On Burnett coefficients in periodic media

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The aim of this work is to demonstrate a curious property of general periodic structures. It is well known that the corresponding homogenized matrix is positive definite. We calculate here the next order Burnett coefficients associated with such structures. We prove that these coefficients form a tensor which is negative semidefinite. We also provide some examples showing degeneracy in multidimension.

I. INTRODUCTION

The aim of this paper is to demonstrate a curious property of general periodic structures. Let us consider acoustic wave propagation in a periodic medium with a small period denoted by ε. To first order, we can approximate this medium by the associated homogenized medium. It is well known that the acoustic wave propagation in the homogenized medium provides good approximation to the propagation of sufficiently long waves in the original periodic medium (Refs. 1 and 9).

Here, we are interested in higher order approximation. From our previous work,[2] it is known that the homogenized medium is described by the second order derivatives of the first Bloch eigenvalue $\lambda_1(\eta)$ at $\eta=0$ and the next order approximation to the periodic medium is provided by a fourth order tensor, namely the fourth derivatives of $\lambda_1(\eta)$ at $\eta=0$. In the physics literature such higher order derivatives are known as *Burnett coefficients* and they are of great interest (Refs. 14 and 8). It is known that the homogenized matrix is positive definite.[1] This work is devoted to the study of the next order approximation.

The approximation of the periodic medium comes about from the asymptotic expansion of $\lambda_1(\eta)$ near $\eta=0$,

$$\lambda_1(\eta) = \frac{1}{2!}\lambda_1''(0)\,\eta^2 + \frac{1}{4!}\lambda_1^{(4)}(0)\,\eta^4 + \cdots .$$

Substituting $\eta=\varepsilon\xi$, we get the asymptotic expansion of the first eigenvalue associated to the ε-periodic operator A^{ε} near $\eta=0$,

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$$\lambda_1^\varepsilon(\xi) = \varepsilon^{-2}\lambda_1(\varepsilon\xi) = \frac{1}{2!}\lambda_1''(0)\xi^2 + \frac{1}{4!}\varepsilon^2\lambda_1^{(4)}(0)\xi^4 + \cdots .$$

It is already established that the Hessian matrix $\lambda_1''(0)/2$ coincides with the homogenized matrix denoted by $(q_{k\ell})$ (defined for instance in Ref. 1) and hence is positive definite. In this write-up, we show that

$$\frac{1}{4!}\lambda_1^{(4)}(0)\,\eta^4 \leqslant 0 \qquad \forall\ \eta \in \mathbb{R}^N.$$

Further, we show that there can be directions $\eta \neq 0$ such that

$$\frac{1}{4!}\lambda_1^{(4)}(0)\,\eta^4 = 0.$$

However, in one dimension, as we show below, such a degeneracy cannot happen unless the medium is homogeneous.

Before starting our computations, let us interpret the above result in terms of acoustic wave propagation in the original ε-periodic medium. From the above expansion, it is clear that

$$\lambda_1^\varepsilon(\xi) \approx \frac{1}{2!}\lambda_1''(0)\xi^2 \quad \text{if } \varepsilon^2|\xi|^4 \text{ is small}.$$

This shows that the usual homogenized medium, as remarked above, provides a good description provided the waves are long. However, for short waves the above approximation is poor. Indeed if $\varepsilon^2|\xi|^4 = O(1)$ and $\varepsilon^4|\xi|^6 = o(1)$ then we have

$$\lambda_1^\varepsilon(\xi) \approx \frac{1}{2!}\lambda_1''(0)\xi^2 + \frac{1}{4!}\varepsilon^2\lambda_1^{(4)}(0)\xi^4.$$

The above picture shows that long waves experience hyperbolic effects while short waves in question undergo some dispersion too. This dispersive nature medium is described by the fourth order tensor $\lambda_1^{(4)}(0)$. However, the dispersion is not classical. It has a negative sign and we may then call it *negative dispersion*. Strictly speaking, the corresponding initial value problem modeling the propagation of such short waves is not well posed. We would like to bring to the attention of homogenization community that some curious materials (with negative refraction, negative reflection coefficients) are being conceived and produced (see Refs. 5, 10, 11, and 13). Viewed in this light, our result says that a fine periodic structure provides one such curious material as far as short wave propagation is concerned. At this point, we would like to emphasize other features of our result. It came as a surprise to us to see a definite sign for the fourth order derivative, as it was not expected. Though our computations follow a general pattern, it is not clear to us whether higher order derivatives too have a definite sign. The next remark is concerned with the level of generality at which we are working: we have no restriction whatsoever on the original periodic medium except the classical ones. In other words, the material components used in mixing are arbitrary, their proportions are arbitrary and the microgeometry of mixing is also arbitrary. This contrasts sharply with the current efforts in producing curious or smart materials using particular components and following a particular design, for example, photonic crystals (see Refs. 7, 6, and 12).

