175 Years of Linear Programming

Part 2. Pivots in Column Space

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The simplex method has been the veritable workhorse of linear programming for five decades now. An elegant geometric interpretation of the simplex method can be visualised by viewing the animation of the algorithm in a *column space representation*. In fact, it is this interpretation that explains why it is called the simplex method. The extreme points of the feasible region (polyhedron) of the linear programme can be shown to correspond to an arrangement of simplices in this geometry and the pivoting operation to a physical pivot from one simplex to an adjacent one in the arrangement. This paper introduces this vivid description of the simplex method as a tutored dance of simplices performing "pivots in column space".

1 Introduction

In part one of this series (The French Connection, Resonance, October 1998), we saw the simple but powerful idea in Joseph Fourier's syntactic rule for elimination of a variable from a system of linear inequalities. Recursive application of the rule computes the projections of convex polyhedra to lower dimensional subspaces and thus solves linear programming problems. We also saw how this could be used to develop the duality theory of linear programming problems. We ended with a technique for generating all the extreme points and extreme rays of a polyhedron using the Fourier elimination method.

Consider a polyhedron $\mathcal{K} = \{x \in \Re^n : Ax = b,\ x \geq 0\}$. Now $\mathcal{K}$ cannot contain an infinite (in both directions) line since it lies within the non-negative orthant of $\Re^n$. Such a polyhedron is called a *pointed* polyhedron since its underlying recession cone is pointed (has an apex). Given a pointed polyhedron $\mathcal{K}$ we observe that

- If $\mathcal{K} \neq \emptyset$ then $\mathcal{K}$ has at least one extreme point.
- If $\min\{cx : Ax = b,\ x \geq 0\}$ has an optimal solution then it has an optimal extreme point solution.

These observations together are sometimes called the fundamental theorem of linear programming since they suggest simple finite tests for both solvability and optimisation by examining the finite number of extreme points. To generate all extreme points of $\mathcal{K}$, in order to find an optimal solution, is an impractical idea (see Box 1 for a reason).

However, we may try to run a partial search of the space of extreme points for an optimal solution. A simple local improvement search strategy of moving from one extreme point to an adjacent extreme point until we get to a local optimum is nothing but the simplex method of linear programming. Note that if none of the edges leaving the extreme point are improving directions of movement then there can be no direction of movement from the extreme point, pointing into the feasible region, that is improving in objective value. Hence this extreme point is a true local optimum of the linear programme.

The local optimum also turns out to be a global optimum because of the convexity of the polyhedron $\mathcal{K}$ and linearity of the objective function cx. The proof is really simple. If a local optimum is not global then all solutions on the line segment, between this local solution and any global solution, strictly improve on the objective value of the local optimum. Since, by convexity, the line segment is entirely within the polyhedron, we have a contradiction of the local optimality assumption.

2 The Simplex Method

Let us consider the following linear programming problem:

$$
\begin{array}{cll}
 & maximise & 15x_1 + 5x_2 + 13.5x_3 + 8x_4 + 11x_5 \\
(P) & s.t & 3x_1 + 6x_2 + 4.5x_3 + x_4 + 8x_5 + x_6 = 5 \\
 & & x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 1 \\
 & & x_1, x_2, x_3, x_4, x_5, x_6 \geq 0
\end{array}
$$

The special feature of (P) is that $\sum_{j=1}^{6} x_j = 1$; $x_j \geq 0$ $(j = 1, ..., 6)$ is satisfied by all feasible solutions. (`More about this later.`) Therefore, we can multiply the r.h.s coefficient 5, of the first constraint, with $\sum_{j=1}^{6} x_j$ to obtain

$$
\begin{array}{cll}
(P) & max & 15x_1 + 5x_2 + 13.5x_3 + 8x_4 + 11x_5 \\
 & s.t. & 2x_1 - x_2 + 0.5x_3 + 4x_4 - 3x_5 + 4x_6 = 0
\end{array}
$$

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 1$$
$$x_j \geq 0 (j = 1, \cdots, 6)$$

