

# Generalised Chern-Simons Theory of Composite Fermions in Bilayer Hall Systems

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## Abstract

We present a field theory of Jain's composite fermion model [1], as generalised to the bilayer quantum Hall systems. We define operators which create composite fermions and write the Hamiltonian exactly in terms of these operators. This is seen to be a complexified version of the familiar Chern Simons theory. In the mean-field approximation, the composite fermions feel a modified effective magnetic field exactly as happens in usual Chern Simons theories, and plateaus are predicted at the same values of filling factors as Lopez and Fradkin [2] and Halperin [3]. But unlike normal Chern Simons theories, we obtain all features of the first-quantised wavefunctions including its phase, modulus and correct gaussian factors at the mean field level. The familiar Jain relations for monolayers and the Halperin wavefunction for bilayers come out as special cases.

## I. INTRODUCTION

With the development of techniques for growing GaAs heterostructures containing two separated layers of two-dimensional electron gas, experimental work on the quantum Hall effect has been extended to such bilayer systems as well (see for example references [4], [5], [6], [7]). New plateaus in Hall conductivity have been observed at filling fractions not seen in single layers such as at  $\nu = \frac{1}{2}$ . On the theoretical front, a large body of work has already been done on bilayer systems. An extensive list of references to this literature has been given in the lucid review of this subject by Girvin and MacDonald [8] and in the paper by Moon et al [9]. In particular, as background and motivation for our present work let us recapitulate the following theoretical developments :

(i) A major step which helped in the study of quantum Hall effect was the proposal of elegant and yet very accurate first-quantised N-particle wavefunctions for the ground state and quasiparticle excitations at the Hall conductivity plateaus. This was pioneered by Laughlin for the mono layer case with his famous wavefunction for the filling fractions of  $\nu = \frac{1}{2m+1}$  [10]. For double layer systems (viewed as a two component system carrying a pseudo-spin layer index ) a generalisation of the Laughlin wavefunction was proposed long ago by Halperin [3] . The Halperin wavefunction  $\psi_{m_1, m_2, n}$  is labelled by three integers  $m_1, m_2$  and  $n$  – of which  $m_1$  and  $m_2$  must be odd – which determine the filling fraction  $\nu$ . For example, it was proposed by Yoshioka, McDonald and Girvin [11] that the plateau at  $\nu = \frac{1}{2}$  seen in the bilayer system corresponded to the wavefunction  $\psi_{3,3,1}$ .

(ii) For mono-layers , the fractional quantum Hall effect plateaus and their phenomenologically very successful wavefunctions were derived or justified from underlying composite particle formation postulates. Jain [1] presented a theory of flux-electron composite fermion formation. Jain's theory related Hall plateaus and their electron wavefunctions at fractional fillings to corresponding plateaus and wavefunctions of the composite fermions at integral fillings.

(iii) On a different front, the fractional effect was studied in field theoretic formulations.

Based on the observations of Girvin and MacDonald [12] that these systems seem to exhibit off-diagonal long range order, Zhang, Hansson and Kivelson [13] constructed a Chern Simons field theory of a Landau Ginsberg order parameter field for quantum Hall effect at fractional fillings  $\nu = \frac{1}{2m+1}$ . Their bosonic order parameter field corresponds to composites of electrons with an odd number of fluxons. A similar Chern Simons field theory, but with even -integer coupling, was studied by Lopez and Fradkin [14]. This corresponds to having composites of even number of fluxons with electrons and gives a field theoretic formulation of Jain's theory. Subsequently, Lopez and Fradkin [2] also extended their fermionic Chern Simons field theories to the bilayer case and predicted possible Hall plateaus for a large family of filling fractions in each layer. For related work on partially polarised electrons see Mandal and Ravishankar [15]. Ezawa and Iwasaki [16] also studied bilayer systems through a bosonic Chern Simons theory. They solved it using self-duality equations which hold in the absence of Coulomb interactions, which they treat perturbatively in the short distance limit.

(iii) All these papers in references [13] , [16], [14] and [2] extracted first quantised N-particle wavefunctions from their field theoretic ground states. But in the mean field approximation, only some aspects of the Laughlin wavefunction emerged. In the Zhang et al theory [13], the Landau Ginsberg field incorporates only the correct phases of the electronic correlations and not their modulus. The all important zeroes in the Jastrow correlation factors as well as the gaussian factors in the Laughlin wave functions emerge only upon including fluctuations about the mean field theory. In ref. [14] the modulus of the Laughlin wavefunction is derived by a very different and ingenious method starting from the low- $q^2$  limit of correlation functions. But such a result holds only in the long distance limit. The same is also true of the Lopez -Fradkin bilayer work [2] where again they have obtained the Halperin-Jain wavefunction for some cases, but only in the long distance limit. The wavefunctions obtained in the Ezawa-Iwasaki work hold only at short distances and are in the absence of Coulomb interactions.

