial x^\nu}=0$$

which means that $\partial\psi/\partial x^\mu$ is a null-vector.

It follows that

$$\left(\frac{\partial\psi}{\partial x^\alpha}g^{\alpha\epsilon}\right)_\beta\frac{\partial\psi}{\partial x^\sigma}g^{\sigma\beta}=0. \tag{5·9}$$

For a line-element of the form (3·1), we have from (5·9) that if

$$\frac{d\psi}{d\tau}=0,\quad \frac{d}{d\tau}(e^{-\lambda}\psi')=0 \tag{5·10}$$

which will show that (4·10) is a consequence of (4·4).

The actual values of v^1 and v^4 may now be deduced. From (5·1) and (5·3) we have

$$\frac{\partial v^1}{\partial r} - \frac{m'}{\dot{m}}\frac{\partial v^1}{\partial t} = 0. \tag{5·11}$$

Hence

$$v^1 = \phi(m),\ v^4 = -\frac{m'}{\dot{m}}\phi(m). \tag{5·12}$$

$\phi(m)$ is now to be obtained by using any one of the equations

$$-T_1^{\ 1} = T_4^{\ 4} = \frac{m'}{4\pi r^2},\ T_1^{\ 4} = \frac{m'^2}{4\pi \dot{m} r^2},\ T^4_{\ 1} = -\frac{\dot{m}}{4\pi r^2}.$$

Thus $4\pi r^2 \rho v_4 v^1 = -\dot{m}$

or

$$\{\phi(m)\}^2 = \frac{m'}{4\pi r^2 \rho}\left(1 - \frac{2m}{r}\right)$$

or

$$v^1 = \left\{\frac{f(m)}{4\pi r^2 \rho}\right\}^{1/2},\ v^4 = -\frac{m'}{\dot{m}}\left\{\frac{f(m)}{4\pi r^2 \rho}\right\}^{1/2}. \tag{5·13}$$

6. The Boundary of the Radiation Zone

For the field of the radiation zone of a star we have two boundaries: (1) the boundary separating radiation from the material contents (or the internal) of the star and (2) the outer expanding boundary of the radiation zone separating it from 'empty' space beyond. We shall try to find the conditions at these boundaries which will ensure a unique solution.

The line-element under discussion is

$$ds^2 = -\left(1 - \frac{2m}{r}\right)^{-1} dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) + \frac{\dot{m}^2}{f^2}\left(1 - \frac{2m}{r}\right)dt^2, \tag{6·1}$$

$$m'\left(1 - \frac{2m}{r}\right) = f(m). \tag{6·2}$$

It contains two arbitrary functions. $f(m)$ is one of them. The other is an arbitrary function of t, say $\phi(t)$, which appears when we solve the partial differential equation (6·2) for m.

The expanding bounding surface of the radiation zone has been taken to be a sphere of radius $r = r_1$ at a time $t = t_1$. Obviously r_1 and t_1 are interconnected. We shall now say that this bounding surface is a sphere of variable radius $r = R(t)$ which would, of course, mean that $R(t_1) = r_1$.

Beyond the bounding sphere of the radiation zone the space is 'empty' and the line-element is

$$ds^2 = -\left(1 - \frac{2M}{r}\right)^{-1} dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\phi^2) + \left(1 - \frac{2M}{r}\right) dt^2. \quad (6 \cdot 3)$$

We show now, that the continuity of $g_{\mu\nu}$ at $r = R(t)$ will be sufficient, firstly, to locate the boundary at any time t, *i.e.*, to determine the function $R(t)$, secondly, to find out the arbitrary function $\phi(t)$ and thirdly to ensure that the total energy of the distribution is M.

Let

$$V(m, r) = \phi(t) \quad (6 \cdot 4)$$

be the general solution of the equation

$$\frac{\partial m}{\partial r}\left(1 - \frac{2m}{r}\right) = f(m)$$

the condition for which is

$$\frac{\partial V}{\partial r}\left(1 - \frac{2m}{r}\right) = -f(m)\frac{\partial V}{\partial m}. \quad (6 \cdot 5)$$

The value of $\dot{m}$ is given by

$$\frac{\partial V}{\partial m}\dot{m} = \dot{\phi}. \quad (6 \cdot 6)$$

Continuity of $g_{\mu\nu}$ gives, at $r = R(t)$

$$m = M$$

$$\dot{m} = -f(M).$$

(6·4) and (6·6) then give

$$V(M, R) = \phi(t) \quad (6 \cdot 7)$$

$$-f(M)\frac{\partial V}{\partial M} = \dot{\phi}. \quad (6 \cdot 8)$$

Here $\frac{\partial V}{\partial M}$ and $\frac{\partial V}{\partial R}$ denote the values of the derivatives $\frac{\partial V}{\partial m}$ and $\frac{\partial V}{\partial r}$ respectively at $m = M$, $r = R$; which is equivalent to saying that they denote the corresponding partial derivatives of V when the variables m and r in V are replaced by M and R. (6·7) and (6·8) are the equations to determine the two functions $R(t)$ and $\phi(t)$.