We now have a linear programme of the form

$$(P) \quad \max\{\sum_j c_j x_j \ : \ \sum_j a_j x_j = 0, \ \sum_j x_j = 1, \ x_j \geq 0 \ \forall j\}$$

which can have an arbitrary number of variables, with two equality constraints and all variables constrained to be non-negative. The feasible region is a convex polytope with extreme points defined by a suitable number of hyperplanes of the form $x_j = 0$ such that their common intersection with the affine set defined by the two equations is a unique point, the extreme point. If there are n variables and the equations are reduced to be of full linear rank m, then every extreme point, of the feasible region, is determined by setting exactly $(n - m)$ variables, called the *Non-Basic Variables*, to zero. If we were to arbitrarily choose the $(n - m)$ non-basics, the remaining m variables, called the *Basic Variables*, are evaluated by solving the non-singular residual $m \times m$ linear system. If the basic variables evaluate to non-negative values, we have an extreme point and we call it a *Basic Feasible Solution*. If not, the basic solution (with non-basics set to zero) is not feasible in the linear programme.

Now let us apply all this to our example. Here n is six and m is two. If we were to take $\{x_1, x_3, x_4, x_5\}$ to be the non-basic set and $\{x_2, x_6\}$ to be the basics, we get $x_2 = 0.8$ and $x_6 = 0.2$. We display the solved form of this basic feasible solution in a tableaux or a dictionary as follows. The first two equations display the basic variables in solved form. The objective function, displayed below the line, is expressed only in terms of the non-basics since the basics have been substituted by their solved forms.

$$\begin{array}{rrrrr} x_2 = & 0.8 - 0.4x_1 - 0.7x_3 & & & -1.4x_5 \\ x_6 = & 0.2 - 0.6x_1 - 0.3x_3 & -x_4 & & +0.4x_5 \\ \hline & & & & (1) \\ z = & 4.0 + 13x_1 + 10x_3 & +8x_4 & & +4x_5 \end{array}$$

From the bottom row of the above dictionary it is evident that the objective value of the current basic feasible solution is 4 and if we increase the value of the non-basic x_1, the objective value would increase. (Here this would be

true of any of the non-basics, not just x_1. Dantzig's Rule dictates that we choose x_1 because it has the largest positive coefficient in the bottom row of the dictionary.) So we increase x_1 from its slumber at value 0 and the first two equations indicate that as soon as x_1 reaches $\frac{1}{3}$ the values of x_2 hits $\frac{2}{3}$ and x_6 hits 0. We declare $\{x_2, x_1\}$ the basics, i.e. we have swapped x_1 and x_6. This is called a *pivot* and the new dictionary after the pivot, representing the new basic feasible solution, is given by

$$\begin{aligned} x_2 &= \frac{2}{3} - \frac{1}{2}x_3 + \frac{2}{3}x_4 - \frac{5}{3}x_5 + \frac{2}{3}x_6 \\ x_1 &= \frac{1}{3} - \frac{1}{2}x_3 - \frac{5}{3}x_4 + \frac{2}{3}x_5 - \frac{5}{3}x_6 \\ \hline z &= 8\frac{1}{3} + 3\frac{1}{2}x_3 - 13\frac{2}{3}x_4 + 12\frac{2}{3}x_5 - 21\frac{2}{3}x_6 \end{aligned} \tag{2}$$

Now the objective value has improved to $8\frac{1}{3}$ and we check if we can improve further. Dantzig's Rule picks x_5, the non-basic with the largest positive coefficient in the objective row of the dictionary, as the variable to enter the basis and x_2 leaves. The resulting dictionary is then,

$$\begin{aligned} x_5 &= 0.4 - 0.6x_2 - 0.3x_3 + 0.4x_4 + 0.4x_6 \\ x_1 &= 0.6 - 0.4x_2 - 0.7x_3 - 1.4x_4 - 1.4x_6 \\ \hline z &= 13.4 - 7.6x_2 - 0.3x_3 - 8.6x_4 - 16.6x_6 \end{aligned}$$

We have found an extreme point solution $(0.6, 0, 0, 0, 0.4, 0)$ with a corresponding basis $\{x_5, x_1\}$ such that none of the non-basics are worth increasing any more. Thus we have located a local maximum, and by arguments made above, a global maximum for the linear programme.

Our illustration of the simplex method with dictionaries of an example should motivate the reader to generalise the method to work on any linear programming problem. A few assumptions implicit in the example need some comment.