In this paper we present a modified Chern Simons field theory for bilayer systems, which yields naturally at the mean field level Jain's model of composite fermion formation as

generalised to bilayer systems as well as the corresponding wavefunctions. (Jain's well known monolayer results also come out as a special case.) The present work is an adaptation to bilayer fermionic composite operators of our earlier work with Sondhi [17] where we had presented an exact field theory of the Read operator [18]. There we had employed a complexified version of the Chern Simons Landau Ginsberg field theory which enabled us to reproduce all the features of the Laughlin wavefunction already at the mean field level. We use an appropriate generalisation of the same method here. We explicitly construct operators that create bilayer composite fermions, using a non-unitary transformation acting on the parent electron field operator. Exact anti-commutation rules and an exact Hamiltonian are written in terms of these composite fermion operators. In the mean field approximation this Hamiltonian relates electrons at fractional fillings to composite fermions at integer fillings. These fractions and integers are seen to be related precisely by the formulae given by Lopez and Fradkin. Equations are also obtained akin to Jain's, but generalised to bilayers, which relate electron wavefunctions to composite fermion wavefunctions. The Halperin wavefunctions are obtained as specific examples.

In our theory, all aspects of these wavefunctions – the phases and moduli of their Jastrow correlations, and appropriate gaussian factors – emerge in the lowest order of the mean-field approximation. No short distance or long distance approximation is used nor any lowest Landau level restriction put in by hand. That our method of reference [17] can be used to generate composite fermions was also pointed out by Wu and Yu [19] for the monolayer case.

## II. COMPOSITE FERMION OPERATORS

Consider a double layer of two-dimensional electrons of mass  $\mu$  and charge  $e$ , placed in a uniform perpendicular magnetic field of strength  $B$  which corresponds to a vector potential in the symmetric gauge of  $\vec{A}(\vec{x}) = \frac{1}{2}B \hat{k} \times \vec{x}$ , where  $\hat{k}$  is the unit vector perpendicular to the plane. We take the electrons to be fully spin polarised along the  $B$  field and hence suppress spin for simplicity. Suppose their interaction potential is  $V(\vec{x} - \vec{x}')$  and the scalar potential

$A_0$  represents any uniform and/or random impurity electric fields in the problem. Let  $\Psi_\alpha(\vec{x})$  denote the bilayer electron quantum field with  $\alpha = 1, 2$  standing for the layer index. It obeys the equal-time anticommutation relations,

$$\begin{aligned}\{\Psi_\alpha(\vec{x}), \Psi_\beta(\vec{x}')\} &= \{\Psi_\alpha^\dagger(\vec{x}), \Psi_\beta^\dagger(\vec{x}')\} = 0 \\ \{\Psi_\alpha(\vec{x}), \Psi_\beta^\dagger(\vec{x}')\} &= \delta_{\alpha,\beta} \delta^{(2)}(\vec{x} - \vec{x}') .\end{aligned}\tag{2.1}$$

Clearly  $\Psi_\alpha$  is the full electron field and not just its lowest Landau level projected part. Note that unless explicitly specified, it is understood throughout this article that repeated indices  $\alpha, \beta$  etc. are not to be summed over. The second quantized Hamiltonian that describes our system is,

$$H = \int d^2x \sum_\alpha \left[ \Psi_\alpha^\dagger(\vec{x}) \left( \frac{-\hbar^2}{2\mu} \vec{D}^2 + eA_0 \right) \Psi_\alpha(\vec{x}) \right] + \sum_{\alpha,\beta} \frac{1}{2} \int \int d^2x d^2x' \delta\rho_\alpha(\vec{x}) V_{\alpha\beta}(\vec{x} - \vec{x}') \delta\rho_\beta(\vec{x}') .\tag{2.2}$$

Here,  $\vec{D} \equiv \vec{\nabla} - i\frac{e}{\hbar c}\vec{A}$  is the covariant derivative inclusive of the vector potential of the uniform magnetic field  $B$ .  $\rho_\alpha(\vec{x}) \equiv \Psi_\alpha^\dagger(\vec{x})\Psi_\alpha(\vec{x})$  is the electron density operator in the layer  $\alpha$  whose deviation from its mean value  $\bar{\rho}_\alpha$  is  $\delta\rho_\alpha(\vec{x})$ .

Next we define our operators for the composite fermion field  $\chi_\alpha(\vec{x})$  and its canonical conjugate field  $\Pi_\alpha(\vec{x})$ , by

$$\begin{aligned}\chi_\alpha(\vec{x}) &\equiv e^{-J_\alpha(\vec{x})}\Psi_\alpha(\vec{x}) \\ \Pi_\alpha(\vec{x}) &\equiv \Psi_\alpha^\dagger(\vec{x})e^{J_\alpha(\vec{x})},\end{aligned}\tag{2.3}$$

where,

$$J_\alpha(\vec{x}) \equiv \sum_\beta R_{\alpha\beta} \int d^2x' [\rho_\beta(\vec{x}') \log(z - z')] - r_\alpha \frac{|z|^2}{4l^2},\tag{2.4}$$

In the above equation  $r_\alpha$  and the  $2 \times 2$  matrix  $R_{\alpha\beta}$  consist of some numbers which will be fixed shortly,  $z \equiv x_1 + ix_2$  is the complex coordinate on the plane and  $l = \sqrt{\frac{\hbar c}{eB}}$  is the magnetic length. Notice that  $\chi_\alpha$  and  $\Pi_\alpha$  are not hermitian conjugates of each other since

$J_\alpha$  has both hermitian and anti-hermitian pieces. Nevertheless, as we now show, they form a pair of canonically conjugate Fermi fields, one in each layer.

