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INPUT-OUTPUT STABILITY OF A BROAD CLASS OF LINEAR TIME-INVARIANT MULTIVARIABLE SYSTEMS*

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Abstract. The input-output stability of closed loop control systems, which are not necessarily open loop stable, is considered. The type of stability considered is very broad, and encompasses bounded-input-bounded-output stability. Both continuous-time and discrete-time systems are considered. It is shown that the Desoer–Wu condition [2] is both necessary and sufficient for a large class of closed loop systems to be stable. The criterion is applicable to multivariable systems as well as to single-input-single-output systems.

1. Introduction. One of the best known and most widely used criteria for the stability of closed loop control systems is the Nyquist criterion [1]. The class of systems to which the Nyquist criterion can be applied contains feedback systems with unit feedback gain. The open loop system is assumed to be linear and time invariant, so that the transfer characteristics of the "gain box" are completely specified by a Laplace transform $\hat{g}(s)$. Originally, Nyquist considered scalar gain functions $\hat{g}(s)$ with the property that $\mathscr{L}^{-1}[\hat{g}(s)] = \hat{g}(\cdot) \in L_\infty \cap L_1$, and showed that the closed loop system is stable if and only if $1 + \hat{g}(s)$ had no zeros in the closed half-plane $\operatorname{Re} s \geqq 0$.

It is natural to attempt an extension of Nyquist's results to handle systems which are not open loop stable. Recently, Desoer and Wu [2], [3], and Baker and Vakharia [4] have obtained a sufficient condition for the stability of a large class of such control systems. To describe these results, it is convenient to introduce the sets $\mathscr{A}(\sigma)$ and $\hat{\mathscr{A}}(\sigma)$. The set $\mathscr{A}(\sigma)$ consists of generalized functions of the form

$$g(t) = g_0(t) + \sum_{i=1}^{\infty} g_i\delta(t - t_i), \tag{1}$$

where $g_0(t)$ is a measurable function, δ denotes the unit delta distribution, and it is further true that

$$\int_0^\infty |g_0(t)|\, e^{-\sigma t}\, dt < \infty, \tag{2}$$

$$\sum_{i=1}^{\infty} |g_i|\, e^{-\sigma t_i} < \infty, \tag{3}$$

where σ is a prespecified real number. The set $\mathscr{A}(\sigma)$ becomes a Banach algebra if the norm of an element $g(\cdot) \in \mathscr{A}(\sigma)$ is defined as

$$\|g(\cdot)\| = \int_0^\infty |g_0(t)|\, e^{-\sigma t}\, dt + \sum_{i=1}^{\infty} |g_i|\, e^{-\sigma t_i} \tag{4}$$

and the product of two elements in $\mathscr{A}(\sigma)$ is defined as their convolution. It is easy to verify that $\delta(t)$ is of unit norm and is the unit element for the Banach algebra

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$\mathscr{A}(\sigma)$. The set $\hat{\mathscr{A}}(\sigma)$ consists of Laplace transforms of the elements in $\mathscr{A}(\sigma)$. The set $\hat{\mathscr{A}}(\sigma)$ also becomes a Banach algebra if the norm of an element in $\hat{\mathscr{A}}(\sigma)$ is defined as the norm of the corresponding element in $\mathscr{A}(\sigma)$, and the product of two elements in $\hat{\mathscr{A}}(\sigma)$ is defined as their pointwise product. The set $\mathscr{A}^n(\sigma)$ ($\mathscr{A}^{n\times n}(\sigma)$) consists of all vector-valued (matrix-valued) generalized functions, each of whose components belongs to $\mathscr{A}(\sigma)$. The sets $\hat{\mathscr{A}}^n(\sigma)$ and $\hat{\mathscr{A}}^{n\times n}(\sigma)$ are defined analogously.

In [2], Desoer and Wu consider gain functions $\hat{g}(s)$ of the form $\hat{g}(s) = r/(s-\sigma) + \hat{g}_1(s)$, where r is an arbitrary constant and $\hat{g}_1(s) \in \hat{\mathscr{A}}(\sigma)$; in [4], Baker and Vakharia consider gain functions of the form $\hat{g}(s) = \hat{g}_0(s) + \hat{g}_1(s)$, where $\hat{g}_0(s)$ is a rational function and $\hat{g}_1(s) \in \hat{\mathscr{A}}(\sigma)$. In both cases, the authors show that the transfer function of the closed loop system, namely, $\hat{g}(s)/(1+\hat{g}(s))$ is a member of $\hat{\mathscr{A}}(\sigma)$ if the condition

