PRAMANA
—journal of
physics

Vol. 48, No. 1,
January 1997
pp. 87–98

Dynamics of simple one-dimensional maps under perturbation

SOMDATTA SINHA and PARICHAY K DAS*
Centre for Cellular & Molecular Biology, Uppal Road, Hyderabad 500 007, India
*Indian Institute of Chemical Technology, Uppal Road, Hyderabad 500 007, India

Abstract. It is known that the one-dimensional discrete maps having single-humped nonlinear functions with the same order of maximum belong to a single class that shows the universal behaviour of a cascade of period-doubling bifurcations from stability to chaos with the change of parameters. This paper concerns studies of the dynamics exhibited by some of these simple one-dimensional maps under constant perturbations. We show that the "universality" in their dynamics breaks down under constant perturbations with the logistic map showing different dynamics compared to the other maps. Thus these maps can be classified into two types with respect to their response to constant perturbations. Unidimensional discrete maps are interchangeably used as models for specific processes in many disciplines due to the similarity in their dynamics. These results prove that the differences in their behaviour under perturbations need to be taken into consideration before using them for modelling any real process.

Keywords. Bifurcation; chaos; ecology; one-dimensional map; perturbation.

PACS Nos 05.45; 80.89.60; 03.20

1. Introduction

The field of dynamical systems and, especially the study of chaotic systems, has been hailed as one of the important breakthroughs in science in this century. Past three decades have seen an explosion of interest in the study of nonlinear dynamical systems in almost all disciplines, and chaotic and random behaviour of solutions of deterministic systems is now understood to be an inherent feature of many nonlinear systems. This has been possible due to infusion of interest and techniques from a variety of fields, from physics and chemistry to biology and social sciences as most natural systems seem to be complex and nonlinear. With renewed interest in low dimensional discrete dynamical systems in physics and other disciplines such as mathematical ecology and economics, it was shown that even very simple mathematical models can actually exhibit very complicated dynamics. In his seminal paper, May [1] urged that simple one-dimensional maps should be part of early mathematical education since on one hand it is simple to analyse, yet on the other hand, the range of complex dynamics exhibited by it can introduce the concepts of variability, chance and imbalance as perfectly natural behaviour expected from natural systems.

The general form of a one-dimensional discrete equation (map) is

$$X_{t+1} = f(X_t), \tag{1}$$

where X_{t+1} is a function of the discrete variable X at the earlier (i.e. tth) iteration and some parameters. Historically [1, 2], X_t has been studied as the population density of organisms (whose generations do not overlap) at the tth generation. If the function $f(X_t)$ is such that it has the propensity to increase monotonically when X is small and to decrease monotonically when it is large due to density-dependent processes, then it assumes a "hump"-like shape with a unique maximum.

More than two decades ago [1, 3–5] the numerous discrete models with single-hump growth functions existing in the ecological literature were placed under a general class of unidimensional maps based on the similarity in their dynamic behaviour with increasing growth rates. It was shown that all these maps display a "universal" bifurcation structure where, with the increase in a parameter (e.g., the growth rate), the system dynamics changes from stable equilibrium point to chaos through the period-doubling route, and the chaotic regime contains an ordered sequence of periodic windows [1, 2, 6]. This similarity in bifurcation structure of these one-dimensional maps initiated the use of these models interchangeably to describe experimental population data in ecology [7–12].

Though it is known that all one-hump maps having the same order of maximum belong to the same universality class [13], the precise conditions that determine if a single-hump nonlinear function would belong to the "universal" class are not clear [14]. Recent studies have shown that slight perturbations of a map can cause breakdown or distortion of universal behaviour [14–18]. These studies, from mathematical ecology again, describe the different responses of two maps (logistic and exponential) belonging to the same universality class when an additive disturbance (can be termed as perturbation or bias) is imposed on the nonlinear functions.

In this paper we study a few representative maps that are used commonly in theoretical studies and also to model real systems [1, 19, 20] under such perturbation and show the similarities and differences of their response. The major results are:

(a) the logistic map shows very different response to perturbations compared to other maps of the same universality class which have long 'tail' or 'plateaus' at high X; and

(b) the response dynamics depends on the details of the shape of the hump – positive perturbation enhances the effect of the 'tail', whereas, negative perturbation exposes the gradient of the upper part of the hump. Thus, the logistic map stands out among other maps whose 'plateaus' and asymmetric shapes give rise to a variety of interesting, unexpected and unusual dynamics. Since these latter type of maps are considered to be more realistic, one may anticipate many unexpected dynamics in nature.

The paper is organized as follows:

(i) Three commonly used one-dimensional maps and the types of perturbations used for this study are introduced.

(ii) The effect of the perturbations on the dynamics of these maps is shown.

(iii) The results are discussed with a view to delineate the differences in these maps.

(i) *One-dimensional maps*

We choose the following three one-dimensional equations with the specific single hump

Table 1. Properties of one dimensional single hump maps.