Note that the only operator appearing in  $J_\alpha(\vec{x})$  is the electron density  $\rho_\beta(\vec{x}) = \Psi_\beta^\dagger(\vec{x})\Psi_\beta(\vec{x})$ , which obeys the commutation relation,

$$[\rho_\alpha(\vec{x}), \Psi_\beta(\vec{x}')] = -\delta_{\alpha\beta} \Psi_\alpha(\vec{x}) \delta^2(\vec{x} - \vec{x}') \quad (2.5)$$

Therefore the following identities follow :

$$\begin{aligned} e^{-J_\alpha(\vec{x})} \Psi_\beta(\vec{x}') &= (z - z')^{R_{\alpha\beta}} \Psi_\beta(\vec{x}') e^{-J_\alpha(\vec{x})} \\ \Psi_\beta^\dagger(\vec{x}') e^{-J_\alpha(\vec{x})} &= (z - z')^{R_{\alpha\beta}} e^{-J_\alpha(\vec{x})} \Psi_\beta^\dagger(\vec{x}') . \end{aligned} \quad (2.6)$$

Using these identities one can verify that

$$\chi_\alpha(\vec{x})\chi_\beta(\vec{x}') = (-1)^{R_{\alpha\beta}+1} (z - z')^{R_{\alpha\beta} - R_{\beta\alpha}} \chi_\beta(\vec{x}')\chi_\alpha(\vec{x}') \quad (2.7)$$

We can see that the explicit  $z-z'$  dependence in (2.7) drops out if  $R_{\alpha\beta}$  is chosen to be a symmetric matrix. Further, if its diagonal elements are even intergers ( $R_{11} = 2s_1$  and  $R_{22} = 2s_2$ ) then the field  $\chi$  in each layer anticommutes with itself , as desired of composite fermions. The off diagonal element  $R_{12} = R_{21}$  can be taken to be an integer  $n$ . Depending on whether  $n$  is odd (even), the fields at two different layers will commute (anticommute). In short the requirement that the composite fields defined in eq (2.3) be fermi fields restricts the matrix  $R_{\alpha\beta}$  to have the form

$$R_{\alpha\beta} = \begin{pmatrix} 2s_1 & n \\ n & 2s_2 \end{pmatrix} \quad (2.8)$$

exactly in accordance with ref [2].

The same choice of  $R_{\alpha\beta}$  also yields the canonical anti-commutator between  $\chi_\alpha$  and  $\Pi_\beta$ . We have, upon using the identities (2.6),

$$\begin{aligned} \chi_\alpha(\vec{x})\Pi_\beta(\vec{x}') &= e^{-J_\alpha(\vec{x})}\Psi_\alpha(\vec{x})\Psi_\beta^\dagger(\vec{x}')e^{J_\beta(\vec{x}')} \\ &= (-1)^{R_{\alpha\beta}+1} \Pi_\beta(\vec{x}')\chi_\alpha(\vec{x}) + \delta_{\alpha\beta}\delta^2(\vec{x} - \vec{x}') . \end{aligned} \quad (2.9)$$

Thus, since  $R_{11}$  and  $R_{22}$  have been even integers, the fields  $\chi_\alpha$  and  $\Pi_\alpha$  form a pair of mutually conjugate Fermi fields for each layer  $\alpha$ . This is despite the presence of non-unitary factors in their definition in Eq. (2.3). However, in contrast to standard fermi field theories, here  $\Pi_\alpha$  is not equal to  $\chi_\alpha^\dagger$ . Instead they obey the more complicated relation

$$\Pi_\alpha(\vec{x}) = \chi_\alpha^\dagger(\vec{x}) e^{J_\alpha(\vec{x}) + J_\alpha^\dagger(\vec{x})} . \quad (2.10)$$

This has to be borne in mind in doing manipulations with these composite fermion fields.

We will define the composite fermion density  $\rho_\alpha$  by

$$\rho_\alpha(\vec{x}) = \Pi_\alpha(\vec{x}) \chi_\alpha(\vec{x}) . \quad (2.11)$$

The corresponding number operator  $\hat{N}_\alpha \equiv \int d^2x \rho_\alpha$  satisfies

$$[ \hat{N}_\alpha, \Pi_\beta(\vec{x}) ] = \delta_{\alpha\beta} \Pi_\alpha(\vec{x}), \quad (2.12)$$

i.e. the operator  $\Pi_\alpha(\vec{x})$  creates one extra composite fermion in the  $\alpha^{th}$  layer. Notice that we have used the same symbol  $\rho_\alpha(\vec{x})$  for the this composite fermion density as we did for the original electron density since the definition given in (2.11) also satisfies

$$\rho_\alpha(\vec{x}) = \Psi_\alpha^\dagger(\vec{x}) \Psi_\alpha(\vec{x}) \quad (2.13)$$

It should be emphasized that our composite fermion operators  $\Pi_\alpha(\vec{x})$  and  $\chi_\alpha(\vec{x})$  are defined over the same space-time domain as the original electron field, as is natural in any field operator transformation in a field theory. This means that the area of the Hall sample is the same, whether we consider electrons or composite fermions. Hence, since the densities of both types of fermions has been shown to be the same, the total number of composite fermions is also the same as the number of the original fermions. Note that in obtaining the identities (2.6) we have used the expression (2.13) for the density in terms of the electron field and the associated commutator. If we were instead to use the expression (2.11) in terms the composite fermion operators and the associated commutators, we can analogously obtain the identities

$$\begin{aligned}
e^{-J_\alpha(\vec{x})} \chi_\beta(\vec{x}') &= (z - z')^{R_{\alpha\beta}} \chi_\beta(\vec{x}') e^{-J_\alpha(\vec{x})} \\
\Pi_\beta(\vec{x}') e^{-J_\alpha(\vec{x})} &= (z - z')^{R_{\alpha\beta}} e^{-J_\alpha(\vec{x})} \Pi_\beta(\vec{x}') .
\end{aligned} \tag{2.14}$$