II. PRELIMINARIES

Let us now introduce the problem to be studied in this work. First, we remark that the summation with respect to the repeated indices is understood throughout this paper. We consider the operator

$$A \stackrel{\text{def}}{=} -\frac{\partial}{\partial y_k}\left(a_{k\ell}(y)\frac{\partial}{\partial y_\ell}\right), \quad y \in \mathbb{R}^N, \tag{2.1}$$

where the coefficients satisfy

$$\begin{aligned}
&a_{k\ell} \in L^\infty_\#(Y) \quad \text{where } Y = \,]0,2\pi[^N, \text{ i.e., each } a_{k\ell} \text{ is a} \\
&Y\text{-periodic bounded measurable function defined on } \mathbb{R}^N, \text{ and} \\
&\exists \alpha > 0 \quad \text{such that } a_{k\ell}(y)\eta_k\eta_\ell \geqslant \alpha|\eta|^2 \quad \forall\, \eta \in \mathbb{R}^N, \quad y \in Y \text{ a.e.,} \\
&a_{k\ell} = a_{\ell k} \quad \forall\, k, \ell = 1, \ldots, N.
\end{aligned} \tag{2.2}$$

For each $\varepsilon > 0$, we consider also the ε-periodic operator A^ε where

$$A^\varepsilon \stackrel{\text{def}}{=} -\frac{\partial}{\partial x_k}\left(a^\varepsilon_{k\ell}(x)\frac{\partial}{\partial x_\ell}\right) \quad \text{with } a^\varepsilon_{k\ell}(x) \stackrel{\text{def}}{=} a_{k\ell}\left(\frac{x}{\varepsilon}\right), \quad x \in \mathbb{R}^N. \tag{2.3}$$

In homogenization theory, it is usual to refer to x and y as the slow and the fast variables, respectively. They are related by $y = x/\varepsilon$.

Our results are based on Bloch waves ψ associated with the operator A which we define now. Let us consider the following spectral problem parametrized by $\eta \in \mathbb{R}^N$: find $\lambda = \lambda(\eta) \in \mathbb{R}$ and $\psi = \psi(y;\eta)$ (not zero) such that

$$\begin{aligned}
&A\psi(\cdot;\eta) = \lambda(\eta)\psi(\cdot;\eta) \quad \text{in } \mathbb{R}^N, \quad \psi(\cdot;\eta) \text{ is } (\eta;Y)\text{-periodic, i.e.,} \\
&\psi(y+2\pi m;\eta) = e^{2\pi i m\cdot\eta}\psi(y;\eta) \quad \forall\, m \in \mathbb{Z}^N, \quad y \in \mathbb{R}^N.
\end{aligned} \tag{2.4}$$

Next, by the Floquet theory, we define $\phi(y;\eta) = e^{-iy\cdot\eta}\psi(y;\eta)$ and (2.4) can be rewritten in terms of ϕ as follows:

$$A(\eta)\phi = \lambda\phi \quad \text{in } \mathbb{R}^N, \quad \phi \quad \text{is} \quad Y\text{-periodic}. \tag{2.5}$$

Here, the operator $A(\eta)$ is called the translated operator and is defined by

$$A(\eta) = e^{-iy\cdot\eta}Ae^{iy\cdot\eta}.$$

It is well known (see Refs. 1 and 3) that for each $\eta \in Y' = [-1/2, 1/2[^N$, the above spectral problem (2.5) admits a discrete sequence of eigenvalues and their eigenfunctions (referred to as *Bloch waves*) introduced above enable us to describe the spectral resolution of A [an unbounded self-adjoint operator in $L^2(\mathbb{R}^N)$] in the orthogonal basis $\{e^{iy\cdot\eta}\phi_m(y;\eta) \mid m \geqslant 1, \eta \in Y'\}$.

