Gödel's Explorations in Terra Incognita

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This paper presents an introduction to and an overview of the celebrated incompleteness theorems of Kurt Gödel. Starting with Richard's paradox, the logical antimony that motivated Gödel to look at encoding meta-mathematics in the arithmetic of integers, this overview traces the highlights of the encoding and the gist of Gödel's final arguments.

Introduction

In 1931, a young mathematician at the University of Vienna, Kurt Gödel, published a relatively short paper in German with the title translating as, 'On formally undecidable propositions of *Principia Mathematica* and related systems [1]'. The paper represents one of the most important advances in logic in modern times. It attacked a central problem in the foundations of mathematics. Gödel showed that the axiomatic method of deductive reasoning has certain inherent limitations. In particular, he proved that even the ordinary arithmetic of integers can never be fully axiomatized.

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Gödel actually proved two substantive results. He showed that it is impossible to give a meta-mathematical proof of the consistency of the arithmetic of integers unless we assume rules of inference that are essentially different from the transformation rules used in deriving theorems in arithmetic. Incidentally, meta-mathematics was defined by Hilbert to be the language that is about mathematics. The statement "$0 \neq 0$" is a mathematical statement, but the statement "$0 \neq 0$ is not a theorem" is meta-mathematical. "Arithmetic is consistent" is a more interesting meta-mathematical statement.

Gödel's second incompleteness result is even more compelling. He proved that given any consistent set of arithmetical axioms, there are statements in arithmetic that are true but cannot be derived from the axioms. Thus he demonstrated a fundamental limitation in the power of the axiomatic method. Even if the axioms of arithmetic were augmented by an indefinite number of other true statements, there will always be further arithmetical truths that are not formally derivable from the augmented set of axioms.

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The original 1931 paper is difficult to read. Forty-six definitions together with several important preliminary results have to be comprehended before the main results can be accessed. A number of recent popular books like *Gödel, Escher, Bach: An Eternal Golden Braid* by Douglas Hofstadter (1979) and *Emperor's New Mind* by Roger Penrose (1990), describe these results of Gödel and some of their consequences. The presentation here closely follows that of Nagel and Newman in their classical 1958 book on Gödel's paper (see [2]). We will only try to get to the gist of the arguments in this overview. But first let us understand how Gödel was motivated to think about encoding meta-mathematics. This came from examining the reasoning involved in the presentation of Richard's paradox.

Richard's Paradox

Many of you may be familiar with the 'Liar paradox'. Let s be the sentence 'This sentence is false'. Since the phrase 'this sentence' refers to s, we have the following chain of equivalences: s iff 'this sentence is false' iff 's is false' iff not s. Thus we have shown that "truth" for English sentences is not definable in English.

In 1905, Jules Richard (a French mathematician) attempted to mathematize the liar paradox. He argued as follows. Consider a language in which purely arithmetical properties of the natural numbers can be formulated and defined. Since each such definition will contain only a finite number of words from a finite alphabet, we can sort the definitions and put them in a serial order (using for example, lexical ordering). Hence we can

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associate a unique positive integer (indicating its position) with each definition in the list.

Suppose 15 corresponds to the definition "is a multiple of 5" and 17 to "is a composite number". Notice that 15 actually satisfies the property correlated with it while 17 does not. This suggests a new definition:

"n is Richardian as shorthand for 'n does not have the property designated by the n^{th} definition in our list of definitions'.

Let x denote the number assigned to the above definition (of being Richardian). The paradox reveals itself in the question "Is x Richardian?" since x is Richardian iff x does not have the property (of being Richardian). Thus the statement "x is Richardian" is both true and false.

The observant reader may have caught the fallacy in the arguments above. The definition of the property of being Richardian (in the box above) is a property that involves notation and is therefore meta-mathematical and our list of definitions were restricted to purely mathematical properties. Hence it does not belong in our list at all and the paradox evaporates. The construction however motivated Gödel to consider how it may be possible to map meta-mathematical statements within the arithmetic of integers.

Gödel Numbering

The first step was to choose a formal calculus within which all the customary arithmetic notations can be expressed. Gödel used an adaptation of such a system from the *Principia Mathematica* (the magnum opus of Russell and Whitehead). The formulae are built up from a set of elementary symbols. A set of formulae is identified as the axioms and the theorems are just new formulae derivable from the axioms with the help of a set of rules of inference.

Gödel's first task was to encode all this within the arithmetic of integers. He showed that it was possible to assign a unique number to each elementary symbol, each formula (a finite sequence of symbols) and each proof (a finite sequence of formulae). This number is called the Gödel number of a symbol, formula, or proof. The numbering is quite straightforward and several alternate schemes are possible. Gödel's original scheme ran as follows:

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1. The numbers 1, 2, ..., 10 were reserved for the elementary symbols.

"~" not
"Ú" or
"É" If ... then
"$" There is an
"=" equals
"0" zero
"s" Immediate successor of
"(" punctuation
")" punctuation
"," punctuation

Note that universal quantifier "for all x" is denoted by "(x)" in this notation.

2. Numerical variables, or variables for which numbers or numerical expressions can be substituted, are assigned distinct prime numbers larger than 10.

3. Logical variables, or variables for which formulae may be substituted, are assigned squares of prime numbers larger than 10.