Having defined the composite fermion fields and obtained their commutation relations, let us next rewrite the Hamiltonian (2.2) in terms of them. First consider the covariant derivative on the electron field. We have,

$$\begin{aligned}
\vec{D}\Psi_\alpha(x) &= \vec{D}(e^{J_\alpha(\vec{x})}\chi_\alpha(\vec{x})) \\
&= (\vec{\nabla} - i\frac{e}{\hbar c}\vec{A}(\vec{x})) (e^{J_\alpha(\vec{x})}\chi_\alpha(\vec{x})) \\
&= e^{J_\alpha(\vec{x})} (\vec{\nabla} - i\frac{e}{\hbar c}\vec{A}(\vec{x}) + \vec{\nabla}J_\alpha(\vec{x})) \chi_\alpha(\vec{x}) \\
&= e^{J_\alpha(\vec{x})}(\vec{D} - \frac{ie}{\hbar c}\vec{v}_\alpha(\vec{x})) \chi_\alpha(\vec{x})
\end{aligned} \tag{2.15}$$

where,

$$\vec{v}_\alpha(\vec{x}) \equiv i\frac{\hbar c}{e}\vec{\nabla}J_\alpha(\vec{x}) . \tag{2.16}$$

Hence,

$$D^2\Psi_\alpha = e^{J_\alpha} (\vec{D} - i\frac{e}{\hbar c}\vec{v}_\alpha)^2 \chi_\alpha \tag{2.17}$$

Inserting this into the starting Hamiltonian (2.2), and using Eqs. (2.3) and (2.11) we get,

$$\begin{aligned}
H &= \int d^2x \left[ \Pi_\alpha(\vec{x}) \left( \frac{-\hbar^2}{2\mu} (\vec{\nabla} - \frac{ie}{\hbar c}(\vec{A} + \vec{v}_\alpha))^2 + eA_0 \right) \chi_\alpha(\vec{x}) \right] \\
&+ \frac{1}{2} \int \int d^2x d^2x' \delta\rho_\alpha(\vec{x}) V_{\alpha\beta}(\vec{x} - \vec{x}') \delta\rho_\beta(\vec{x}')
\end{aligned} \tag{2.18}$$

This Hamiltonian in terms of the composite fermion fields defined in (2.3) is exactly equal to that of our original electron problem. No approximations have been made so far. Clearly this is very similar to a Chern Simons theory but it is more than just a direct generalisation to bilayers. As in normal Chern Simons theories, the vector field  $\vec{v}_\alpha$  appearing in (2.18) above is also constrained in terms of the density by Eq. (2.16), where  $J_\alpha(\vec{x})$  is defined in (2.4). But since this  $J_\alpha(\vec{x})$  involves more than just the phase of  $(z - z')$ , the field  $\vec{v}$  is not the bilayer statistical Chern-Simon gauge field used, for instance, in [2]. Because  $J_\alpha(\vec{x})$  has

real parts,  $\vec{v}_\alpha$  is a complex vector field. However,  $\vec{v}_\alpha$  will turn out to be simply related to Chern Simons fields.

Let us define a Chern-Simons field for each layer index  $\alpha$  by

$$\vec{a}_\alpha(\vec{x}) = \frac{-\hbar c}{e} \vec{\nabla}_x \int d^2 x' \rho_\alpha(\vec{x}') \text{Im} \log(z - z') , \quad (2.19)$$

or equivalently

$$b_\alpha(\vec{x}) \equiv \nabla \times \vec{a}_\alpha(\vec{x}) = -\phi_0 \rho_\alpha(\vec{x}) \quad (2.20)$$

where  $\phi_0 \equiv \frac{\hbar c}{e}$  is the flux quantum. Following reference [17], use the Cauchy-Riemann condition

$$\vec{\nabla}(\text{Re} \log z) = \vec{\nabla}(\text{Im} \log z) \times \hat{k} \quad (2.21)$$

where  $\hat{k}$  is a unit vector perpendicular to the plane. Using this we get,

$$\begin{aligned} \vec{v}_\alpha(\vec{x}) &= \frac{i\hbar c}{e} \vec{\nabla} J_\alpha(\vec{x}) \\ &= \frac{i\hbar c}{e} \vec{\nabla}_x \left\{ \sum_\beta R_{\alpha\beta} \int d^2 x' [\rho_\beta(\vec{x}') (\text{Re} \log(z - z') + i \text{Im} \log(z - z'))] - r_\alpha \frac{|z|^2}{4l^2} \right\} \\ &= \sum_\beta R_{\alpha\beta} \left( \vec{a}_\beta(\vec{x}) + i \hat{k} \times \vec{a}_\beta(\vec{x}) \right) - i\phi_0 r_\alpha \frac{\vec{x}}{4\pi l^2} . \end{aligned} \quad (2.22)$$