Maps	x-Domain	r-Range	Equilibrium points $f(x) = x$	Critical point $f'(x) = 0$
Logistic map	$0 \leq x \leq 1$	$1 < r \leq 4$	0 and $(r-1)/r$	$1/2$
Exponential map	$0 \leq x < \infty$	$0 < r$	0 and 1	$1/r$
Bellows map	$0 \leq x < \infty$	$1 < r$	0 and $(r-1)^{1/b}$	$\left(\frac{1}{b-1}\right)^{1/b}$

functions for our study:

(I) Logistic model:

$$X_{t+1} = rX_t(1 - X_t). \tag{2a}$$

(II) Exponential model:

$$X_{t+1} = X_t \exp[r(1 - X_t)]. \tag{2b}$$

(III) Bellows model:

$$X_{t+1} = \frac{rX_t}{(1 + X_t^b)}. \tag{2c}$$

Here 'r' and 'b' are parameters. These three models were chosen because they are most commonly used, both theoretically and experimentally [1, 3–5, 11, 12, 14, 19–22]. Though eq. (2a) is the simplest among the non-linear difference equation models, the other two equations (2b), (2c) have more secure provenance in the biological literature [1, 12, 14, 21]. Table 1 summarizes the detailed features of these models, and figures (1a–f) show the return maps (X_t versus X_{t+1} plot) and the bifurcation diagrams with increasing r for each model. Figure 1 clearly describes the similar "inverted U-shape" (hump) nature of $f(X)$–the nonlinear function–in the return maps, and the commonality in their dynamics of progression from stable point, to a hierarchy of stable cycles of period 2^n, to chaos with increasing r. These properties are observed in all models from this class of unidimensional single-hump equations [1, 2, 5].

Constant perturbation. Under an additive constant perturbation the one-dimensional equation (1) takes the form

$$X_{t+1} = f(X_t) + L = F(X_t) \tag{3}$$

where L is a parameter which can take both positive and negative values. The effect of a fixed positive and negative L on $f(X)$ is to move the humps shown in the return maps in figures (1a, 1c and 1e) above or below the X-axis respectively. As examples this is shown for the logistic model for $r = 3.9$ in figure 2a, and for the Bellows model for $r = 5$ and $b = 6$ in figure 2b. The continuous curves represent the unperturbed return maps (eqs (2a) and (2c)) and the dashed and dotted curves are for the perturbed maps ($+L$ and $-L$ respectively) in eq. (3). The exponential model is similar to the Bellows model.

It is clear from figure 2 that the perturbations change the intersection of the return maps (dashed curve and dotted curve) with the $X_{t+1} = X_t$ line (solid line), thus changing the