To eliminate $\phi(t)$ between (6·7) and (6·8) we differentiate (6·7) with respect to t, to get

$$\frac{\partial V}{\partial R}\dot{R} = \dot{\phi}$$

which with (6·8) gives

$$\frac{\partial V}{\partial R}\dot{R}=-f(M)\frac{\partial V}{\partial M}. \tag{6·9}$$

We shall now compare (6·9) with (6·5). (6·5) is a relation in m and r. When $m=M$, $r=R$, it becomes

$$\frac{\partial V}{\partial R}\left(1-\frac{2M}{R}\right)=-f(M)\frac{\partial V}{\partial M}. \tag{6·10}$$

Comparing (6·9) and (6·10) we find

$$\dot{R}=1-\frac{2M}{R}.$$

The general solution of this last differential equation is

$$R+2M\log(R-2M)-t=\text{a contsant},$$

which in our former notation, would mean that if the boundary of the radiation zone is $r=r_1$ at a time $t=t_1$,

$$r_1+2M\log(r_1-2M)-t_1=\text{a constant}. \tag{6·11}$$

The function $\phi(t)$ is now given by (6·7). It is interesting to note that the boundary radius $r=R(t)$ is determined independently of the nature of the function $V(m, r)$.

Before we proceed further let us study the condition $\dot{m}=-f(M)$ at $r=R(t)$. It says that, at all times, on the boundary of the radiation zone, $\dot{m}$ is a constant. But $\dot{m}$ is not conserved along a line of flow. Using the explanations of the last section, we say that the radiating star goes on emitting a series of wave-fronts. As $\dot{m}$ is not a conserved function, it is not constant for each one of these wave-fronts. But as m contains an arbitrary function of t, it is possible to select this function in such a way that $\dot{m}$ takes up a constant value on a particular wave-front. And this is what we have done by the condition (6·8). Note that the continuity of $g_{\mu\nu}$ at $r=R$ ensures that at the start, $r=r_0$, $t=t_0$, the line-eliment is again (6·3).

The conditions at the boundary $r=R(t)$ have left $f(m)$ undetermined. We expect that $f(m)$ will be determined by the conditions at the inner boundary of the star. It is clear that $f(m)$ is governed by the conditions in the interior of the star, different stellar models giving different forms of $f(m)$. That this will be the case, can be very easily seen from the definition of $f(m)$:

$$f(m)=m'\left(1-\frac{2m}{r}\right)$$

or approximately $f(m) = m'$ or again $f(m) = -\dot{m}$, because m' is almost equal to $-\dot{m}$. Thus $f(m)$ measures the luminosity of the star, at the Newtonian level of approximation.

Lastly we may now verify that the principle of conservation of energy holds good. The line-element (3·1) can be expressed in the form

$$ds^2 = -\{(dx)^2 + (dy)^2 + (dz)^2\} - \frac{e^\lambda - 1}{r^2}(xdx + ydy + zdz)^2 + e^\nu dt^2 \tag{6·12}$$

By using the well-known formulæ[1] the energy content of (6·12) is found to be

$$E = \lim_{r \to \infty} \{\tfrac{1}{2} r (e^\lambda - 1) e^{(\nu - \lambda)/2}\}. \tag{6·13}$$

Hence for all distributions for which the line-element (3·1) goes off continuously over some boundary to the Schwarzschild's form (6·3), the principle of conservation

$$E = M$$

holds good.

7. The Electro-Magnetic Field

The outside of a radiating star is the seat of electro-magnetic phenomena. So the field which we have considered above must be capable of being obtained from an electro-magnetic potential K_μ. That this is the case, has been already shown elsewhere (Narlikar and Vaidya, 1947, 1948).

8. Particular Solutions of the Equation $m'\left(1 - \frac{2m}{r}\right) = f(m)$

We shall here solve the equation

$$\frac{\partial m}{\partial r}\left(1 - \frac{2m}{r}\right) = f(m) \tag{8·1}$$

under different assumptions for $f(m)$.

Case (1): Let $f(m)$ be a constant.

$$f(m) = k < \tfrac{1}{8}.$$

m is given by the algebraic equation $(m - \alpha r)^A (m - \beta r)^B = \phi(t)$.

Here

$$\alpha, \beta = \tfrac{1}{4}\{1 \pm (1 - 8k)^{1/2}\};$$

$$A, B = \tfrac{1}{2}\{1 \mp (1 - 8k)^{-1/2}\}.$$

$\phi(t)$ is an arbitrary function of t.