$$\inf_{\operatorname{Re} s \geqq \sigma} |1 + \hat{g}(s)| > 0 \tag{5}$$

holds. In [3], Desoer and Wu consider n-input-n-output multivariable systems where the transfer function matrix $\hat{G}(s)$ is of the form $\hat{G}(s) = R/s + \hat{G}_1(s)$, where R is a type of $n \times n$ matrix and $G_1(s) \in \hat{\mathscr{A}}^{n\times n}(\sigma)$. They show that the transfer function matrix of the closed loop system is a member of $\hat{\mathscr{A}}^{n\times n}(\sigma)$ if the condition

$$\inf_{\operatorname{Re} s \geqq \sigma} |\det (I + \hat{G}(s))| > 0, \tag{6}$$

holds, where I is the $n \times n$ identity matrix. Finally, Desoer and Vidyasagar [5] show that the conditions (5) and (6) are both necessary and sufficient for closed loop stability. The conditions (5) and (6) are referred to here as the Desoer–Wu conditions.

While the results of [2]–[4] are very useful, it is natural to attempt to expand still further the class of gain functions $\hat{G}(s)$ which are allowed in the forward loop. In this paper, we study the input-output stability of closed loop systems where the forward path transfer function $\hat{G}(s)$ is of the form

$$\hat{G}(s) = \hat{P}(s)\hat{Q}^{-1}(s), \tag{7}$$

where $\hat{P}(s)$ and $\hat{Q}(s)$ belong to $\hat{\mathscr{A}}^{n\times n}(\sigma)$, and the ordered pair $(\det \hat{Q}(s), \det (\hat{P}(s) + \hat{Q}(s)))$ satisfies Condition N, defined below.

Condition N. An ordered pair of scalar-valued functions $(a(s), b(s))$ is said to satisfy Condition N if, whenever $\{s_k\}$ is a sequence in the closed half-plane $\operatorname{Re} s \geqq \sigma$ such that $\lim_{k\to\infty} a(s_k) = 0$, we have $\liminf_{k\to\infty} |b(s_k)| > 0$.

In the case that $a(s)$ and $b(s)$ are meromorphic functions, Condition N amounts to requiring that $a(s)$ and $b(s)$ have no common zeros in the half-plane $\operatorname{Re} s \geqq \sigma$, even at infinity (common poles are allowed, however). This class of gain functions contains the classes considered in [2]–[4] (this is proved later). We show in § 2 that the closed loop transfer function $\hat{H}(s) = \hat{G}(s)(I + \hat{G}(s))^{-1}$ is a member of $\hat{\mathscr{A}}^{n\times n}(\sigma)$ *if and only if* the condition

$$\inf_{\operatorname{Re} s \geqq \sigma} |\det (I + \hat{G}(s))| > 0 \tag{8}$$

is satisfied. This result is extended to discrete-time systems in § 3. Section 4 contains an illustrative example and § 5 comprises the concluding remarks.

2. Continuous-time systems. In this section, the main theorems of the paper are stated and proved.

THEOREM 1. *Suppose a function* $\hat{G}(s)$ *of the form*

$$\hat{G}(s) = \hat{P}(s)\hat{Q}^{-1}(s), \tag{9}$$

where $\hat{P}(s) \in \hat{\mathscr{A}}^{n \times n}(\sigma)$, $\hat{Q}(s) \in \hat{\mathscr{A}}^{n \times n}(\sigma)$, *and the ordered pair* (det $\hat{Q}(s)$, det $(\hat{P}(s) + \hat{Q}(s))$) *satisfies Condition* N, *is given. Then the function* $\hat{H}(s) = \hat{G}(s)(I + \hat{G}(s))^{-1}$ *is a member of* $\hat{\mathscr{A}}^{n \times n}(\sigma)$ *if and only if*

$$\inf_{\operatorname{Re} s \geqq \sigma} |\det (I + \hat{G}(s))| > 0. \tag{10}$$