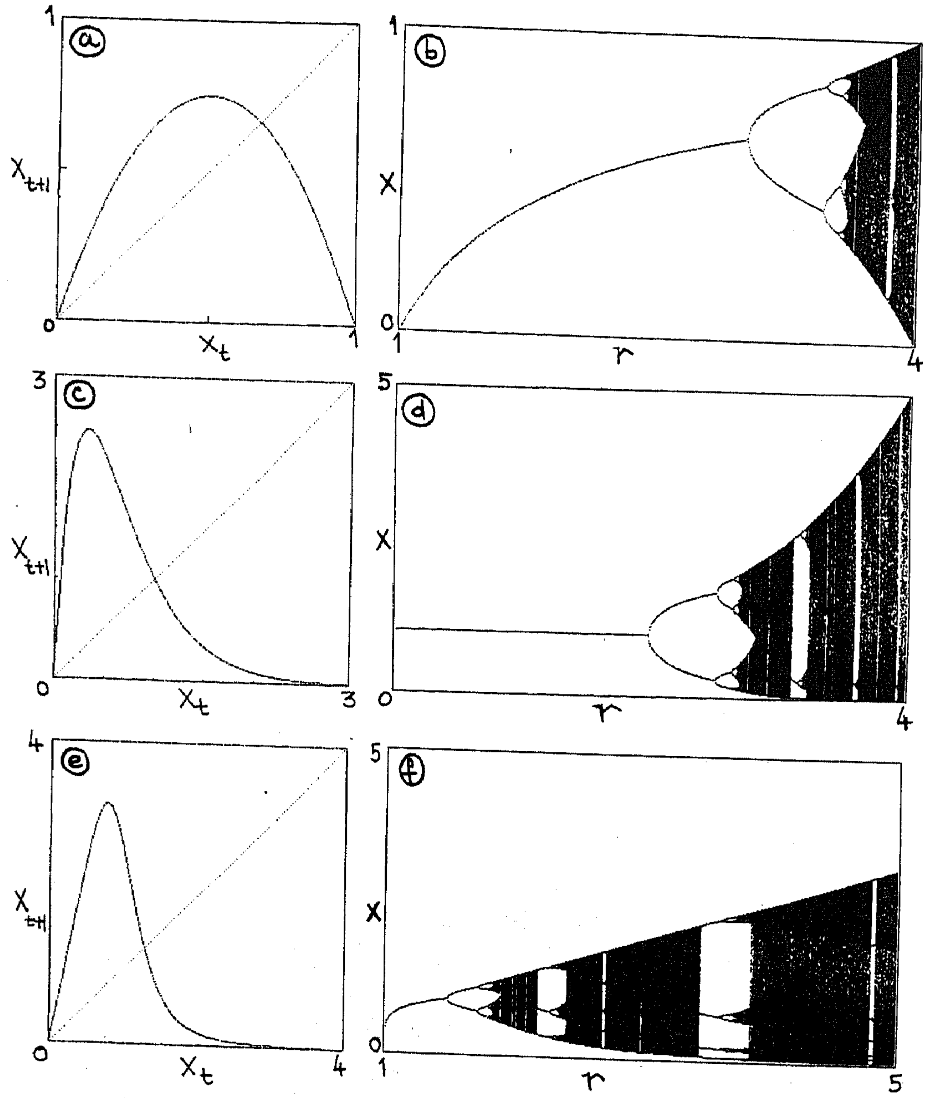


Figure 1. Return maps and bifurcation diagrams, respectively, for the (**a**, **b**) logistic, (**c**, **d**) exponential, and (**e**, **f**) Bellows maps. Unless otherwise stated the initial conditions are chosen to be X at which $f'(X) = 0$ in all figures.

number of positive equilibrium points and their stability. With negative L, there are two equilibrium points (x_1, x_2 in figure 2), and the first equilibrium point (x_1) is unstable leading all iterations below that critical value to zero. The dynamics exhibited by these maps depend on the gradient of the hump at the second equilibrium point (x_2). The other major change that negative perturbation introduces in the exponential and Bellows maps is that these two maps are no longer valid for all $X > 0$, thus introducing limits in both r and X within which the iterations will remain positive. In the case of the exponential and the Bellows maps (see figure 2b), positive L creates a higher 'floor' as the tail of the maps are elevated. This does not happen in the case of the logistic map and it continues to be valid only within a range of X. In this paper whenever X assumes zero or negative values it is termed as 'escape'.

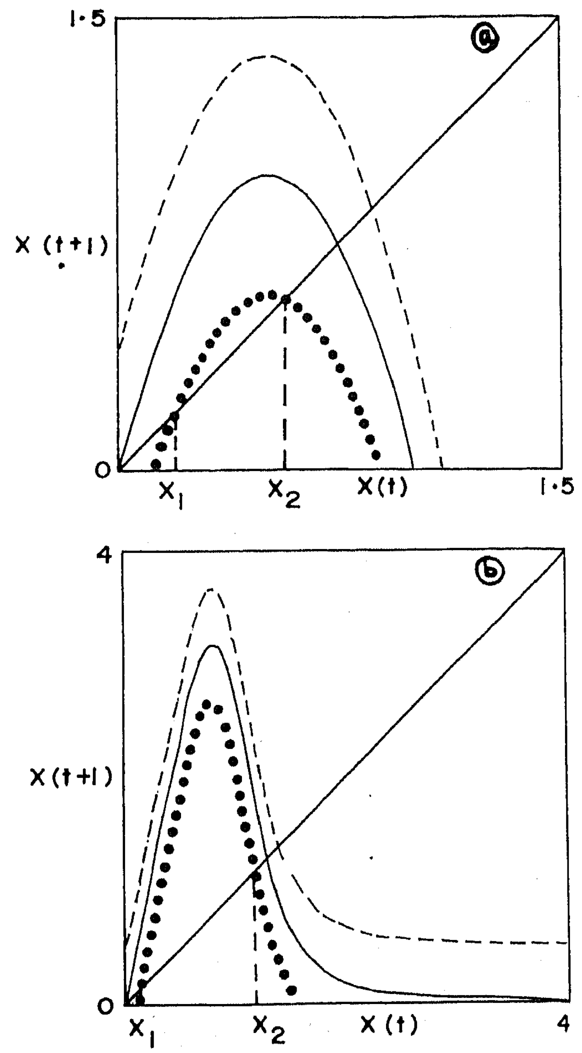


Figure 2. Return maps of the (**a**) logistic, and (**b**) Bellows maps. Continuous curve – unperturbed map, Dashed curve – maps with positive perturbation ($+L$). Dotted curve – maps with negative perturbation ($-L$).

(ii) *The dynamics of the perturbed maps*

The dynamics exhibited by the three models with positive and negative L are shown through bifurcation and stability diagrams in the $r-L$ parameter space ($r-L$ diagrams). Interestingly whatever be the magnitude of L (except when it pushes the system to "escape"), the dynamical system always returns to its unperturbed state on withdrawal of the perturbation. Therefore, we have studied the response of the maps to large ranges of L values. Figures 3, 4 and 5 are the bifurcation diagrams showing the dynamics of the perturbed maps with increasing L at two r values corresponding to stable and chaotic dynamics at the unperturbed state.