4. Predicates are assigned cubes of prime numbers larger than 10.

5. A formula is a finite sequence of symbols, each of which has an assigned Gödel number. The Gödel number of the formula is taken to be the product of the first k primes, each raised to the power equal to the Gödel number of the associated symbol, where k is the length of the formula. So the formula (\$$x$)(s$x = y$),

read as "there is an x whose immediate successor is y" is encoded as:

$$2^8 \times 3^4 \times 5^{11} \times 7^9 \times 11^8 \times 13^7 \times 17^{11} \times 19^5 \times 23^{13} \times 29^9,$$

where 11 and 13 are the Gödel numbers of x and y, respectively.

6. Similarly, a finite sequence of formulae $F_1, F_2, \ldots$ is assigned the Gödel number

$2^{g_1} \times 3^{g_2} \times \ldots$, where $g_1,\ g_2, \ldots$ are the Gödel numbers of $F_1, F_2, \ldots$

The above scheme gives us a unique arithmetic encoding of the formal calculus. The reader should convince herself that the above scheme does in fact give a unique Gödel number for each entity in the calculus and that by the simple use of the fundamental theorem of arithmetic (unique prime factorization of a number) we can reconstruct the symbol/variable/formula/sequence of formulae, as the case may be.

Arithmetization of Meta-mathematics

The second step in the encoding achieved by Gödel was very subtle and ingenious. Since every expression in the calculus is associated with a unique Gödel number, a meta-mathematical statement about expressions and their relations could perhaps be represented by a statement about their Gödel numbers and their arithmetic relations. If this could be pulled off, we would have an arithmetization of meta-mathematics. This is exactly what Gödel did. The details are far too intricate to be presented here. But we should observe that the arithmetization of two specific meta-mathematical relations were critical to Gödel's triumph.

Since every expression is associated with a unique Gödel number, a meta-mathematical statement about expressions and their relations could perhaps be represented by a statement about their Gödel numbers and their arithmetic relations.

- Dem(x, z), shorthand for an arithmetic relation (of the form $f(x,z)=0$) which models the meta-mathematical statement "The sequence of formulae with Gödel number x is a proof of the formula with Gödel number z". Similarly $\sim$Dem(x, z) is the formal contradictory of Dem(x, z).

- sub$(m, 13, m)$, shorthand for an arithmetic function (of the form h $(m, 13)$) which represents the value, "The Gödel number of the formula obtained from the formula with Gödel number m, by substituting the numeral m for the variable with Gödel number 13. Note that we could have used any prime larger than 10 instead of 13 and this definition is still meaningful. Let y denote the variable symbol with Gödel number 13. Then, sub$(y, 13, y)$ is an interesting special case.

The Assault

Equipped with the encoding machinery, Gödel's results were derived as follows:

1. Construct an arithmetic formula G that represents the meta-mathematical statement "The formula G is not provable".
2. Prove that G is provable iff ~G is.
3. G is a true arithmetic formula, even though it is not provable. Hence, the axioms of arithmetic are incomplete.
4. Even if additional axioms were assumed (making the truth of G decidable), another true but formally undecidable formula can always be constructed.
5. Construct an arithmetic formula A that represents the meta-mathematical statement "Arithmetic is consistent" and prove that the formula "A É G" is formally provable. Deduce that A is not provable.

The real import of Gödel's result is that if arithmetic is consistent, its consistency cannot be established by any meta-mathematical reasoning that can be represented within the formalism of arithmetic.

Notice the similarity of steps 1 and 2 above with the approach in the discussion of Richard's paradox.

The constructions needed in this scheme (of G and A) are carried out using Dem and sub, which were defined above. G is given by the formula

$$(x) \sim \mathrm{Dem}(x, \mathrm{sub}(n, 13, n)),$$

where n is the Gödel number of the formula

$$(x) \sim \mathrm{Dem}(x, \mathrm{sub}(y, 13, y)).$$

Note that sub(n, 13, n) is the Gödel number of G.

Finally, A is given by the formula

$$(\$y)(x) \sim \mathrm{Dem}(x, y),$$

which says that there must be some formula y that is not provable from any sequence of formulae x. Note that from an inconsistent axiom system, everything is provable.

The real import of Gödel's result is that if arithmetic is consistent, its consistency cannot be established by any meta-mathematical reasoning that can be represented within the formalism of arithmetic. This led to the conclusion that the prospect of deductive systems with an absolute proof of consistency from finite axiomatic bases as envisioned by the Hilbert programme, was extremely unlikely to be realized.

In contrast with arithmetic of the integers, the theories of real numbers, complex numbers and Euclidean geometry do have complete axiomatizations. The reason they escape the clutch of incompleteness is that they cannot encode and compute with finite sequences.

A theory is said to be adequate if it is strong enough to encode finite sequences of numbers and define simple sequence operations such as concatenation. Set theory is adequate since numbers can be defined in set theory. Even certain weak number theories are adequate. Gödel's theorem extends to such theories and the statement roughly translates as: Any adequate axiomatizable theory is incomplete.

There are several philosophical aspects to the interpretations of Gödel's results. This is not the appropriate place to delve into them. An interested reader is directed to the book by Hao Wang listed in the Suggested Reading [3].

Suggested Reading

[1] Kurt Gödel, On Formally Undecidable Propositions of *Principia Mathematica* and Related Systems, Translated by B Meltzer, Basic Books, 1962.

[2] Ernst Nagel and James & James Newman, *Gödel's Proof*, New York University Press, 1958.

[3] Hao Wang, *A Logical Journey: From Gödel to Philosophy*, MIT Press, 1996.

[4] John W Dawson, Gödel and the Limits of Logic, *Scientific American*, June 1999.

[5] Dale Myers, *Gödel's Incompleteness Theorem*, http://www.math.hawii.edu/~dale/godel/godel.html

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