Proof. Sufficiency of (10). Suppose (10) is satisfied. We wish to show that $\hat{H}(s) \in \hat{\mathscr{A}}^{n \times n}(\sigma)$. We achieve this in two steps: first, we show that (10) implies that

$$\inf_{\operatorname{Re} s \geqq \sigma} |\det (\hat{P}(s) + \hat{Q}(s))| > 0; \tag{11}$$

secondly, we show that (11) implies that $\hat{H}(s) \in \hat{\mathscr{A}}^{n \times n}(\sigma)$. To prove (11), we show that there exists no sequence $\{s_k\}$, with $\operatorname{Re} s_k \geqq \sigma$, such that $|\det (\hat{P}(s_k) + \hat{Q}(s_k))| \to 0$ as $k \to \infty$. Since $\hat{P}(s) + \hat{Q}(s) = (I + \hat{G}(s))\,\hat{Q}(s)$, it follows that if $\lim_{k \to \infty} |\det (\hat{P}(s_k) + \hat{Q}(s_k))| = 0$, then either $\liminf_{k \to \infty} |\det (I + \hat{G}(s_k))| = 0$ or $\liminf_{k \to \infty} |\det \hat{Q}(s_k)| = 0$. The first possibility is ruled out since (10) is assumed to hold. However, the second possibility can also be ruled out. If $\liminf_{k \to \infty} |\det \hat{Q}(s_k)| = 0$, then there is a subsequence $\{s_{k_i}\}$ of $\{s_k\}$ such that $\lim_{i \to \infty} |\det \hat{Q}(s_{k_i})| = 0$. By Condition N, $\liminf_{i \to \infty} |\det (\hat{P}(s_{k_i}) + \hat{Q}(s_{k_i}))| > 0$, which contradicts the earlier assumption that $|\det (\hat{P}(s_k) + \hat{Q}(s_k))| \to 0$. Hence (11) is proved.

Now we proceed to the second step. It is easy to see that

$$\hat{H}(s) = \hat{P}(s)(\hat{P}(s) + \hat{Q}(s))^{-1}. \tag{12}$$

Let $\hat{D}(s) = \hat{P}(s) + \hat{Q}(s)$; then $D(s) \in \hat{\mathscr{A}}^{n \times n}(\sigma)$ and $\hat{H}(s) = \hat{P}(s)\hat{D}^{-1}(s)$. The matrix $\hat{D}^{-1}(s)$ is given by the cofactor matrix of $\hat{D}(s)$ divided by det $\hat{D}(s)$. Since sums and products of elements in $\hat{\mathscr{A}}(\sigma)$ once again lie in $\hat{\mathscr{A}}(\sigma)$, the cofactor matrix of $\hat{D}(s)$ is a member of $\hat{\mathscr{A}}^{n \times n}(\sigma)$. Similarly det $\hat{D}(s) \in \hat{\mathscr{A}}(\sigma)$. So condition (11), together with the results of [6, p. 150], imply that $[1/\det \hat{D}(s)] \in \hat{\mathscr{A}}(\sigma)$, whence $\hat{D}^{-1}(s) \in \hat{\mathscr{A}}^{n \times n}(\sigma)$. This shows that $\hat{H}(s) \in \hat{\mathscr{A}}^{n \times n}(\sigma)$.

Necessity of (10). It is a direct consequence of [5, Theorem 1] that (10) is a necessary condition for $\hat{H}(s)$ to belong to $\hat{\mathscr{A}}^{n \times n}(\sigma)$. Hence the theorem is proved.

Even though Theorem 1 is stated for systems with unity feedback, it can be readily modified to handle systems with any nonsingular constant feedback. Theorem 2 below presents a necessary and sufficient condition for the input-output stability of a class of systems with nonconstant feedback.

THEOREM 2. *Suppose a function* $\hat{G}(s)$ *is of the form*

$$\hat{G}(s) = \hat{P}(s)\hat{Q}^{-1}(s), \tag{13}$$

where $\hat{P}(s) \in \hat{\mathscr{A}}^{n \times n}(\sigma)$ *and* $\hat{Q}(s) \in \hat{\mathscr{A}}^{n \times n}(\sigma)$. *Suppose* $\hat{K}(s) \in \hat{\mathscr{A}}^{n \times n}(\sigma)$, *and that the ordered pair* (det $\hat{Q}(s)$, det $(\hat{Q}(s) + \hat{K}(s)\hat{P}(s))$) *satisfies Condition* N. *Then the function*

$\hat{H}(s) = \hat{G}(s)(I + \hat{K}(s)\hat{G}(s))^{-1}$ *is an element of* $\hat{\mathscr{A}}^{n \times n}(\sigma)$ *if and only if*

$$\inf_{\operatorname{Re} s \geqq \sigma} |\det(I + \hat{K}(s)\hat{G}(s))| > 0. \tag{14}$$