REMARKS:

1. The form of the linear programme assumed here is non-standard but we now show that this form is completely general. If a linear programme has a finite optimum, the values of the variables are finite.

Consequently, the constraint $\sum_{j=1}^{n} x_j \leq M$ where M is a sufficiently large positive constant, can be appended to the original linear programme, to obtain a bounded problem. Now consider a bounded linear programme:

$$\max\{\sum_{j=1}^{n} c_j x_j \ : \ \sum_{j=1}^{n} a_{ij} x_j = b_i \ \forall i; \ \sum_{j=1}^{n} x_j + x_0 = M; \ x_j \geq 0 \ \forall j\}$$

If the artificial variable x_0 goes to zero at optimality in the linear programme, it is an indication that the original problem might have no finite optimum. The transformation $u_j = \frac{1}{M} x_j, \ j = 1, 2, \cdots, n+1$ together with the right hand side coefficients b_i replaced by $b_i \times \sum_{j=1}^{n+1} u_j$ gives the form of the linear programme assumed here.

2. We assumed that an extreme point (a basic feasible solution) of the polyhedron is available. This presupposes that the solvability of the constraints has been established. These assumptions are reasonable since we can formulate the solvability problem as an optimisation problem, with a self-evident extreme point, whose optimal solution either establishes unsolvability of $Ax = b, \ x \geq 0$, or provides an extreme point of $\mathcal{K}$. Such an optimisation problem is usually called a Phase I model.The point being, of course, that the simplex method, as described above, can be invoked on the Phase I model and if successful, can be invoked once again to carry out the intended maximisation of cx. There are several different formulations of the Phase I model that have been advocated. Here is one.

$$\min\{v_0 \ : \ Ax + bv_0 = b, \ x \geq 0, \ v_0 \geq 0\}$$

The solution $(x, v_0)^T = (0, \cdots, 0, 1)$ is a self-evident extreme point and $v_0 = 0$ at an optimal solution of this model is a necessary and sufficient condition for the solvability of $Ax = b, \ x \geq 0$.

3 The Column Space Geometry

The simplex method after its initial formulation was shelved by George Dantzig because he felt that it would be terribly inefficient as it wandered about the boundary of higher dimensional polyhedra (see Figure 1). However, a fresh look at the simplex method in the "column space geometry" indicated to Dantzig that the simplex method may be efficient after all.

To realise the column space interpretation in our example we identify the column $\binom{a_j}{c_j}$ with each variable x_j. Thus,

$$\begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\ \binom{2}{15} & \binom{-1}{5} & \binom{0.5}{13.5} & \binom{4}{8} & \binom{-3}{11} & \binom{4}{0} \end{array}$$

which can be plotted as six points {[1], [2], ..., [6]} on (a, c) plane as indicated by the annular marks in Figure 2. For any feasible solution $\bar{x}$ we know that

$$\sum_{j=1}^{6} \bar{x_j} = 1;\ \bar{x_j} \geq 0\ (j = 1, 2, \cdots, 6);\ \sum_{j=1}^{6} a_j \bar{x_j} = 0$$

Since $\{\bar{x_j}\}$ is a set of convex multipliers, the convex combination of the points [1], [2], ..., [6] must result in a point on the **c axis**. The value of the ordinate (c-intercept) indicates the objective value attained by this particular convex combination. If some of the $\bar{x_j}$ are zero, it just means that the corresponding points are not involved.

In a basic feasible solution all but two of the $\bar{x_j}$ are zero. The two basic variables define a line segment in Figure 2 that must cross the **c axis**. A line segment is also called a 1-simplex since it is the affine hull of two affinely independent points. Thus the pairs ([2],[6]), ([2],[1]) and ([5],[1]) correspond to three basic feasible solutions - exactly the same three that we saw as dictionaries in the previous section.

Now it is easy to bring to life the machinations of the simplex method. We grasp the simplex ([2],[6]), notice that [1] is the point most vertically above this simplex. let go of [6] (since [6] and [1] are on the same side of the **c-axis**) and pivot to the simplex ([2],[1]). Now, [5] is the point most vertically above ([2],[1]). We drop [2] and pivot to ([5],[1]). We need go no further since all points are below ([5],[1]) and we declare the basis $\{x_5, x_1\}$ optimal. The c-intercept (ordinate) of the optimal simplex is 13.4 which is the optimal value of the linear programme.