### III. MEAN FIELD APPROXIMATION

Thus far everything is exact. Let us now introduce the mean field (MF) approximation, by replacing the actual space dependent density operator  $\rho_\beta(\vec{x})$  in (2.22) by its average value  $\bar{\rho}_\beta = \frac{\nu_\beta B}{\phi_0}$  where  $\nu_\beta$  is the filling factor in the layer  $\beta$ . Under this approximation eq.(2.20) becomes

$$\begin{aligned} (b_\alpha)_{MF} &\equiv \nabla \times (\vec{a}_\alpha)_{MF} \\ &= -\phi_0 \bar{\rho}_\alpha \\ &= -\nu_\alpha B \end{aligned} \quad (3.1)$$

or equivalently,

$$(\vec{a}_\alpha)_{MF} = -\nu_\alpha \vec{A} \quad (3.2)$$

where  $\vec{A}$  is the applied external vector potential. Then, (2.22) reduces to

$$\vec{v}_\alpha = -\vec{A} \sum_\beta R_{\alpha\beta} \nu_\beta + i (\hat{k} \times \vec{A}) \left( -\sum_\beta R_{\alpha\beta} \nu_\beta + r_\alpha \right) \quad (3.3)$$

Here we have used the fact that in our symmetric gauge

$$\phi_0 \frac{\vec{x}}{4\pi l^2} = -\hat{k} \times \vec{A} \quad (3.4)$$

Then the covariant derivative in the composite fermion Hamiltonian (2.18) becomes, upon using the MF approximation (3.3),

$$\begin{aligned} \vec{D}_\alpha &= \vec{\nabla} - \frac{ie}{\hbar c} (\vec{A} + \vec{v}_\alpha) \\ &= \vec{\nabla} - \frac{ie}{\hbar c} \vec{A} \left( 1 - \sum_\beta R_{\alpha\beta} \nu_\beta \right) + i (\hat{k} \times \vec{A}) \left( -\sum_\beta R_{\alpha\beta} \nu_\beta + r_\alpha \right) \end{aligned} \quad (3.5)$$

Recall that the numbers  $r_\alpha$  were introduced in our definition of the composite field operators (2.3) and (2.4). Thus far we had left them unspecified, but let us now choose them to satisfy

$$r_\alpha = \sum_\beta R_{\alpha\beta} \nu_\beta \quad (3.6)$$

Then the covariant derivative simplifies to

$$\vec{D}_\alpha = \vec{\nabla} - \frac{ie}{\hbar c} \vec{A} \left( 1 - \sum_\beta R_{\alpha\beta} \nu_\beta \right) \quad (3.7)$$

Therefore in the MF approximation, our composite fermion field experiences a Hamiltonian

$$\begin{aligned} H &= \int d^2x \left[ \Pi_\alpha(\vec{x}) \left( \frac{-\hbar^2}{2\mu} (\vec{\nabla} - \frac{ie}{\hbar c} (\vec{A}_\alpha))^2 + eA_0 \right) \chi_\alpha(\vec{x}) \right] \\ &\quad + \frac{1}{2} \int \int d^2x d^2x' \delta\rho_\alpha(\vec{x}) V_{\alpha\beta}(\vec{x} - \vec{x}') \delta\rho_\beta(\vec{x}') \end{aligned} \quad (3.8)$$

where

$$\vec{A}_\alpha \equiv \left( 1 - \sum_\beta R_{\alpha\beta} \nu_\beta \right) \vec{A}. \quad (3.9)$$

$$B_\alpha^* \equiv \text{curl} \vec{\mathcal{A}}_\alpha = \left( 1 - \sum_\beta R_{\alpha\beta} \nu_\beta \right) B \quad (3.10)$$

is the effective magnetic field felt by the composite fermions of the  $\alpha$ th layer. Since the filling factor is inversely proportional to the magnetic field, this reduced effective magnetic field amounts to a correspondingly enhanced filling factor for the composite fermions given by

$$p_\alpha \equiv \left( \frac{\nu_\alpha}{1 - \sum_\beta R_{\alpha\beta} \nu_\beta} \right) \quad (3.11)$$

When these equations are inverted we get

$$\begin{aligned} \nu_1 &= \frac{1}{\Delta} \left( n - \frac{1}{p_2} - 2s_2 \right) \\ \nu_2 &= \frac{1}{\Delta} \left( n - \frac{1}{p_1} - 2s_1 \right) \end{aligned} \quad (3.12)$$

where

$$\Delta \equiv n^2 - \left( \frac{1}{p_1} + 2s_1 \right) \left( \frac{1}{p_2} + 2s_2 \right) \quad (3.13)$$

These are precisely the filling factors obtained for the bilayer system by Lopez and Fradkin [2].

#### IV. FIRST-QUANTISED WAVEFUNCTIONS

The whole purpose of our defining composite fermion operators is to give a field theoretic version of Jain's theory of fractional quantum Hall effect as generalised to bilayer systems. Following Jain's philosophy, if the effective filling factors  $p_\alpha$  are integers then one may intuitively expect the composite fermions to form incompressible quantum Hall ground states. In terms of electron coordinates the same quantum Hall state would appear at fractional filling factor  $\nu_\alpha$  related to the  $p_\alpha$  by equations (3.12).