Proof. Sufficiency of (14). Suppose (14) holds. Inequality (14) can be equivalently expressed as

$$\inf_{\operatorname{Re} s \geqq \sigma} |\det(I + \hat{K}(s)\hat{P}(s)\hat{Q}^{-1}(s))| > 0. \tag{15}$$

Since $(\det \hat{Q}(s), \det(\hat{Q}(s) + \hat{K}(s)\hat{P}(s)))$ satisfies Condition N, it follows from reasoning analogous to that in the proof of Theorem 1 that (15) implies

$$\inf_{\operatorname{Re} s \geqq \sigma} |\det(\hat{Q}(s) + \hat{K}(s)\hat{P}(s))| > 0. \tag{16}$$

Hence, from [6, p. 150], it follows that $(\hat{Q}(s) + \hat{K}(s)\hat{P}(s))^{-1} \in \hat{\mathscr{A}}^{n \times n}(\sigma)$. Since $\hat{H}(s)$ is also equal to $\hat{P}(s)(\hat{Q}(s) + \hat{K}(s)\hat{P}(s))^{-1}$, we see that $\hat{H}(s) \in \hat{\mathscr{A}}^{n \times n}(\sigma)$. This proves the sufficiency of (14).

Necessity of (14). This part of the proof closely follows that of [5, Theorem 1]. Suppose $\hat{H}(s) \in \hat{\mathscr{A}}^{n \times n}(\sigma)$, i.e., that $\hat{G}(s)(I + \hat{K}(s)\hat{G}(s))^{-1} \in \hat{\mathscr{A}}^{n \times n}(\sigma)$. Since $\hat{K}(s) \in \hat{\mathscr{A}}^{n \times n}(\sigma)$, this implies that $\hat{K}(s)\hat{H}(s) = \hat{K}(s)\hat{G}(s)(I + \hat{K}(s)\hat{G}(s))^{-1} \in \hat{\mathscr{A}}^{n \times n}(\sigma)$. From this it follows that $I - \hat{K}(s)\hat{H}(s) = (I + \hat{K}(s)\hat{G}(s))^{-1} \in \hat{\mathscr{A}}^{n \times n}(\sigma)$, whence $\det(I + \hat{K}(s)\hat{G}(s))^{-1} = 1/\det(I + \hat{K}(s)\hat{G}(s)) \in \hat{\mathscr{A}}(\sigma)$. It can be easily shown that any element in $\hat{\mathscr{A}}(\sigma)$ is bounded over the half-plane $\operatorname{Re} s \geqq \sigma$. Hence the reciprocal of any element in $\hat{\mathscr{A}}(\sigma)$ is bounded away from zero over the half-plane $\operatorname{Re} s \geqq \sigma$, which implies (14). The theorem is proved.

We now proceed to show that the class of systems considered in [2]–[4] is in fact contained in the class of systems covered by Theorems 1 and 2. Consider first of all the scalar case, and suppose $\hat{g}(s)$ is of the form considered by Baker and Vakharia in [4]; namely, let $\hat{g}(s) = \hat{g}_0(s) + \hat{n}(s)/\hat{d}(s)$, where $\hat{g}_0(s) \in \hat{\mathscr{A}}(\sigma)$, and $\hat{n}(s)$ and $\hat{d}(s)$ are polynomials in s with no common factors. To express $\hat{g}(s)$ in the form required by Theorems 1 and 2, let r be the larger of the degrees of $\hat{n}(s)$ and $\hat{d}(s)$, and define $\hat{p}_0(s) = \hat{n}(s)/(s + a)^r$, $\hat{q}_0(s) = \hat{d}(s)/(s + a)^r$, where a is any real number satisfying $a > -\sigma$. It is easy to see that $\hat{p}_0(s) \in \hat{\mathscr{A}}(\sigma)$, $\hat{q}_0(s) \in \hat{\mathscr{A}}(\sigma)$, and that $(\hat{q}_0(s), \hat{p}_0(s))$ satisfies Condition N. Also $\hat{g}(s) = \hat{g}_0(s) + \hat{p}_0(s)/\hat{q}_0(s) = (\hat{g}_0(s)\hat{q}_0(s) + \hat{p}_0(s))/\hat{q}_0(s)$, which is of the form $\hat{p}(s)/\hat{q}(s)$ with $\hat{p}(s) = \hat{g}_0(s)\hat{q}_0(s) + \hat{p}_0(s)$, and $\hat{q}(s) = \hat{q}_0(s)$. It only remains to show that $(\hat{q}(s), \hat{p}(s))$ satisfies Condition N. This is very easy to show, once it is observed that $\hat{g}_0(s)$ is bounded over the half-plane $\operatorname{Re} s \geqq \sigma$.