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[1] Formula (91·1) on p. 232 of Relativity Thermodynamics and Cosmology by R. C. Tolman (1934) was used.

Next let $f(m)=k=\frac{1}{8}$.

m is given by the algebraic equation $r=4(m-r)\{\log(4m-r)+\phi(t)\}$.

Finally let $f(m)=k>\frac{1}{8}$. m is given by the equation

$$2\tan^{-1}\left\{\left(\frac{4m}{r}-1\right)(8k-1)^{-1/2}\right\}=(8k-1)^{\frac{1}{2}}\log(2m^2-mr+kr^2)+\phi(t).$$

Case (2): Some more particular solutions may be obtained by trying the following method. The condition that the total differential equation

$$m'dr+\dot{m}dt-dm=0 \tag{8·2}$$

should be exact is given by

$$\frac{dt}{0}=\frac{dr}{1}=\frac{(r-2m)\,dm}{rf(m)}=\frac{(r-2m)\,d(\dot{m})}{r\dot{m}\left\{\frac{df}{dm}+2f(r-2m)^{-1}\right\}}. \tag{8·3}$$

Solving (8·3) we try to obtain $\dot{m}$ as a function of m, r and t. Then this $\dot{m}$ and m' from (8·1) will make (8·2) exact. The solution of (8·2) will therefore give us the final solution for m. It can now be verified that the following is a solution of (8·3)

$$\dot{m}(r-2m)(m-\alpha r)^n(m-\beta r)^n\gamma=\phi(t) \tag{8·4}$$

where
$$(\alpha+\beta)2n=-1,\quad \alpha^n\beta^n\gamma\cdot f(m)=k^n \tag{8·5}$$

$$\alpha=\alpha(m),\ \beta=\beta(m),\ \gamma=\gamma(m),\ k=\text{a constant}, \tag{8·6}$$

$$\left(\frac{f(m)}{2}+\frac{4n+1}{n^2}\right)=k(3+2n)\Big(f(m)\,\gamma\Big)^{-n} \tag{8·7}$$

and $f(m)$ is to be taken as

$$\left(\frac{f(m)}{m}+\frac{1}{m}\,\frac{1+n}{2n^2}\right)^{4(1+n)}=c\left(4f(m)+\frac{1+2n}{2n}\right)^{1+2n}. \tag{8·8}$$

n, c are constants and $1+n\neq 0$, $3+2n\neq 0$. Various cases follow from this solution for different values of n.

Case (3): $1+2n=0$, $f(m)=cm-1$, c a constant. Then

$$m'\left(1-\frac{2m}{r}\right)=cm-1,$$

$$4c\dot{m}^2(r-2m)^2=\dot{\phi}^2(cm-1)^2(cr^2-4r+4n).$$

The complete solution for m is

$$2\left(r^2-\frac{4r}{c}+\frac{4m}{c}\right)^{1/2}+\frac{4}{c}\log\left\{\left(r-\frac{2}{c}\right)+\left(r^2-\frac{4r}{c}+\frac{4m}{c}\right)^{1/2}\right\}$$
$$-\frac{2}{c}\log(cm-1)=\phi(t).$$

Case (4): Let $5 + 6n = 0$ then $50f(m) = (625\, cm + 36)^{1/2}$, c a constant.

$$\frac{\dot{m}\,(r-2m)}{f(m)}\,X\,(m, r) = \dot{\phi},\ \left(\frac{3f}{8} + \frac{9}{200}\right)^{5/6} = X.\ \left(m^2 - \frac{3}{5}\,mr + \frac{3f}{8} + \frac{9}{200}\right)^{5/6}.$$

The final (m, r) relation is given by

$$V = \int\left(\frac{\partial V}{\partial m} + \frac{r-2m}{f}\,X\right)dm,$$

where the integrand on the right hand side will be a function of m only and

$$V = \int Xr\,dr + \phi\,(t).$$

In the last expression while performing the integration with respect to r, m is to be treated as a constant.

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Summary

A star of mass M and radius r_0 is supposed to start radiating at time t_0. The zone of radiation extends to $r = r_1$ at a later instant $t = t_1$. The energy tensor for the radiation zone, describing the directed flow of radiation, is evaluated and a relativistic line-element representing the field of radiation for $r_0 \leqslant r \leqslant r_1$ and corresponding $t_0 \leqslant t \leqslant t_1$ is obtained. It is shown that certain quantities m, v^1, $r^2\rho$, etc., are conserved in the field along a world-line of flow. At $r = r_0$, $t = t_0$ and at $r = r_1$, $t = t_1$, the line-element reduces to Schwarzschild's static form for a mass M. The conservation of energy is verified. The electro-magnetic potential K_μ of this field has already been obtained elsewhere.

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