Detecting Embedded Horn Structure in Propositional Logic *

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Abstract

We show that the problem of finding a maximum renamable Horn problem within a propositional satisfiability problem is NP-hard but can be formulated as a set packing and therefore a maximum clique problem, for which numerous algorithms and heuristics have been developed.

1 Introduction

Horn clauses are widely used because, for them, satisfiability and inference problems are soluble in linear time. "Renamable Horn" problems (which are Horn up to a rescaling of variables) are also soluble in linear time. We address the problem of obtaining a renamable Horn problem by removing as few variables as possible from a given non-Horn satisfiability problem. One can then solve the original problem by enumerating truth assignments to the removed variables and solving a renamable Horn problem for each assignment.

We show that finding a maximal renamable Horn subproblem can be formulated as a maximum clique problem, for which numerous algorithms and heuristics have been developed and tested [2, 3, 6, 7, 8, 9]. We also observe that finding such a subproblem is NP-hard.

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2 Embedded Renamable Horn Sets

A set of clauses is *Horn* when each clause in it contains at most one positive literal. To *scale* a set of clauses is to replace every occurrence of x_j with $\neg x_j$ and every occurrence of $\neg x_j$ with x_j, for zero or more variables x_j. A *renamable Horn set* (RHS) is a clause set that can be scaled to obtain a Horn set.

The time required to check the satisfiability of a Horn set is linear in the number of literals [5]. There are also linear-time algorithms that determine whether a given clause set is renamable Horn and find an appropriate scaling when one exists [1, 4]. It is therefore possible to solve the satisfiability problem for an RHS in linear time.

Given a clause set S, let X be a subset of the variables occurring in S, and let $\overline{X}$ contain the variables not in X. Define $S(X)$ to be the result of removing from S all occurrences of variables in $\overline{X}$. If $S(X)$ is renamable Horn, we say that it is an *embedded RHS* of S. An embedded RHS of maximum size is a *maximum* embedded RHS.

Let $v : \overline{X} \to \{T, F\}$ be a mapping that assigns truth values to variables in $\overline{X}$. Then $S(X, v)$ is the clause set that results when each x_j occurring in $\overline{X}$ is fixed to the value $v(x_j)$. That is, $S(X, v)$ is the result of removing from S every clause containing a negated variable x_j for which $v(x_j) = F$, every clause containing a posited x_j for which $v(x_j) = T$, every negated occurrence of a variable x_j for which $v(x_j) = T$, and every posited occurrence of a variable x_j for which $v(x_j) = F$.

Since $S(X, v) \subset S(X)$ for any assignment v, $S(X, v)$ is renamable Horn if $S(X)$ is. Also S is satisfiable if and only if $S(X, v)$ is satisfiable for some v. Thus if $S(X)$ is renamable Horn, we can check S for satisfiability in $O(2^{|\overline{X}|}L)$ time, where L is the number of literals in $S(X)$, by enumerating the $2^{|\overline{X}|}$ assignments v. We naturally prefer $S(X)$ to be a maximum embedded RHS of S, so that $|\overline{X}|$ is as small as possible.

3 Finding a Maximum Embedded RHS

The problem of finding a maximum embedded RHS of a set S of m clauses containing n variables can be formulated as the following set packing problem.

$$\begin{array}{rl} \max & \sum_j y_j + \overline{y}_j \\ \text{s.t.} & Ay + B\overline{y} \leq e \\ & y_j + \overline{y}_j \leq 1, \ \text{all } j \\ & y_j, \overline{y}_j \in \{0, 1\}, \ \text{all } j, \end{array} \tag{1}$$

Here e is a vector of 1's and A and B are 0-1 $m \times n$ matrices. We define A by letting $a_{i,j} = 1$ precisely when the literal x_j occurs in clause i, and B by letting $b_{ij} = 1$ precisely when $\neg x_j$ occurs in clause i.

We interpret $y_j = 1$ as indicating a positive scaling for x_j and $\overline{y}_j = 1$ as indicating a negative scaling. If $y_j = \overline{y}_j = 0$, we omit variable x_j altogether. Thus problem (1) finds a largest set of variables that, when rescaled in some fashion, yield a Horn set. In other words, it finds a maximum embedded RHS. We have shown the following.

Theorem 1 *If $(y, \overline{y})$ solves (1), then $S(\{x_j \mid y_j + \overline{y}_j = 1\})$ is a maximum embedded RHS of S.*

It is well known that a set packing problem,

$$\begin{array}{rl} \max & \sum_j z_j \\ \text{s.t.} & Qz \leq e \quad z_j \in \{0,1\}, \end{array} \tag{2}$$

can be formulated as a maximum clique problem on a graph. The graph contains a node for each z_j and an arc (z_j, z_k) whenever columns j and k of Q are orthogonal. A clique is a set of nodes in which every pair is connected by an arc. If C is a clique of maximum size, then z given by $z_j = 1$ if node $z_j \in C$, and $z_j = 0$ otherwise, is an optimal solution of the set packing problem. In the present case $z = (y, \overline{y})$ and $Q = \begin{bmatrix} A & B \\ I & I \end{bmatrix}$.

We can not only solve the maximum embedded RHS problem as a set packing problem but can do the reverse as well. Given an $m \times n$ set packing problem (2), consider the clause set,

$$\begin{array}{l} \bigvee_{q_{ij}=1} x_j, \quad i = 1, \ldots, m, \\ \neg x_1 \vee \ldots \vee \neg x_n \vee y_1 \vee \neg y_2, \\ \neg x_1 \vee \ldots \vee \neg x_n \vee \neg y_1 \vee y_2. \end{array} \tag{3}$$

None of the x_j's can be negatively scaled in the maximum embedded RHS of (3). Clearly, at most one can be negatively scaled, and if one is, y_1 and y_2 must be deleted. In this case one could do better by deleting the negatively scaled variable and retaining y_1 and y_2. Thus the maximum number of variables retained in (3) is the maximum number of z_j's equal to 1 in a solution of (2). Since (2) is NP-hard, we have,

Theorem 2 *Finding a maximum embedded RHS is NP-hard.*

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