Logistic map. Figures 3a–d show the dynamics exhibited by the logistic map with increasing positive and negative perturbations for stable r ($r = 2.0$) and for chaotic r ($r = 3.9$) respectively. It is clear that positive perturbation induces (figure 3a) or

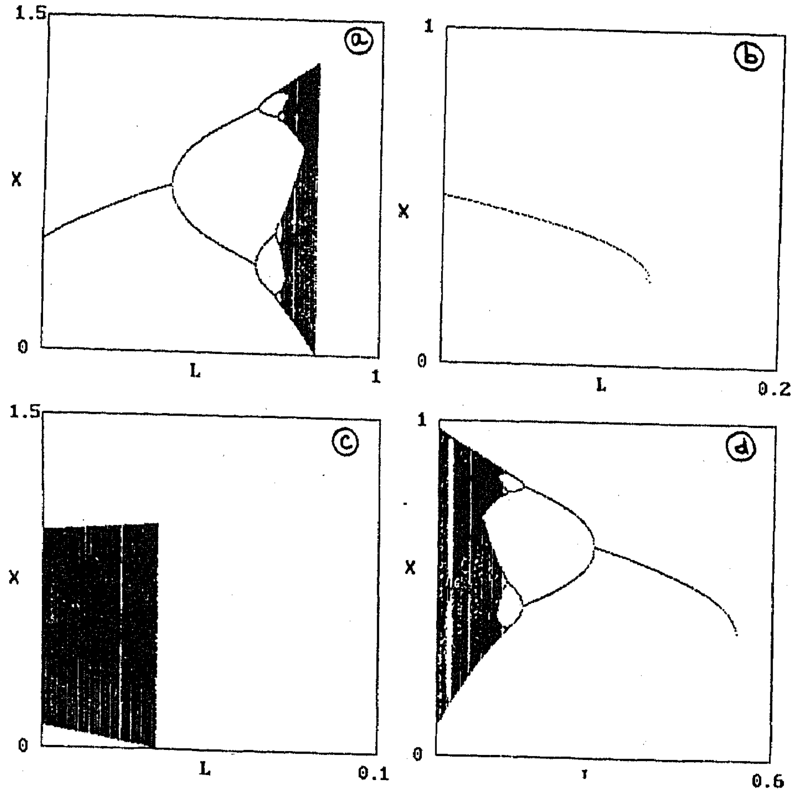


Figure 3. Bifurcation diagrams with L for the logistic map for stable $r = 2.0$. (**a**) positive L, (**b**) negative; and, L chaotic $r = 3.9$ (**c**) positive L, (**d**) negative L.

maintains (figure 3c) chaotic dynamics for all r, though 'escape' occurs for very small perturbations in a chaotic logistic map. On the other hand, negative perturbation induces stability in a chaotic map (figure 3d), though 'escape' ensues for stable r at relatively low negative L (figure 3b). Thus the final response of the logistic map to both types of perturbations is opposite and also independent of r.

Exponential map. Figures 4a–d show the effects of increasing positive and negative perturbations in the exponential map for stable $(r = 1.8)$ and chaotic r $(r = 3)$ respectively. In contrast to the logistic map (figure 3a), a stable exponential map remains stable under both types of perturbations (figures 4a and 4b), and a small amount of positive perturbation can induce stability in a chaotic map also (figure 4c). The response of the chaotic map to negative perturbation is however unusual. Though, like the logistic map in this case also increasing negative perturbation shows overall period reversal to stability from chaos, yet this map shows an intervening band of L values at which 'escape' occurs, but positive iterates recur for a range of higher L beyond which iterates 'escape' for all L. Interestingly, though in this map both types of perturbations finally render the map stable for all r, 'escape' is observed with $-L$ only for parameter (r) values corresponding to higher period $(P \geq 4)$ dynamics.

Bellows map. It is clear from figures 5a–d that the response of the bellows map is very similar to the exponential map for stable r $(r = 1.2)$ under negative L, and for chaotic r