Further, since the composite fermion field  $\chi_\alpha$  can be expressed in terms of the original electron field  $\Psi_\alpha$  through equation (2.3), if one knows the first quantised wavefunction of

composite fermions in a given state, the corresponding wavefunction of electrons in the same state is directly obtainable.

Let  $\phi_{p_1, p_2}(z_1, z_2, \dots, z_{N_1}; w_1, w_2, \dots, w_{N_2})$  represent the composite fermion wavefunction at composite fermion filling fractions  $p_\alpha = \frac{\rho_\alpha \hbar c}{e B_\alpha^*}$ , where  $B_\alpha^*$  is the effective magnetic field felt by the composite fermions in layer  $\alpha$  as given in in eq (3.10). The  $z_i$  are complex coordinates of the composite fermions on the first layer and the  $w_i$  the complex coordinates in the second layer. Note that  $N_1$  and  $N_2$ , which stand for the number of composite particles in each of the two layers respectively, are also the number of electrons in each of the two layers, since the density and Number operators of the electrons and those of the composite fermions are equal to one another (see eq 2.13). We recall that both the electron and the composite fermion operators are defined in the same two-dimensional space with the same area (sample size). This first quantised wavefunction can be written in terms of the field theoretic states and field operators  $\chi_\alpha$  as follows :

$$\begin{aligned} \phi_{p_1, p_2}(z_1, z_2, \dots, z_{N_1}; w_1, w_2, \dots, w_{N_2}) \\ \equiv \langle O | \chi_1(z_1) \chi_1(z_2) \dots \chi_1(z_{N_1}) \chi_2(w_1) \chi_2(w_2) \dots \chi_2(w_{N_2}) | MF \rangle \end{aligned} \quad (4.1)$$

where  $|MF\rangle$  stands for the mean-field ground state of the composite fermion system at filling factor  $p_\alpha$  and  $\langle O |$  is the vacuum state. Meanwhile corresponding to the same state  $|MF\rangle$  the first quantised wavefunction of the electrons (whose filling fractions in the two layers are respectively  $\nu_1, \nu_2$ ) is given by

$$\begin{aligned} \psi_{\nu_1, \nu_2}(z_1, z_2, \dots, z_{N_1}; w_1, w_2, \dots, w_{N_2}) \\ \equiv \langle O | \Psi_1(z_1) \Psi_1(z_2) \dots \Psi_1(z_{N_1}) \Psi_2(w_1) \Psi_2(w_2) \dots \Psi_2(w_{N_2}) | MF \rangle \end{aligned} \quad (4.2)$$

To relate the two wavefunctions, one only needs to write the operator  $\psi_\alpha$  in terms of  $\chi_\alpha$  using (2.3) . We get

$$\begin{aligned} \psi_{\nu_1, \nu_2}(z_1, z_2, \dots, z_{N_1}; w_1, w_2, \dots, w_{N_2}) \\ = \langle O | e^{J_1(z_1)} \chi_1(z_1) e^{J_1(z_2)} \chi_1(z_2) \dots e^{J_1(z_{N_1})} \chi_1(z_{N_1}) \\ e^{J_2(w_1)} \chi_2(w_1) e^{J_2(w_2)} \chi_2(w_2) \dots e^{J_2(w_{N_2})} \chi_2(w_{N_2}) | MF \rangle \end{aligned} \quad (4.3)$$

Next bring all the  $e^J$  factors in the above expression to the left by commuting them across the operators  $\chi$  using the commutators (2.14). We get

$$\begin{aligned}
& \psi_{\nu_1, \nu_2}(z_1, z_2, \dots, z_{N_1} ; w_1, w_2, \dots, w_{N_2}) \\
&= \prod_{i>j}^{N_1} (z_i - z_j)^{R_{11}} \prod_{k>l}^{N_2} (w_k - w_l)^{R_{22}} \prod_r^{N_1} \prod_s^{N_2} (z_r - w_s)^{R_{12}} \\
&\langle O | \exp\left(\sum_1^{N_1} J_1(z_i) + \sum_1^{N_2} J_2(w_j)\right) \\
&\chi_1(z_1)\chi_1(z_2)\dots\chi_1(z_{N_1}) \chi_2(w_1)\chi_2(w_2)\dots\chi_2(w_{N_2}) | MF \rangle \quad (4.4)
\end{aligned}$$

Next apply the operators  $e^J$  to the left on the vacuum state. Notice from its definition in eq.(2.4) that the only nontrivial operator contained in  $J_\alpha$  is the density, in the first term. The second term in  $J$  is just a c-number. In an interacting field theory, the vacuum is not generally an eigenstate of the density operator. However in the spirit of the mean field approximation being employed in this section, one can replace the density operator by its mean value. The mean density of the vacuum is zero. Thus when the operators  $e^{J_\alpha}$  in the above equation act on the left on the vacuum state the density dependent first term of  $J_\alpha$  can be taken as zero and only the second ( c-number ) survives , giving gaussian factors. Therefore

$$\begin{aligned}
& \psi_{\nu_1, \nu_2}(z_1, z_2, \dots, z_{N_1} ; w_1, w_2, \dots, w_{N_2}) \\
&= \prod_{i>j}^{N_1} (z_i - z_j)^{2s_1} \prod_{k>l}^{N_2} (w_k - w_l)^{2s_2} \prod_r^{N_1} \prod_s^{N_2} (z_r - w_s)^n \\
&\exp\left[\frac{-1}{4l^2}\left(r_1 \sum_1^{N_1} |z_i|^2 + r_2 \sum_1^{N_2} |w_i|^2\right)\right] \\
&\langle O | \chi_1(z_1)\chi_1(z_2)\dots\chi_1(z_{N_1}) \chi_2(w_1)\chi_2(w_2)\dots\chi_2(w_{N_2}) | MF \rangle \quad (4.5)
\end{aligned}$$