Where are Dual multipliers?

Recall from Part 1 of this series that associated with any linear programme is a dual linear programme and the two attain the same objective value. In our example, this turns out to be

$$(P) \quad z = max \quad 15x_1 + 5x_2 + 13.5x_3 + 8x_4 + 11x_5$$

$$
\begin{aligned}
& s.t. && 2x_1 - x_2 + 0.5x_3 + 4x_4 - 3x_5 + 4x_6 = 0 \\
& && x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 1 \\
& && x_1, x_2, x_3 x_4, x_5, x_6 \geq 0 \\
& && x^* = (0.6, 0, 0, 0, 0.4, 0)\,;\ z = 13.4 \ \texttt{optimal} \\
(D) \quad & w = min && y_2 \\
& s.t. && 2y_1 + y_2 \geq 15 \\
& && -y_1 + y_2 \geq 5 \\
& && 0.5y_1 + y_2 \geq 13.5 \\
& && 4y_1 + y_2 \geq 8 \\
& && -3y_1 + y_2 \geq 11 \\
& && 4y_1 + y_2 \geq 0 \\
& && y_1, y_2, unrestricted \\
& && y^* = (0.8, 13.4)\,;\ w = 13.4 \ \texttt{optimal}
\end{aligned}
$$

A natural question to ask is if the column space representation of the simplex method allows us to infer the optimal solution to the dual problem (D) as well. Indeed it does. The equation of line determined by the final simplex ([5],[1]) is given by

$$\boxed{c = 0.8a + 13.4 = y_1^* a + y_2^*}$$

Notice that the optimal solution and objective value of (D) can be read off as coefficients of this equation.

Remark:

We note that this simplex interpretation of the simplex method carries over to linear programmes that have an arbitrary (say m) equality constraints along with the simplex constraints. In this case we can visualise in $(m+1)$- dimensional space coordinatised by $a_1, a_2, \cdots, a_m, c$. Each column x_j gives a point $(a_{1j}, \cdots, a_{mj}, c_j)^T$ in this coordinate frame. The correspondences are as indicated in the table:

Algebraic		**Geometric**
Basic Feasible Solution	$\longleftrightarrow$	m-simplex spanning **c-axis**
Updated Objective Coefficients	$\longleftrightarrow$	Vertical (c-direction) distances to m-Simplex $\begin{cases} +\text{ve if } \texttt{point} \text{ above } \texttt{simplex} \\ \text{-ve if } \texttt{point} \text{ below } \texttt{simplex} \end{cases}$
Simplex Pivot	$\longleftrightarrow$	Choose point `above` the hyperplane defined by the simplex. Pivot to a new simplex while continuing to span **c-axis**.
Dual Variables	$\longleftrightarrow$	Coefficients of hyperplane equation determined by m-simplex.
Optimality Test	$\longleftrightarrow$	All points on or below the hyperplane.
Degeneracy	$\longleftrightarrow$	A proper face of the m-simplex spans the **c-axis**.

Degeneracy in a linear programme occurs when in a basic feasible solution, one or more of the basic variables evaluate to zero. The next section is devoted to the study of degeneracy.

4 Degenerate Linear Programmes

It is possible to have linear programmes for which an extreme point is geometrically over-determined (degenerate) i.e., there are more than d inequalities of the linear programme which hold as equalities at the extreme point, where d is the dimension of the feasible polyhedron, and several combinations of the equations are of rank d. In such a situation, there would be several feasible bases corresponding to the same extreme point. When this happens, the linear programme is said to be *degenerate*.

Consider the following example.

$$\begin{aligned} max \quad & x_1 + x_2 + 4x_3 \\ s.t. \quad & x_1 + 4x_3 + x_4 = 4 \\ & x_2 + 4x_3 + x_5 = 4 \\ & x_1, x_2, \cdots, x_5 \geq 0 \end{aligned}$$

Figure 3 is the feasible polyhedron visualised in $x_1x_2x_3$-space. The four planes at the extreme point $(0, 0, 1)$ correspond to setting x_1, x_2, x_4 and x_5 to zero respectively. This extreme point is overdetermined. It can be shown that the simplex method with Dantzig's Rule would pivot from basis $\{x_3, x_5\}$ to $\{x_3, x_2\}$ while standing at this extreme point (a degenerate pivot).