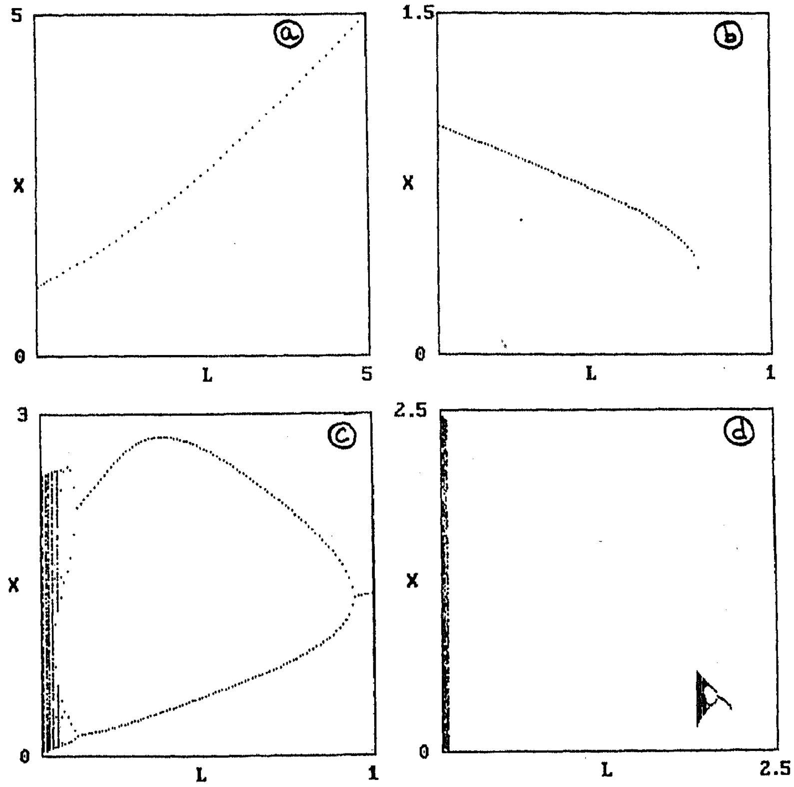


Figure 4. Bifurcation diagrams with L for the exponential map for stable $r = 1.8$. (**a**) positive L, (**b**) negative; and, L chaotic $r = 3.0$ (**c**) positive L, (**d**) negative L.

($r = 3.0$) under both positive and negative L. The stable-unstable-stable 'bubble' in figure 5a with increasing positive perturbation in a stable map is a reminder of the logistic map (see figure 3a) except that further period doubling bifurcations are curtailed here and period reversal gives rise to stable dynamics at higher L. It is also interesting to note that, like the logistic map (figure 3b), stable bellows map (figure 5b) also shows 'escape' for small negative perturbation. This map has an additional parameter 'b' which has a role in deciding the shape of the hump and hence the bifurcation scenario. We have taken $b = 6$ here which imposes strong nonlinearity in the maps, though similar behaviour is observed for lower b (> 2) also.

To get a complete picture of the effect of the constant perturbations on the dynamics of the maps for a range of values of the nonlinear parameter r, we have used a combined analytical and numerical approach to obtain the stability regions in the $r - L$ parameter space. Figure 6 shows three $r - L$ diagrams which provide a description of the different dynamical behaviour such as, stable fixed point ($P = 1$), stable oscillations of period 2 ($P = 2$), and complex oscillations of higher periods including chaos ($P \geq 4$) that are exhibited by the models for different values of r and L. The space outside the demarcated region constitutes the r and L values that lead to 'escape'. Since positive and negative L moves the hump vertically up and down, the range of X and r values for which the logistic map is valid also changes with L. For negative L, the exponential and Bellows maps both resemble the logistic map since they now do not have the extended tail and hence exist

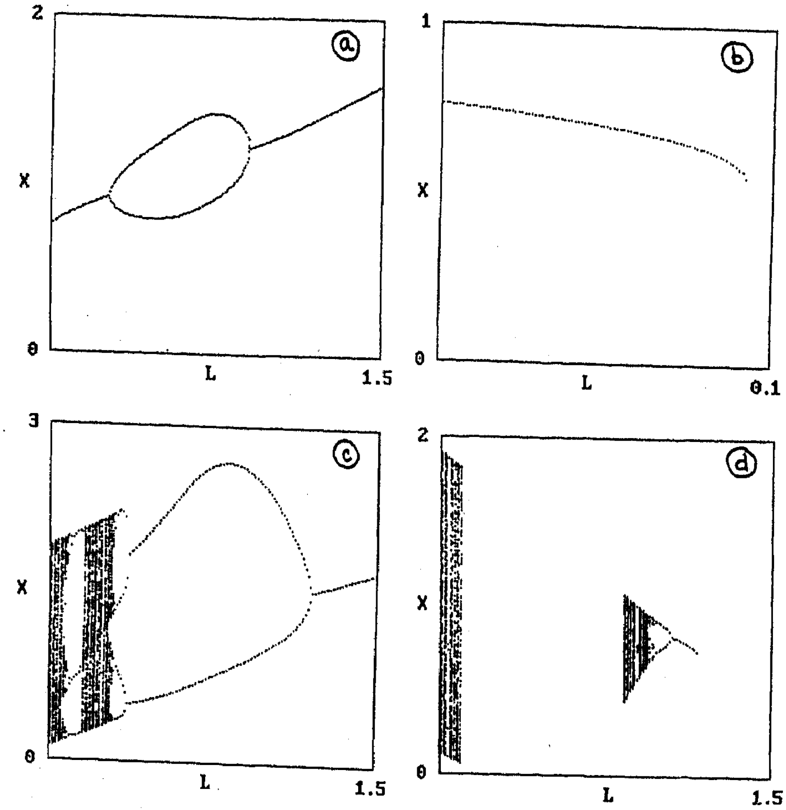


Figure 5. Bifurcation diagrams with L for the bellows map for stable $r = 1.2$. (**a**) positive L; and, (**b**) negative L; and, chaotic $r = 3.0$ (**c**) positive L, (**d**) negative L.

only within a certain range of r. Therefore they also show the 'escape' region for higher negative perturbations.