In the above equation we have also inserted the matrix elements of  $R_{\alpha\beta}$  from eq (2.8). In terms of the composite fermion wavefunction defined in eq.(4.1) we thus get

$$\begin{aligned}
& \psi_{\nu_1, \nu_2}(z_1, z_2, \dots, z_{N_1} ; w_1, w_2, \dots, w_{N_2}) \\
&= \prod_{i>j}^{N_1} (z_i - z_j)^{2s_1} \prod_{k>l}^{N_2} (w_k - w_l)^{2s_2} \prod_r^{N_1} \prod_s^{N_2} (z_r - w_s)^n
\end{aligned}$$

$$\exp\left[\frac{-1}{4l^2}\left(r_1 \sum_1^{N_1} |z_i|^2 + r_2 \sum_1^{N_2} |w_i|^2\right)\right] \phi_{p_1, p_2}(z_1, z_2, \dots, z_{N_1}; w_1, w_2, \dots, w_{N_2}) \quad (4.6)$$

This is just the generalisation to double layers of Jain's formula relating wavefunctions of electrons at certain fractional fillings to corresponding wavefunctions of composite fermions at other related fillings.

## V. DISCUSSION

Several features of the result (4.6) are worth pointing out.

(a) Although this relation holds for any electronic filling  $\nu_\alpha$  and the corresponding composite fermion filling  $p_\alpha$ , Jain's theory pertains to cases where the  $p_\alpha$  are integers. Then, in the non-interacting limit, the composite fermions will completely fill an integer number of Landau levels, giving rise to an energy gap. Therefore the usual arguments in the quantum Hall literature can be invoked to expect that even in the presence of e-e interactions, impurities etc., an incompressible ground state will be obtained.

(b) The case  $\nu_2 = 0$  corresponds to no electrons at all in the second layer, i.e. to the single layer case for which Jain proposed his ideas originally [1]. For this case  $N_2 = 0$  and the second-layer coordinates  $w_i$  will be absent. Then (4.6), (3.6) and (3.11) reduce to

$$\psi_{\nu_1}(z_1, z_2, \dots, z_{N_1}) = \prod_{i>j}^{N_1} (z_i - z_j)^{2s_1} \exp\left[\frac{-1}{4l^2}(2s_1\nu_1 \sum_1^{N_1} |z_i|^2)\right] \phi_{p_1}(z_1, z_2, \dots, z_{N_1}) \quad (5.1)$$

with  $\nu_1 = \frac{p_1}{(1+2p_1s_1)}$  This is just Jain's well known formula for the single layer spinless problem. That our procedure for constructing non-unitary transformations to get flux-electron composites as developed in [17] will yield Jain's wavefunctions has also been pointed out by Wu and Yu [19] for the single layer case. Notice that the right hand side of eq.(5.1) contains a gaussian factor not included in Jain's version of this formula. Whether such a factor should be there or not just depends on the relative conditions under which the composite fermion

system is being compared to the electron system. The way Jain writes such an equation, the electron wavefunction  $\psi$  and the composite fermion wave function  $\phi$  correspond to the same magnetic field. Hence there is no relative gaussian factor between them. They also carry the same number of particles  $N_1$ . However they do correspond to different filling factors  $p_1$  and  $\nu_1$  and hence different densities from one another. This tacitly implies that in Jain's way of writing this relationship the two sides of the equation correspond to different areas (sample sizes). By contrast, we have defined the electron operator  $\Psi_\alpha$  and the composite fermion operator  $\chi_\alpha$  in the same domain, as is natural in a field theory. The sample areas are thus taken as equal. The total number of particles (and therefore the density) is also the same. The difference in filling factors is caused by the difference in effective magnetic fields, namely,  $B$  for the electron wavefunction  $\psi$  and  $B^*$  (as given in eq(3.10)) for the composite fermion state  $\phi$ . Hence the gaussian factors (whose exponent is proportional to the magnetic field) will be different in  $\psi$  and  $\phi$ . The additional gaussian factor in (5.1) compensates for this difference. To verify this note that eq(3.10) reduces for the single layer case, to

$$\Delta B \equiv B - B^* = 2s_1\nu_1 B \quad (5.2)$$

Recalling that  $B \propto \frac{1}{l^2}$  we see that this difference  $\Delta B$  corresponds precisely to the relative gaussian factor in (5.1).