There are two sources of non-determinism in the primal simplex procedure. The first involves the choice of the entering variable in a pivot. At a typical iteration there may be many candidates that are improving in the sense that the coefficient in the objective row of the corresponding dictionary is of the right sign. Dantzig's Rule, Maximum Improvement Rule, and Steepest Descent Rule are some of the many rules that have been used to make this choice of entering variable in the simplex method. There is, unfortunately, no clearly dominant rule and successful implementations exploit the empirical and analytic insights that have been gained over the years to resolve the edge selection (entering variable) nondeterminism in the simplex method.

The second source of non-determinism arises from degeneracy. When there are multiple feasible bases corresponding to an extreme point, the simplex method has to pivot from basis to adjacent basis by picking an entering basic variable (a psuedo edge direction) and by dropping one of the old ones. The degeneracy may force the entering variable to enter the basis but take a value of zero, indicating that we are still at the same extreme point. Now, if there are several old basic variables at value zero, there is a choice as to which one should leave the basis. Pathological examples have been constructed to show that a wrong choice of the leaving variables may lead to cycling in the sequence of feasible bases generated at this extreme point.

Near-cycling is a serious problem when linear programmes are highly degenerate as in the case of linear relaxations of many combinatorial optimisation problems. The Lexicographic Rule (Perturbation Rule) for choice of leaving variables in the simplex method is a provably finite method in which small (symbolic) perturbations of the r.h.s. cofficients force out the geometric overdetermination of a degenerate extreme point.A clever method proposed by Robert Bland preorders the columns of the matrix A. In case of non-determinism in either entering or leaving variable choices, Bland's Rule just picks the lowest index candidate. All cycles are avoided by this rule also.

5 Long Isotonic Simplex Paths

V. Klee and G.L. Minty exploited the sensitivity of the original simplex method of Dantzig, to projective scaling of the data, and constructed exponential examples for it. These examples were simple projective distortions of the hypercube to embed long isotonic (improving objective value) paths in the graph. Scaling is used in the Klee-Minty construction, to trick the choice of entering variable (based on most positive objective coefficient) in the simplex method and thus keep it on an exponential path. Later, several variants of the entering variable choice (best improvement, steepest descent, etc.) were all shown to be susceptible to similar constructions of exponential examples. Figure 4 shows the projectively distorted unit cube of the Klee-Minty construction with the long isotonic path marked on it. The figure corresponds to the formulation:

$$\begin{aligned} max \quad & 100x_1 + 1000x_2 + 10000x_3 \\ s.t. \quad & x_1 \leq 1 \\ & 0.2x_1 + x_2 \leq 1 \\ & 0.02x_1 + 0.2x_2 + x_3 \leq 1 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

While known variants of the simplex method can be tricked into following long paths, all these polyhedra also exhibit short paths (in the distorted cube, for example, a clairvoyant choice would move to the optimal corner in one step). An interesting question is whether they exist polyhedra which have no short simplex paths at all. This question is related the unresolved diameter conjecture for convex polytopes (see Box 2).

Despite its worst-case behaviour, the simplex method has been the veritable workhorse of linear programming for five decades now. This is because both empirical and probabilistic analyses indicate that the "average" number of iterations of the simplex method is just slightly more than linear in the dimension of the polyhedron. Also, hundreds of man-years have been devoted to optimising the engineering details in implementations of the simplex method - exploiting sparsity, matrix factorisations, parallelism and pivot rules (see the article by Bixby for some details).

The ellipsoid method was devised to overcome poor scaling in convex programming problems and therefore turned out to be the natural choice of an algorithm to first establish polynomial-time solvability of linear programming. Later, a young scientist of Indian origin, N.K. Karmarkar took

care of both projection and scaling simultaneously and arrived at a superior algorithm. His algorithm will be subject of the next article in this series.