The boundaries in the $r - L$ plot in figure 6 can be obtained as follows:

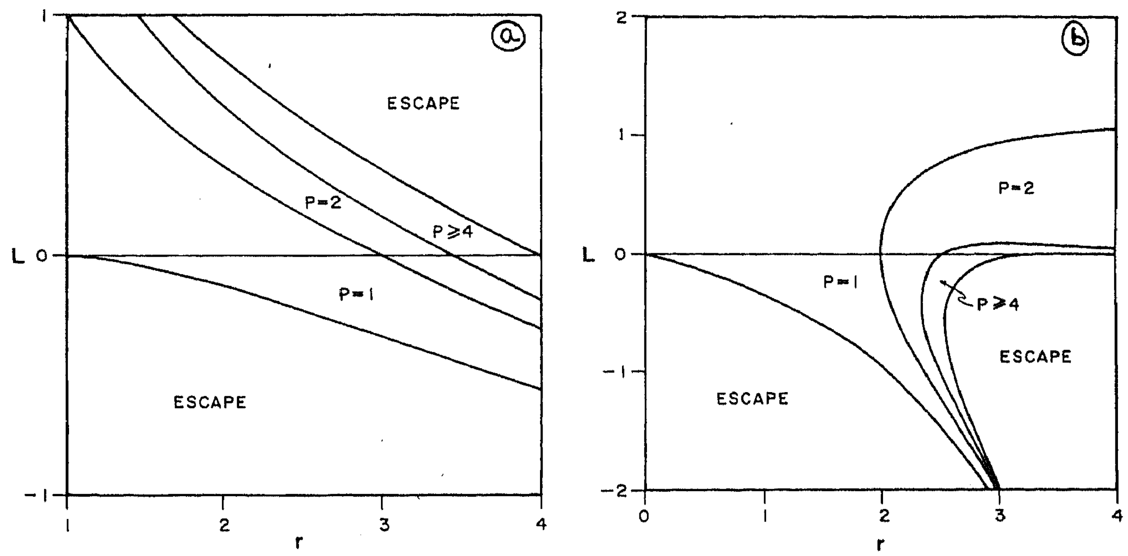
(a) The curve that forms the lower bound for all maps in the $r - L$ plot for negative L is obtained by finding the values of L for every r at which a saddle-node bifurcation takes place [20].

(b) The intervening 'escape' region observed in the exponential and bellows maps at higher r occurs for intermediate values of L at which, $F(X_{\max}) - x_1 < 0$ where x_1 is the first unstable equilibrium point (cf. figure 2b). Thus, the boundary in the $r - L$ parameter space can be obtained by considering values of r and L which satisfy the expression

$$F(X_{\max}) - x_1 = 0.$$

This condition is never satisfied in the case of the logistic map.

(c) The uppermost curve for $+L$ separating the 'escape' region is observed only for the logistic map (figure 6a). In the logistic map, there is one L for every r at which $F^2(X_{\max}) \leq 0$, i.e. the iterate falls outside the hump on the X-axis. Both exponential and Bellows maps do not have this region since they have 'tails' for high X (for $+L$) which force the iterates to be positive.



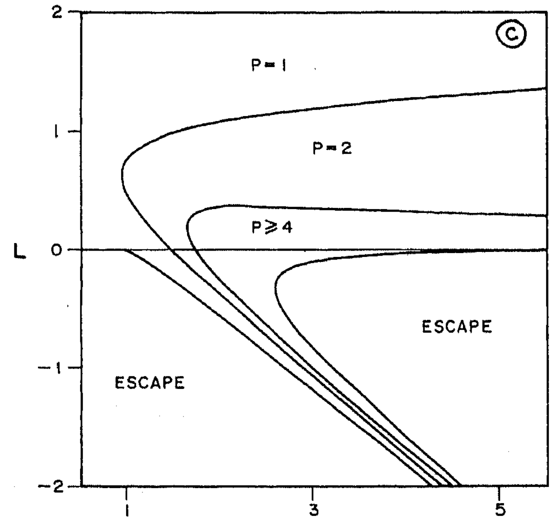


Figure 6. Stability diagrams in $r-L$ parameter space for the (**a**) logistic map, (**b**) exponential map, and (**c**) Bellows map.