In [3], Desoer and Wu consider gain functions of the form $\hat{G}(s) = \hat{G}_0(s) + R/(s - \sigma)$, where $\hat{G}_0(s) \in \hat{\mathscr{A}}^{n \times n}(\sigma)$ and R is a nonsingular $n \times n$ matrix. Such a $\hat{G}(s)$ can be expressed as $\hat{P}(s)\hat{Q}^{-1}(s)$, where $\hat{Q}(s) = I \cdot (s - \sigma)/(s + a)$, $\hat{P}(s) = \hat{G}_0(s)Q(s) + R/(s + a)$, and a is any real number satisfying $a > -\sigma$. It is easy to show that the nonsingularity of R implies that $(\det \hat{Q}(s), \det(\hat{P}(s) + \hat{Q}(s)))$ satisfies Condition N. Hence the results contained in Theorems 1 and 2 are actually generalizations of the results in [2]–[4].

In fact, the class of functions studied in [2]–[4] is a proper subset of the class of functions covered by Theorems 1 and 2. For example, consider $\hat{g}(s) = 1/\cosh(s - \sigma)$. Since $\hat{g}(s)$ has an infinite number of poles in the half-plane $\operatorname{Re} s \geqq \sigma$ (namely at

$s = \sigma \pm j(2k + 1)\pi/2$, $k = 0, 1, \cdots, \pm\infty$), it does not fall into the class of functions studied in [2]–[4]. However, if we express $\hat{g}(s)$ as $2e^{-(s-\sigma)}/(1 + e^{-2(s-\sigma)})$, we see that it can be handled by Theorems 1 and 2.

Now we have a remark concerning the application of Theorems 1 and 2 to single-input-single-output systems. If $\hat{g}(s)$, $\hat{p}(s)$ and $\hat{q}(s)$ are all scalar-valued and members of $\hat{\mathscr{A}}(\sigma)$, then the requirement that $(\hat{q}(s), \hat{p}(s) + \hat{q}(s))$ satisfy Condition N can be simplified to: $\hat{p}(s)$ and $\hat{q}(s)$ have no common zeros in the half-plane $\operatorname{Re} s \geqq \sigma$, not even at infinity (note that $\hat{p}(s)$ and $\hat{q}(s)$, being members of $\hat{\mathscr{A}}(\sigma)$, have no singularities in the half-plane $\operatorname{Re} s \geqq \sigma$). Observe, however, that common poles and common zeros in the half-plane $\operatorname{Re} s < \sigma$ are permitted.

3. Discrete-time systems. It is quite clear that analogous versions of Theorems 1 and 2 can be proved for discrete-time systems. In the interest of brevity, the proofs are omitted. Theorem 3 below is a generalization of results due to Desoer and Wu [7], and Desoer and Lam [8].

In what follows, the input and output are both sequences of $n \times 1$ real vectors; l_1 represents the Banach space of absolutely summable sequences, and $l_1^{n \times n}$ is defined in the obvious way; $\mathscr{Z}$ and $\mathscr{Z}^{-1}$ denote z-transformation and inverse z-transformation, respectively; finally "$\tilde{}$" denotes z-transformed quantities.

THEOREM 3. *Given a function $\tilde{G}(z)$ of the form*

$$\tilde{G}(z) = \tilde{P}(z)\tilde{Q}^{-1}(z), \tag{17}$$

where $\{P_i\} = \mathscr{Z}^{-1}[\tilde{P}(z)]$, $\{Q_i\} = \mathscr{Z}^{-1}[\tilde{Q}(z)]$, $\{P_i\} \in l_1^{n \times n}$, $\{Q_i\} \in l_1^{n \times n}$, *suppose the ordered pair* $(\det \tilde{Q}(z), \det(\tilde{P}(z) + \tilde{Q}(z)))$ *satisfies Condition* N. *Then the inverse z-transform of the function $\tilde{H}(z) = \tilde{G}(z)(I + \tilde{G}(z))^{-1}$ is an element of $l_1^{n \times n}$ if and only if*