(c) Returning to the double layer system, the various filling fraction possibilities contained in eq(3.12) have been outlined at length by Lopez and Fradkin [2]. Of particular interest is the case of  $p_1 = p_2 = 1$  with  $s_1, s_2$  and  $n$  being arbitrary integers. Then the filling fractions (3.12) reduce to

$$\begin{aligned} \nu_1 &= \frac{2s_2 + 1 - n}{(2s_1 + 1)(2s_2 + 1) - n^2} \\ \nu_2 &= \frac{2s_1 + 1 - n}{(2s_1 + 1)(2s_2 + 1) - n^2} \end{aligned} \quad (5.3)$$

Our theory then yields for the corresponding electronic wavefunction of this bilayer state with  $\nu_1, \nu_2$  as given above, the formula (see 4.6) :

$$\psi_{\nu_1, \nu_2}(z_1, z_2, \dots, z_{N_1}; w_1, w_2, \dots, w_{N_2})$$

$$= \prod_{i>j}^{N_1} (z_i - z_j)^{2s_1} \prod_{k>l}^{N_2} (w_k - w_l)^{2s_2} \prod_r^{N_1} \prod_s^{N_2} (z_r - w_s)^n \exp\left[\frac{-1}{4l^2} \left( r_1 \sum_1^{N_1} |z_i|^2 + r_2 \sum_1^{N_2} |w_i|^2 \right)\right] \phi_{1,1}(z_1, z_2, \dots, z_{N_1}; w_1, w_2, \dots, w_{N_2}) \quad (5.4)$$

But  $\phi_{1,1}$  is nothing but the wavefunction for unit filling factor in each layer, for each of which we can use the  $\nu = 1$  Laughlin wavefunction, corresponding to an effective magnetic field of  $B_\alpha^* = \left( 1 - \sum_\beta R_{\alpha\beta} \nu_\beta \right) B = (1 - r_\alpha)B$ . This Laughlin wavefunction for unit filling in the first layer is

$$\psi_{Laughlin}^{\nu=1} = \prod_{i>j}^N (z_i - z_j) \exp\left[\frac{-1}{4l_1^2} \left( \sum_1^N |z_i|^2 \right)\right] \quad (5.5)$$

where

$$\frac{1}{l_1^2} \equiv \frac{eB_1^*}{\hbar c} = \frac{(1 - r_1)}{l^2} \quad (5.6)$$

and similarly for the second layer in terms of the coordinates  $w_i$ . Inserting such a  $\phi_{1,1}$  into (5.4) we get

$$\begin{aligned} & \psi_{\nu_1, \nu_2}(z_1, z_2, \dots, z_{N_1}; w_1, w_2, \dots, w_{N_2}) \\ &= \prod_{i>j}^{N_1} (z_i - z_j)^{2s_1+1} \prod_{k>l}^{N_2} (w_k - w_l)^{2s_2+1} \prod_r^{N_1} \prod_s^{N_2} (z_r - w_s)^n \\ & \exp\left[\frac{-1}{4l^2} \left( \sum_1^{N_1} |z_i|^2 + \sum_1^{N_2} |w_i|^2 \right)\right] \end{aligned} \quad (5.7)$$

This is just the Halperin wavefunction for the ground state of the bilayer system [8], [3] with filling factors in the two layers given by (5.3). Such a wavefunction was derived from a Chern-Simons field theoretic model by Ezawa and Iwasaki [16] earlier, but in a very different way. Their results are based on solutions to certain semiclassical self dual equations which require not only a mean-field approximation but also the neglect of e-e Coulomb interactions. They do treat interactions, but perturbatively, in the short distance limit. Our work here does use a mean field approximation in deriving results such as (5.4), but nowhere has the Coulomb term in the Hamiltonian dropped nor short distance approximation made. We also provide exact operator definitions of the composite particles. This wavefunction for

the case  $2s_1 + 1 = 2s_2 + 2$  (which is a special case of the above ) was also obtained by Lopez and Fradkin [2]. They used their fermionic Chern-Simons theory [14] suitably generalised to two-component wavefunctions and obtained the modulus of the wavefunction in the long wavelength limit. Our field theory, using the composite operators defined in (2.3) gives the full wavefunction including its modulus, phase and gaussian factors. No long wavelength approximation has been invoked by us.

(d) Clearly our work is only a variant on the earlier work of Zhang et al, Read and Lopez and Fradkin cited above. The main advantage of our generalisation to non-unitary transformations and complex Chern Simons fields is that our composite operators incorporate all aspects of the Laughlin and Jain wavefunctions including the moduli  $|z_i - z_j|$  of the correlations. These moduli contain the important zeroes which should be there in the presence of Coulomb repulsion, and also restore the wavefunctions to the lowest Landau level. The gaussian factors which should be present in the wavefunctions are also incorporated. Indeed, as our derivation shows these gaussian factors are essential for cancelling the imaginary part of our complex statistical gauge field in the mean field limit. True, some of these factors are obtained in the works of Zhang et al [13] and Lopez and Fradkin [2] but only upon including fluctuations about the mean field. That they are present already at the mean field level in our operators indicates that our operators may be better candidates for flux-electron composite fields.

(e) That we already get the full Laughlin and Jain wavefunctions at the mean field level raises hopes that corrections to these wavefunctions could be obtained even at the lowest order in fluctuations about the mean field. Unfortunately, this is where the non-unitary nature of our transformation (2.3) could create difficulties. Although the imaginary parts of our statistical field  $\vec{v}(\vec{x})$  cancel in the mean field approximation they will be present away from mean field. Of course the full Hamiltonian (2.18) is hermitian as can be verified using (2.10), but its separation into a mean field part and a perturbation does not maintain hermiticity in each part. Standard perturbation techniques would have to be re-examined and modified to take this into account. These remarks hold not only for the present work

but also our earlier work with Sondhi [17]. For a discussion on how to go beyond mean field theory in such cases see Wu and Yu [19].

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