(d) The boundaries of the first two period doubling bifurcations ($P=2$ and $P=4$) in figures 6a–c are obtained by solving the equations

$$F^n(X)=X \quad \text{and} \quad [F^n(X)]'=-1, \quad n=1,2.$$

Figure 6 clearly shows that the response of the logistic map to perturbations is very different when compared to that of the exponential and the Bellows maps. This feature clearly indicates that all these maps can not be treated at par.

Though from above the exponential and the Bellows maps seem to have very similar response to both positive and negative perturbations, comparison of figures 4a and 5a indicate finer differences between them. The 'bubble' of instability in the bellows map (figure 5a) is a feature which arises due to the detailed shape of $f(X)$ near the critical point. Figure 7 exemplifies this for the three maps. In this figure the continuous curve shows the slope of the map functions $f'(X)$ for all X starting after the critical point for the r value at which the first bifurcation from stability to oscillation occurs. The dotted curve represents the quantity $\{f(X)-X\}$.

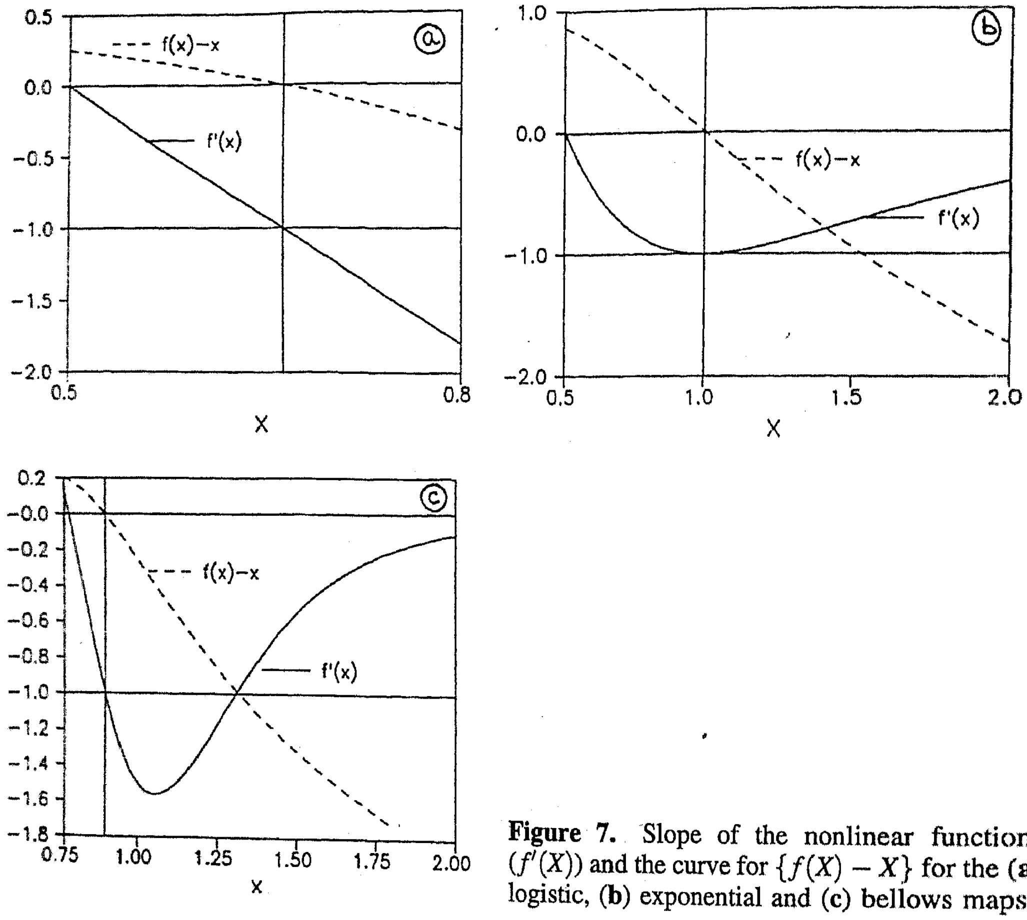


Figure 7. Slope of the nonlinear functions ($f'(X)$) and the curve for $\{f(X)-X\}$ for the **(a)** logistic, **(b)** exponential and **(c)** bellows maps.

It is clear from these figures that the equilibrium point x_2 (cf. figure 2b) where $\{f(X)-X\}=0$ occurs is also the point at which $f'(X)=-1$. The major point to be noted is that there are two values of X at which $f'(X)=-1$ for the Bellows map in figure 7c as opposed to the exponential map in figure 7b. Here (figure 7c), the equilibrium point is closer to the critical point of the map when the first bifurcation (at $r=1.5$) occurs. A lower value of r will obviously show stable dynamics in the unperturbed map and the equilibrium point will be on the left of the first intersection of $f'(X)=-1$. A constant positive perturbation (i.e. $+L$) to $f(X)$ at stable r will move the equilibrium point to the right on the X-axis. The system will then exhibit stable-unstable-stable 'bubble' as L is increased due to the presence of the two intersections of $f'(X)=-1$ in the Bellows map. Thus it is clear from this figure that there exist a range of stable r values in this map where positive perturbations can give rise to "bubble" like dynamics. Such a situation never arises for both the logistic and the exponential maps due to the shape of their humps (see figures 7a, b).