$$\inf_{|z| \geqq 1} |\det(I + \tilde{G}(z))| > 0. \tag{18}$$

THEOREM 4. *Given a function $\tilde{G}(z)$ of the form*

$$\tilde{G}(z) = \tilde{P}(z)\tilde{Q}^{-1}(z), \tag{19}$$

where $\{P_i\} \in l_1^{n \times n}$ and $\{Q_i\} \in l_1^{n \times n}$, suppose $\{K_i\} \in l_1^{n \times n}$ and that the ordered pair $(\det \tilde{Q}(z), \det(\tilde{Q}(z) + \tilde{K}(z)\tilde{P}(z)))$ *satisfies Condition* N. *Then the inverse z-transform of the function $\tilde{H}(z) = \tilde{G}(z)(I + \tilde{K}(z)\tilde{G}(z))^{-1}$ is an element of $l_1^{n \times n}$ if and only if*

$$\inf_{|z| \geqq 1} |\det(I + \tilde{K}(z)\tilde{G}(z))| > 0. \tag{20}$$

4. Example. Consider a feedback system with a forward gain

$$\hat{g}(s) = 1/\cosh s \tag{21}$$

and feedback of constant gain k. Note that $\hat{g}(s)$ is the A-parameter of a uniform LC transmission line, and can also be interpreted as the transfer function of a uniform vibrating string. The gain $\hat{g}(s)$ lies in the class of functions covered by Theorem 1, since $\hat{g}(s) = \hat{p}(s)/\hat{q}(s)$, where

$$\hat{p}(s) = 2e^{-s}, \qquad \hat{q}(s) = 1 + e^{-2s}. \tag{22}$$

Clearly $\hat{p}(s)$ and $\hat{q}(s)$ are members of $\hat{\mathscr{A}}(0)$, and further, whenever $\hat{q}(s) = 0$, we have $p(s) = \pm 2j$, so that Condition N is satisfied. However, $\hat{g}(s)$ has an infinite number of poles in the half-plane $\operatorname{Re} s \geqq 0$, and therefore cannot be handled by any other known methods.

The closed loop system with feedback gain $k \neq 0$ is stable if and only if (cf. Theorem 1).

$$\inf_{\operatorname{Re} s \geqq 0} |1 + k\hat{g}(s)| > 0. \tag{23}$$

We have $1 + k\hat{g}(s) = (1 + 2ke^{-s} + e^{-2s})/(1 + e^{-2s})$, so that if $k \neq 0$, then $1 + k\hat{g}(s) = 0$ whenever

$$e^{-s} = -k \pm \sqrt{k^2 - 1}. \tag{24}$$

The quantities $-k \pm \sqrt{k^2 - 1}$ are both real if $|k| \geqq 1$, and are complex if $0 < |k| < 1$. Now, if $0 < |k| < 1$, then both $-k \pm \sqrt{k^2 - 1}$ are complex numbers of magnitude 1, so that all solutions for s of (24) are purely imaginary. In this case (23) does not hold. If $|k| \geqq 1$, then at least one of the quantities $-k \pm \sqrt{k^2 - 1}$ is less than or equal to 1 in magnitude, so that some solutions of (24) satisfy $\operatorname{Re} s \geqq 0$. In this case once again (23) does not hold. So the closed loop system is unstable for all nonzero values of k.

5. Conclusions. In this paper, necessary and sufficient conditions have been presented for the input-output stability of a broad class of linear time-invariant systems. An interesting question for future researchers is: what class of Laplace transforms $\hat{g}(s)$ are covered by Theorems 1 and 2? Consider the scalar case, in the interest of simplicity. Then it is clear that any $\hat{g}(s)$ for which Theorem 1 applies necessarily satisfies the following conditions: any singularities of $\hat{g}(s)$ in the half-plane $\operatorname{Re} s \geqq \sigma$ must be poles, and these poles must be isolated. However, it is not clear to what extent these conditions are sufficient.

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Added in proof. In view of recent results due to Baker and Nasburg [9] and Desoer and Callier [10], Theorem 1 of the present paper can be strengthened to read as follows:

Let G be an $n \times n$ matrix whose elements are Laplace transformable distributions. Then the function $\hat{H}(s) = \hat{G}(s)(I + \hat{G}(s))^{-1}$ is an element of $\hat{\mathscr{A}}^{n\times n}(\sigma)$ if and only if there exist $\hat{P}(s), \hat{Q}(s) \in \hat{\mathscr{A}}^{n\times n}(\sigma)$ such that

(i) $(\det \hat{Q}(s), \det (\hat{P}(s) + \hat{Q}(s)))$ *satisfies Condition* N;

(ii) $\hat{G}(s) = \hat{P}(s)\hat{Q}^{-1}(s)$;

(iii) $\inf_{\operatorname{Re} s \geqq \sigma} |\det (I + \hat{G}(s))| > 0$.

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