To obtain the spectral resolution of A^ε, let us introduce Bloch waves at the ε-scale,

$$\lambda^\varepsilon_m(\xi) = \varepsilon^{-2}\lambda_m(\eta), \quad \phi^\varepsilon_m(x;\xi) = \phi_m(y;\eta), \quad \psi^\varepsilon_m(x;\xi) = \psi_m(y;\eta),$$

where the variables (x,ξ) and (y,η) are related by $y = x/\varepsilon$ and $\eta = \varepsilon\xi$. Observe that $\phi^\varepsilon_m(x;\xi)$ is εY-periodic (in x) and $\varepsilon^{-1}Y'$ periodic with respect to ξ. In the same manner, $\psi^\varepsilon_m(\cdot;\xi)$ is $(\varepsilon\xi;\varepsilon Y)$ periodic because of the relation $\psi^\varepsilon_m(x;\xi) = e^{ix\cdot\xi}\phi^\varepsilon_m(x;\xi)$. Note that the dual cell at ε-scale is $\varepsilon^{-1}Y'$ and hence we take ξ to vary in $\varepsilon^{-1}Y'$ in the sequel. With these notations, we have (see Ref. 1) the following.

Theorem 2.1: *Let $g \in L^2(\mathbb{R}^N)$. The* mth *Bloch coefficient of g at the ε-scale is defined as follows:*

$$(B^\varepsilon_m g)(\xi) = \int_{\mathbb{R}^N} g(x)e^{-ix\cdot\xi}\bar{\phi}^\varepsilon_m(x;\xi)\mathrm{d}x \quad \forall\, m \geqslant 1, \quad \xi \in \varepsilon^{-1}Y'.$$

Then the following inverse formula and Parseval's identity hold:

$$g(x) = \int_{\varepsilon^{-1}Y'} \sum_{m=1}^{\infty} (B_m^\varepsilon g)(\xi) e^{ix\cdot\xi} \phi_m^\varepsilon(x;\xi) \mathrm{d}\xi,$$

$$\int_{\mathbb{R}^N} |g(x)|^2 \,\mathrm{d}x = \int_{\varepsilon^{-1}Y'} \sum_{m=1}^{\infty} |(B_m^\varepsilon g)(\xi)|^2 \,\mathrm{d}\xi.$$

Finally, for all g in the domain of A^ε, we get

$$A^\varepsilon g(x) = \int_{\varepsilon^{-1}Y'} \sum_{m=1}^{\infty} \lambda_m^\varepsilon(\xi)(B_m^\varepsilon g)(\xi) e^{ix\cdot\xi} \phi_m^\varepsilon(x;\xi) \mathrm{d}\xi.$$

■

In the homogenization process, one can neglect all the modes for $m \geq 2$ (see Refs. 4 and 2). To this end, we consider a sequence of solutions u^ε of the equation

$$A^\varepsilon u^\varepsilon = f \quad \text{in } \mathbb{R}^N. \tag{2.6}$$

We can show that the component of u^ε in the higher Bloch modes are not significant. More precisely, let us consider v^ε defined by

$$v^\varepsilon(x) = \int_{\varepsilon^{-1}Y'} \sum_{m=2}^{\infty} (B_m^\varepsilon u^\varepsilon)(\xi) e^{ix\cdot\xi} \phi_m^\varepsilon(x;\xi) \mathrm{d}\xi. \tag{2.7}$$

which is nothing but the projection of u^ε corresponding to all higher Bloch modes. Then the following estimates on v^ε are derived in Ref. 2.

Proposition 2.2: *Depending on the regularity of the source term f in* (2.6), *we have*

(i) *If* $f \in H^{-1}(\mathbb{R}^N)$: $\|v^\varepsilon\|_{L^2(\mathbb{R}^N)} \leq c\varepsilon \|f\|_{H^{-1}(\mathbb{R}^N)}$.
(ii) *If* $f \in L^2(\mathbb{R}^N)$: $\|v^\varepsilon\|_{L^2(\mathbb{R}^N)} \leq c\varepsilon^2 \|f\|_{L^2(\mathbb{R}^N)}$.
(iii) *If* $f \in L^2(\mathbb{R}^N)$: $|v^\varepsilon|_{H^1(\mathbb{R}^N)} \leq c\varepsilon \|f\|_{L^2(\mathbb{R}^N)}$.
(iv) *If* $f \in H^1(\mathbb{R}^N)$: $\|v^\varepsilon\|_{L^2(\mathbb{R}^N)} \leq c\varepsilon^3 |f|_{H^1(\mathbb{R}^N)}$.
(v) *If* $f \in H^1(\mathbb{R}^N)$: $|v^\varepsilon|_{H^1(\mathbb{R}^N)} \leq c\varepsilon^2 |f|_{H^1(\mathbb{R}^N)}$.

Here, we denote by $|\cdot|_{H^1(\mathbb{R}^N)}$ the seminorm of $H^1(\mathbb{R}^N)$. ■

The above result is at the basis of neglecting higher order Bloch eigenvalues $\{\lambda_m^\varepsilon(\xi)\}_{m\geq 2}$ in the context of our discussion in the Introduction.