6 Suggested Reading

R.E.Bixby, Progress in Linear Programming, *ORSA Journal on Computing*, Vol. 6, No. 1, (1994) 15-22.

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M.W.Padberg and M.R.Rao, The travelling salesman problem and a class of polyhedra of diameter two, *Mathematical Programming*, **7**, (1974) 32-45.

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Box 1: Number of Extreme Points

The simplex method walks along edge paths on the combinatorial graph structure defined by the boundary of convex polyhedra. These graphs are quite dense. Balinski's theorem (cf. Ziegler) states that the graph of d-dimensional polyhedron must be d-connected. A polyhedral graph can also have a huge number of vertices. We know that for any linear programme of dimension d and defined by k inequalities can have no more than $\binom{k}{d}$ extreme points. However, ingenious arguments due to David Gale and McMullen (cf. Ziegler) show that the number of extreme points can be as large as, but no larger than,

$$\left(\begin{array}{c} k - \lfloor \frac{d+1}{2} \rfloor \\ m - n \end{array} \right) + \left(\begin{array}{c} k - \lfloor \frac{d+2}{2} \rfloor \\ m - n \end{array} \right)$$

for a polytope in d dimensions defined by k linear inequalities.

Box 2. The Diameter Conjecture

A polyhedral graph is a graph in which the extreme points of the polyhedron are represented as vertices of the graph and edges of the polytope are represented as edges of the graph. The distance between two vertices of the graph is the minimum number of edges connecting the two vertices. The diameter of the graph and equivalently the diameter of the polytope is the maximum of the distances between all pairs of vertices of the graph. A polynomial bound on the diameter of polyhedral graphs is not known. The best bound obtained to date is $O(k^{1+\log d})$ of a polytope in d dimensions defined by k constraints. Hence it is no surprise that there is no known variants of the simplex method with a worst-case polynomial guarantee on the number of iterations.

> The unresolved *Hirsch conjecture* is that the diameter of a convex polytope is less than or equal to $m - d$ where d is the dimension of the polytope defined by m facets. Recall that facets are proper faces of a polyhedron that are of dimension one less than the polyhedron itself.

The unit hypercube of dimension d has $2d$ facets and hence satisfies the conjecture. D. Naddef has shown that the Hirsch conjecture is true for $0-1$ polytopes, i.e. polytopes whose vertex co-ordinates are 0 or 1. If the diameter of the polytope is large, the simplex method may require a large number of iterations. However, even if the diameter is small, the presence of degeneracy may force a large number of pivots in the simplex method. Polytopes associated with combinatorial problems tend to have a small diameter. For instance, Padberg and Rao have shown that the diameter of the travelling salesman polytope (hull of the incidence vectors of traveling salesman tours) is only two although the problem is very difficult to solve. In fact, given any system of linear inequalities, it is easy to construct the quasi-dual polytope (a pyramid) with diameter 2 such that solvability of the linear inequality system is equivalent to linear optimisation on the quasi-dual.

A Glossary

- `Convex Polyhedron`: The set of solutions to a finite system of linear inequalities on real-valued variables. Equivalently, the intersection of a finite number of linear half-spaces in $\Re^n$.
- `Polyhedral (Convex) Cone`: A special convex polyhedron which is the set of solutions to a finite system of *homogeneous* linear inequalities on real-valued variables.
- `Extreme Ray`: Any direction vector in which we can move and still remain in the polyhedron is called a *ray*. A ray is extreme if it cannot be expressed as a strict positive combination of two or more rays of the polyhedron.
- `Extreme Point`: A point in the polyhedron is extreme if it cannot expressed as a strict convex combination of two or more points of the polyhedron.
- `Dimension`: The dimension of a polyhedron is the affine rank of the polyhedron minus one. Equivalently, it is equal to dimension of the smallest affine space that contains it.
- d-`Simplex`: A simplex of d dimensions is the convex hull of $d + 1$ affinely independent points. A 1-simplex is a line segment, a 2-simplex a triangle, a 3-simplex a tetrahedron, and so on.
- `Basic Feasible Solution`: An algebraic representation of an extreme point for a linear programme with equality constraints and all non-negative variables.
- `Dictionary`: A tableaux of coefficients displaying a basic feasible solution of the linear programme.