2. Discussion

The above-mentioned results clearly demonstrate that small perturbations can delineate the differences among the maps which are considered to belong to the same universality class and have similar bifurcation dynamics. They can now be classified based on their response to constant perturbations which is model-specific. The main features that play important roles in the manifestation of the differences in dynamics in these maps under perturbations are

1. The presence of the 'tails' in the nonlinear functions in the exponential and Bellows maps at higher X. This gives rise to higher 'floors' (plateaus which flattens out at high X) with positive perturbations, leading to period reversals to stability [14].
2. The detail shapes of the nonlinear functions near the maximum at medium values of X. These are highlighted in the form of 'bubbling' under positive perturbations, or, the minimum iterate $(F(X_{\max}))$ falling below the first unstable equilibrium point under negative perturbations.
3. The symmetric, purely concave-down structure of the hump in the logistic model does not allow any of the above processes to occur under constant additive perturbations. In fact the nonlinear parameter r in the logistic map can be easily scaled to include the effect of the perturbation (L). This can explain the changes in the dynamics with positive and negative perturbation in the logistic map in terms of increased or decreased effective r.

These simple one-dimensional discrete equations studied here have been extensively used for describing density-dependent population growth of organisms having non-overlapping generations [3, 5, 9, 23, 24]. In ecological terms, the constant perturbation can be considered as constant migration to/from a population. We have shown that these maps do not behave similarly under perturbations thus exposing the inherent non-equivalence among them. Our results have far-reaching implications not only in the study of one-dimensional non-linear discrete systems in general but also in the area of population dynamics and resource management studies. Thus one can say that though all unidimensional, discrete, single-hump maps show similar bifurcation dynamics [1, 2, 5], the exact nature of the non-linear functions plays an important role in their different dynamic response to simple perturbations.

Acknowledgement

This work is a part of the project funded by Department of Science and Technology, India. We gratefully acknowledge its support.

References

[1] R M May, *Nature* **261**, 459 (1976)
[2] M J Feigenbaum, *J. Stat. Phys.* **19**, 25 (1978)
[3] R M May, *Science* **186**, 645 (1974)
[4] R M May, *J. Theor. Biol.* **51**, 511 (1975)
[5] R M May and G F Oster, *Am. Natur.* **110**, 573 (1976)
[6] N Metropolis, M L Stein and P R Stein, *J. Combin. Theory* **15**, 25 (1973)
[7] C J Krebs, *Ecology: The experimental analysis of distribution and abundance*, 2nd edn (Harper International Edition, Harper and Row, New York, 1978)
[8] J A Yorke, N Nathanson, G Pianigiani and J Martin, *Am. J. Epidem.* **109**, 103 (1979)
[9] C W Clark, *Bioeconomic modeling and fisheries management* (Wiley, New York, 1985)

[10] J D Murray, *Mathematical Biology* (Springer-Verlag, Berlin, 1989)
[11] N V Joshi and M Gadgil, *Theor. Pop. Biol.* **40**, 211 (1991)
[12] H I McCallum, *J. Theor. Biol.* **154**, 277 (1992)
[13] G Ambika and K Babu Joseph, *Pramana – J. Phys.* **39**, 193 (1992)
[14] L Stone, *Nature* **365**, 617 (1993)
[15] R D Holt, in *Population biology – Lecture notes in biomathematics* edited by H I Freedman and C Strobeck (Springer Verlag, Berlin, 1983) vol. 52
[16] S Sinha and S Parthasarathy, *J. Bios.* **19**, 247 (1994)
[17] S Parthasarathy and S Sinha, *Phys. Rev.* **E51** 6239 (1995)
[18] S Sinha and S Parthasarathy, *Proc. Natl. Acad. Sci. (USA)* **93**, 1504 (1996)
[19] T S Bellows, *J. Anim. Ecol.* **50**, 139 (1981)
[20] R L Devaney, *An introduction to chaotic dynamical systems* (Menlo Park, Ca: The Benjamin/Cummings Pub. Co. Inc., 1986)
[21] M P Hassell, J H Lawton and R M May, *J. Anim. Ecol.* **45**, 471 (1976)
[22] K Masutani, *Bull. Math. Biol.* **55**, 1 (1993)
[23] J R Beddington and R M May, *Science* **197**, 463 (1977)
[24] W M Schaffer and M Kot, *J. Theor. Biol.* **112**, 403 (1985)