III. FOURTH ORDER TENSOR $\lambda_1^{(4)}(0)$

In this section, we present the expression for the fourth order tensor $\lambda_1^{(4)}(0)$ and show that it is negative semidefinite. Recall that $\lambda_1(\eta)$ and $\phi_1(\cdot;\eta)$ depend analytically on η in a small neighborhood B_δ of $\eta=0$ (see Ref. 4).

A. Derivatives of the first Bloch eigenvalue and eigenvector

The purpose of this section is to present expressions for derivatives of the first Bloch eigenvalue $\lambda_1(\eta)$ and the first Bloch eigenvector $\phi_1(\cdot;\eta)$ at $\eta=0$ and indicate a systematic method to compute them. For details of these computations, the reader is referred to Ref. 2. Our approach exploits the connection between the Bloch space computation with the multiscale computation.

The derivatives of the first eigenvalue and eigenfunction in $\eta=0$ exist thanks to the regularity property established in Ref. 4. In fact, we know that there exists $\delta > 0$ such that the first eigenvalue $\lambda_1(\eta)$ is an analytic function on $B_\delta = \{\eta \,|\, |\eta| < \delta\}$, and there is a choice of the first eigenvector $\phi_1(y;\eta)$ satisfying

$$\eta \to \phi_1(\cdot\,;\eta) \in H^1_\#(Y) \text{ is analytic on } B_\delta, \quad \phi_1(y;0) = |Y|^{-1/2} = \frac{1}{(2\pi)^{N/2}}.$$

Apart from the above result of regularity on the Bloch spectrum, the following proposition was also proved in Ref. 2.

Proposition 3.1: *We have the relations*

$$\lambda_1(0) = 0, \quad D_k\lambda_1(0) = \frac{\partial \lambda_1}{\partial \eta_k}(0) = 0 \quad \forall\, k = 1, \ldots, N,$$

$$\frac{1}{2}D^2_{k\ell}\lambda_1(0) = \frac{1}{2}\frac{\partial^2 \lambda_1}{\partial \eta_k\, \partial \eta_\ell}(0) = q_{k\ell} \quad \forall\, k, \ell = 1, \ldots, N,$$

where $q_{k\ell}$ are the homogenized coefficients defined by

$$q_{k\ell} = \mathcal{M}_Y\left(a_{k\ell} + a_{km}\frac{\partial \chi_\ell}{\partial y_m}\right) \overset{\text{def}}{=} \frac{1}{|Y|}\int_Y \left(a_{k\ell} + a_{km}\frac{\partial \chi_\ell}{\partial y_m}\right)\mathrm{d}y \quad \forall\, k, \ell = 1, \ldots, N, \tag{3.1}$$

with test function χ_k defined by the following cell problem:

$$A\chi_k = \frac{\partial a_{k\ell}}{\partial y_\ell} \quad \text{in } \mathbb{R}^N,$$

$$\chi_k \in H^1_\#(Y), \quad \mathcal{M}_Y(\chi_k) = 0. \tag{3.2}$$

Moreover, all odd order derivatives of λ_1 at $\eta=0$ vanish, i.e.,

$$D^\beta \lambda_1(0) = 0 \quad \forall\, \beta \in \mathbb{Z}^N_+, \quad |\beta| \text{ odd}.$$

All even order derivatives of λ_1 at $\eta=0$ can be calculated in a systematic fashion. For instance, the fourth order derivatives have the following expressions: for all $k, \ell, m, n = 1, \ldots, N$,

$$\begin{aligned}\frac{1}{4!}D^4_{k\ell mn}\lambda_1(0) = {}& \frac{1}{4}\frac{1}{|Y|}\int_Y \{C_n\chi_{k\ell m} + C_k\chi_{\ell mn} + C_\ell\chi_{mnk} + C_m\chi_{nk\ell}\}\mathrm{d}y \\ & - \frac{1}{3!}\frac{1}{|Y|}\int_Y \{(a_{k\ell} - q_{k\ell})\chi_{mn} + (a_{\ell m} - q_{\ell m})\chi_{nk} + (a_{mn} - q_{mn})\chi_{k\ell} \\ & + (a_{nk} - q_{nk})\chi_{\ell m} + (a_{km} - q_{km})\chi_{\ell n} + (a_{\ell n} - q_{\ell n})\chi_{km}\}\mathrm{d}y.\end{aligned}$$

Here, the operator C_k is defined by

$$C_k\phi \overset{\text{def}}{=} -a_{kj}(y)\frac{\partial \phi}{\partial y_j} - \frac{\partial}{\partial y_j}(a_{kj}(y)\phi) \tag{3.3}$$

(*is skew-adjoint, $C_k^* = -C_k$*), *and $\chi_{k\ell}$, $\chi_{k\ell m}$ are the test functions defined by the following cell problems*:

$$A\chi_{k\ell} = (a_{k\ell} - q_{k\ell}) - \tfrac{1}{2}(C_k\chi_\ell + C_\ell\chi_k) \quad \text{in } \mathbb{R}^N,$$

$$\chi_{k\ell} \in H^1_\#(Y), \quad \mathcal{M}_Y(\chi_{k\ell}) = 0, \tag{3.4}$$

$$A\chi_{k\ell m} = \tfrac{1}{3}[(a_{k\ell} - q_{k\ell})\chi_m + (a_{\ell m} - q_{\ell m})\chi_k + (a_{mk} - q_{mk})\chi_\ell]$$

$$- \tfrac{1}{3}[C_k\chi_{\ell m} + C_\ell\chi_{mk} + C_m\chi_{k\ell}] \quad \text{in } \mathbb{R}^N, \tag{3.5}$$

$$\chi_{k\ell m} \in H^1_\#(Y), \quad \mathcal{M}_Y(\chi_{k\ell m}) = 0.$$

■

The above expressions are obtained by differentiating the eigenvalue problem,

$$(A(\eta) - \lambda_1(\eta))\phi_1(\cdot\,;\eta) = 0,$$

and using that the branch $\eta \mapsto \phi_1(\cdot\,;\eta)$ can be so chosen that the following conditions are satisfied simultaneously:

$$\eta \in B_\delta \mapsto \phi_1(\cdot\,;\eta) \in H^1_\#(Y) \quad \text{is analytic},$$

$$\|\phi_1(\cdot\,;\eta)\|_{L^2(Y)} = 1 \qquad \forall\ \eta \in B_\delta,$$

$$\mathrm{Im}\int_Y \phi_1(y;\eta)\mathrm{d}y = 0 \qquad \forall\ \eta \in B_\delta.$$

B. $D^4\lambda_1(0)$ is negative semidefinite

First, we denote the fourth derivatives as

$$B_{k\ell mn} = \frac{1}{4!}D^4_{k\ell mn}\lambda_1(0).$$

Thus, by the Proposition 3.1, for any $\eta \in \mathbb{R}^N$ we get

$$\begin{aligned}
B_{k\ell mn}\eta_k\eta_\ell\eta_n\eta_m = {} & \frac{1}{4}\frac{1}{|Y|}\int_Y [(C_n\eta_n)(\chi_{k\ell m}\eta_k\eta_\ell\eta_m) + (C_k\eta_k)(\chi_{\ell mn}\eta_\ell\eta_m\eta_n)]\mathrm{d}y \\
& + \frac{1}{4}\frac{1}{|Y|}\int_Y [(C_\ell\eta_\ell)(\chi_{mnk}\eta_m\eta_n\eta_k) + (C_m\eta_m)(\chi_{nk\ell}\eta_k\eta_\ell\eta_n)]\mathrm{d}y \\
& - \frac{1}{3!}\frac{1}{|Y|}\int_Y [\eta_k\eta_\ell(a_{k\ell} - q_{k\ell})(\chi_{mn}\eta_m\eta_n) + \eta_\ell\eta_m(a_{\ell m} - q_{\ell m})(\chi_{nk}\eta_k\eta_n)]\mathrm{d}y \\
& - \frac{1}{3!}\frac{1}{|Y|}\int_Y [\eta_m\eta_n(a_{mn} - q_{mn})(\chi_{k\ell}\eta_k\eta_\ell) + \eta_k\eta_n(a_{nk} - q_{nk})(\chi_{\ell m}\eta_m\eta_\ell)]\mathrm{d}y \\
& - \frac{1}{3!}\frac{1}{|Y|}\int_Y [\eta_m\eta_k(a_{km} - q_{km})(\chi_{\ell n}\eta_\ell\eta_n) + \eta_\ell\eta_m(a_{\ell n} - q_{\ell n})(\chi_{km}\eta_m\eta_k)]\mathrm{d}y.
\end{aligned}$$

Now, we introduce the following notations:

$$C = \eta_n C_n, \quad \chi^{(1)} = \eta_k\chi_k, \quad \chi^{(2)} = \eta_k\eta_\ell\chi_{k\ell}, \quad \chi^{(3)} = \eta_k\eta_\ell\eta_m\chi_{k\ell m},$$

$$a = \eta_k\eta_m a_{km}, \quad q = \eta_k\eta_m q_{km}. \tag{3.6}$$

Then, by the summation, the above expression is simplified to the following:

$$B_{k\ell mn}\eta_k\eta_\ell\eta_m\eta_n = \frac{1}{|Y|}\int_Y C\chi^{(3)}\,\mathrm{d}y - \frac{1}{|Y|}\int_Y (a-q)\chi^{(2)}\,\mathrm{d}y. \tag{3.7}$$

Since the test function χ_k satisfies the cell problem (3.2), we have by the notations (3.6),

$$A\chi^{(1)} = \eta_k \frac{\partial a_{k\ell}}{\partial y_\ell}. \tag{3.8}$$

Analogously, by the cell problems (3.4) and (3.5), we have the following Y-periodic problems:

$$A\chi^{(2)} = (a-q) - C\chi^{(1)}, \tag{3.9}$$

$$A\chi^{(3)} = (a-q)\chi^{(1)} - C\chi^{(2)}. \tag{3.10}$$

By the notation (3.6), since C_k is defined in (3.3), we get

$$\frac{1}{|Y|}\int_Y C\chi^{(3)}\,\mathrm{d}y = -\frac{1}{|Y|}\int_Y \eta_k a_{k\ell}\frac{\partial\chi^{(3)}}{\partial y_\ell}\mathrm{d}y.$$

Using (3.8) and integrating by parts, we have

$$\frac{1}{|Y|}\int_Y C\chi^{(3)}\,\mathrm{d}y = \frac{1}{|Y|}\int_Y A\chi^{(1)}\chi^{(3)}\,\mathrm{d}y = \frac{1}{|Y|}\int_Y \chi^{(1)}A\chi^{(3)}\,\mathrm{d}y = \frac{1}{|Y|}\int_Y [(a-q)(\chi^{(1)})^2 - \chi^{(1)}C\chi^{(2)}]\mathrm{d}y,$$

by (3.10). Therefore, from (3.7), we obtain

$$B_{k\ell mn}\eta_k\eta_\ell\eta_m\eta_n = \frac{1}{|Y|}\int_Y (a-q)(\chi^{(1)})^2\,\mathrm{d}y - \frac{1}{|Y|}\int_Y \chi^{(1)}C\chi^{(2)}\,\mathrm{d}y - \frac{1}{|Y|}\int_Y (a-q)\chi^{(2)}\,\mathrm{d}y.$$

Again, by definition of C, we have $C^* = -C$ and hence

$$B_{k\ell mn}\eta_k\eta_\ell\eta_m\eta_n = \frac{1}{|Y|}\int_Y (a-q)(\chi^{(1)})^2\,\mathrm{d}y - \frac{1}{|Y|}\int_Y [(a-q) - C\chi^{(1)}]\chi^{(2)}\,\mathrm{d}y.$$

Using now (3.9), we get

$$B_{k\ell mn}\eta_k\eta_\ell\eta_n\eta_m = \frac{1}{|Y|}\int_Y (a-q)(\chi^{(1)})^2\,\mathrm{d}y - \frac{1}{|Y|}\int_Y A\chi^{(2)}\chi^{(2)}\,\mathrm{d}y.$$

Substituting,

$$a - q = A\chi^{(2)} + C\chi^{(1)},$$

we can rewrite the above expression as follows:

$$\begin{aligned} B_{k\ell mn}\eta_k\eta_\ell\eta_m\eta_n = {}& -\frac{1}{|Y|}\int_Y A\left[\chi^{(2)} - \frac{1}{2}(\chi^{(1)})^2\right]\cdot\left[\chi^{(2)} - \frac{1}{2}(\chi^{(1)})^2\right]\mathrm{d}y + \frac{1}{4}\frac{1}{|Y|}\int_Y A(\chi^{(1)})^2\cdot(\chi^{(1)})^2\,\mathrm{d}y \\ & + \frac{1}{|Y|}\int_Y C\chi^{(1)}\cdot(\chi^{(1)})^2\,\mathrm{d}y. \end{aligned} \tag{3.11}$$

We show now that the last two terms of the above expression add up to zero. More precisely, we prove

$$\frac{1}{4}\frac{1}{|Y|}\int_Y A(\chi^{(1)})^2\cdot(\chi^{(1)})^2\,\mathrm{d}y+\frac{1}{|Y|}\int_Y C\chi^{(1)}\cdot(\chi^{(1)})^2\,\mathrm{d}y=0. \tag{3.12}$$

To this end, let us first consider the last term. By definition of C,

$$\int_Y C\chi^{(1)}\cdot(\chi^{(1)})^2\,\mathrm{d}y=-2\eta_k\int_Y a_{k\ell}\frac{\partial}{\partial y_\ell}\left(\frac{1}{3}(\chi^{(1)})^3\right)\mathrm{d}y-\eta_k\int_Y\frac{\partial a_{k\ell}}{\partial y_\ell}(\chi^{(1)})^3\,\mathrm{d}y.$$

Via a simple integration by parts, we obtain

$$\int_Y C\chi^{(1)}\cdot(\chi^{(1)})^2\,\mathrm{d}y=-\frac{1}{3}\int_Y A\chi^{(1)}\cdot(\chi^{(1)})^3\,\mathrm{d}y. \tag{3.13}$$

On the other hand, regarding the first term in (3.12), we can establish a more general relation,

$$\frac{1}{(p+1)^2}\int_Y A(\chi^{(1)})^{p+1}\cdot(\chi^{(1)})^{p+1}\,\mathrm{d}y=\frac{1}{2p+1}\int_Y A\chi^{(1)}\cdot(\chi^{(1)})^{2p+1}\,\mathrm{d}y\quad\forall p\in\mathbb{N}. \tag{3.14}$$

This proof is simply obtained by writing the expression

$$a_{k\ell}\frac{\partial\chi^{(1)}}{\partial y_\ell}\frac{\partial\chi^{(1)}}{\partial y_k}(\chi^{(1)})^{2p}$$

in two different ways, namely

$$a_{k\ell}\frac{\partial\chi^{(1)}}{\partial y_\ell}\frac{\partial}{\partial y_k}\left(\frac{1}{2p+1}(\chi^{(1)})^{2p+1}\right)\quad\text{and}\ a_{k\ell}\frac{\partial}{\partial y_\ell}\left(\frac{1}{p+1}(\chi^{(1)})^{p+1}\right)\frac{\partial}{\partial y_k}\left(\frac{1}{p+1}(\chi^{(1)})^{p+1}\right).$$

A simple integration of these expressions leads us to the above relation (3.14). Finally, taking $p=1$ in (3.14) and using (3.13), we get (3.12). Thus, we conclude the proof of the following result.

Proposition 3.2: *The tensor of fourth derivatives of* λ_1 *in* 0 *is negative semidefinite. More precisely, for any* $\eta\in\mathbb{R}^N$, *we get*

$$\frac{1}{4!}D^4_{k\ell mn}\lambda_1(0)\eta_k\eta_\ell\eta_m\eta_n=-\frac{1}{|Y|}\int_Y A\left[\chi^{(2)}-\frac{1}{2}(\chi^{(1)})^2\right]\cdot\left[\chi^{(2)}-\frac{1}{2}(\chi^{(1)})^2\right]\mathrm{d}y\leqslant 0. \tag{3.15}$$

■

C. One-dimensional case

In this case, we get a more simple formula for the form associated with the fourth order derivatives of λ_1. More exactly, we show that, for any $\eta\in\mathbb{R}$, we have

$$\frac{1}{4!}D^4\lambda_1(0)\eta^4=-\frac{q}{2\pi}\int_0^{2\pi}(\chi^{(1)})^2\,\mathrm{d}y\leqslant 0. \tag{3.16}$$

Indeed, by (3.7) we know that

$$\frac{1}{4!}D^4\lambda_1(0)\eta^4=\frac{1}{2\pi}\int_0^{2\pi}(C\chi^{(3)}-(a-q)\chi^{(2)})\mathrm{d}y.$$

Then, we prove (3.16) by showing that

$$\int_0^{2\pi}C\chi^{(3)}\,\mathrm{d}y=\int_0^{2\pi}(-q(\chi^{(1)})^2+(a-q)\chi^{(2)})\mathrm{d}y. \tag{3.17}$$

To prove the above result, we first establish some formulas where we use one-dimensional nature of the problem. Considering (3.8) and integrating, we get

$$a\frac{\mathrm{d}\chi^{(1)}}{\mathrm{d}y} = (q-a)\,\eta. \tag{3.18}$$

Now, considering (3.9) and integrating, we get

$$a\frac{\mathrm{d}\chi^{(2)}}{\mathrm{d}y} = -\,a\chi^{(1)}\eta + c,$$

where c is some constant. Dividing the above relation by a and integrating it over Y, we get $c=0$ and then the following relation results:

$$\frac{\mathrm{d}\chi^{(2)}}{\mathrm{d}y} = -\,\chi^{(1)}\eta. \tag{3.19}$$

Because of the above relations (3.18) and (3.19), there are simplifications in one dimension which can be exploited to establish (3.17). To this end, it is enough to use the equations satisfied by $\chi^{(1)}$ and $\chi^{(3)}$ and integration by parts.

Remark 3.3: An immediate consequence of (3.16) is

$$D^4\lambda_1(0) = 0 \iff a(y) \quad \textit{is constant}. \tag{3.20}$$

Indeed, if $D^4\lambda_1(0)=0$ *then* $\chi^{(1)}=0$ *and by (3.18), the coefficient* $a(y)$ *is constant.* ■

IV. DEGENERATE CASES

Unfortunately, in several space dimensions (3.20) need not be true. It can happen that the coefficients $\{a_{k\ell}\}$ are not constants and yet

$$D^4_{k\ell mn}\lambda_1(0)\,\eta_k\eta_\ell\eta_m\eta_n = 0 \quad \text{for some } \eta \neq 0. \tag{4.1}$$

In this section, we show some examples of such degenerates cases.

First, we propose to prove two equivalent expressions for (4.1).

Proposition 4.1: Let us pose

$$\beta^{(2)} = \beta^{(2)}_{k\ell}\,\eta_k\eta_\ell,$$

where the constants $\beta^{(2)}_{k\ell}$ *are defined by*

$$\beta^{(2)}_{k\ell} = \frac{1}{2!}\frac{1}{|Y|}\int_Y \chi_\ell\chi_k\,\mathrm{d}y. \tag{4.2}$$

Then, if there exists $\eta \in \mathbb{R}^N - \{0\}$ *satisfying (4.1), we get*

$$\chi^{(2)} = \tfrac{1}{2}(\chi^{(1)})^2 - \beta^{(2)}. \tag{4.3}$$

Also (4.1), is equivalent to the following Hamilton-Jacobi type equation for $\chi^{(1)}$*:*

$$a - q + 2\,\eta_k a_{k\ell}\frac{\partial\chi^{(1)}}{\partial y_\ell} + a_{k\ell}\frac{\partial\chi^{(1)}}{\partial y_k}\frac{\partial\chi^{(1)}}{\partial y_\ell} = 0. \tag{4.4}$$

■

Proof: From (3.15) and (4.1), we have immediately that $\chi^{(2)}-\frac{1}{2}(\chi^{(1)})^2=C$, for some constant C. Integrating this relation and using the definition of $\chi^{(1)}$ and $\chi^{(2)}$, we get

$$C|Y| = -\frac{1}{2}\eta_k\eta_\ell \int_Y \chi_k\chi_\ell \, \mathrm{d}y,$$

and by definition (4.2) of $\beta_{k\ell}^{(2)}$, we obtain (4.3).

Applying the operator A on the relation (4.3), we get

$$A\chi^{(2)} = A\left[\tfrac{1}{2}(\chi^{(1)})^2\right].$$

Since $\chi^{(1)}$ and $\chi^{(2)}$ are solutions of (3.8) and (3.9), respectively, we have

$$A\chi^{(2)} = a - q + 2\,\eta_k a_{k\ell}\frac{\partial\chi^{(1)}}{\partial y_\ell} + A\chi^{(1)}\cdot\chi^{(1)}$$

and

$$A\left[\frac{1}{2}(\chi^{(1)})^2\right] = A\chi^{(1)}\cdot\chi^{(1)} - a_{k\ell}\frac{\partial\chi^{(1)}}{\partial y_\ell}\frac{\partial\chi^{(1)}}{\partial y_k}.$$

We subtract the last two expressions and arrive at (4.4). Conversely, we can start from (4.4) and deduce (4.3). This completes the proof. ■

Next, we present some examples of degenerate cases in several space dimensions.

Case of laminates: We place ourselves in two dimensions. Consider the matrix of coefficients $(a_{k\ell}(y))$,

$$\begin{pmatrix} a_{11}(y_1) & 0 \\ 0 & a_{22} \end{pmatrix},$$

with a_{22} being a constant. Taking $\eta=(0,1)$, the following can be easily checked:

$$\chi_2 \equiv 0, \quad \chi^{(1)} \equiv 0, \quad q_{22} = a_{22}, \quad a = q = a_{22}.$$

[Recall that χ_2 is the solution of (3.2)]. Thus, the equation (4.4) and hence the property (4.1) are easily satisfied but $(a_{k\ell}(y))$ is not a constant matrix.

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