175 Years of Linear Programming

Part 1. The French Connection

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"I don't want to bore you," Harvey said, "but you should understand that these heaps of wires can practically think - linear programming - which means that instead of going through all the alternatives, we have a hunch which is the right one."
From *The Billion-Dollar Brain* (1966) by Len Deighton

1 Introduction

The scheme of computing a solution to simultaneous linear equations by sequential elimination of variables followed by back substitution is familiar to most high school students. Elimination in *linear inequalities* (Fourier Elimination), on the other hand, is intimately related to polyhedral theory and aspects of linear programming that are not quite as familiar. In this first part, of a series of articles on linear programming, we will review the remarkable and prescient work on linear inequalities by Joseph Fourier [4, 5] that began 175 years ago. This is yet another shining example of his contributions to the foundations of engineering mathematics.

The two fundamental problems of linear programming (which are closely related) are:

- Solvability: This is the problem of checking if a system of linear constraints on real (rational) variables is solvable or not. Geometrically, we have to check if a (convex) polyhedron, defined by such constraints, is nonempty.

- optimisation: Linear Programming is usually defined as the problem of optimising a linear objective function over a polyhedron described by a system of linear constraints.

Our focus in this article will be largely on the solvability problem. In Section 3, we will see that optimisation is equivalent to solvability as a consequence of the duality theorem of linear programming.

2 Fourier's Elimination Method

Constraint systems of linear inequalities of the form $Ax \leq b$, where A is an $m \times n$ matrix of real numbers are widely used in mathematical modeling. Solving such a system is the essential task of linear programming. We now describe an elegant syntactic method for this problem known as Fourier Elimination.

2.1 Syntactics

Suppose we wish to eliminate the first variable x_1 from the system $Ax \leq b$. Let us denote

$$I^+ = \{i : A_{i1} > 0\} \quad I^- = \{i : A_{i1} < 0\} \quad I^0 = \{i : A_{i1} = 0\}$$

Our goal is to create an equivalent system of linear inequalities $\tilde{A}\tilde{x} \leq \tilde{b}$ defined on the variables $\tilde{x} = (x_2, x_3, \cdots, x_n)$.

1. If I^+ is empty then we can simply delete all the inequalities with indices in I^- since they can be trivially satisfied by choosing a large enough value for x_1. Similarly, if I^- is empty we can discard all inequalities in I^+.
2. For each $k \in I^+$, $l \in I^-$ we add $-A_{l1}$ times the inequality $A_k x \leq b_k$ to A_{k1} times $A_l x \leq b_l$. In these new inequalities the coefficient of x_1 is wiped out, i.e. x_1 is eliminated. Add these new inequalities to those already in I^0.
3. The inequalities $\{\tilde{A}_{i1}\tilde{x} \leq \tilde{b}_i\}$ for all $i \in I^0$ represent the equivalent system on the variables $\tilde{x} = (x_2, x_3, \cdots, x_n)$.

If all the inequalities are deleted in step 1 the equivalent system $\tilde{A}\tilde{x} \leq \tilde{b}$ is undefined and we declare the original system *strongly solvable*. Else, we repeat this construction with $\tilde{A}\tilde{x} \leq \tilde{b}$ to eliminate another x_j and so on until all variables are eliminated. If the final $\tilde{b}$ (after eliminating all n variables) is non-negative we declare the original (and intermediate) inequality systems as being solvable. Otherwise $\tilde{b} \not\geq 0$ and we declare the system unsolvable.

Proposition 2.1 *Fourier Elimination is correct.*

Proof: It suffices to show that $\tilde{x}$ is a solution for $\tilde{A}\tilde{x} \leq \tilde{b}$ if and only if there is some x_1 such that $(x_1, \tilde{x})$ is a solution for $Ax \leq b$. But this is evident from the following reformulation of $Ax \leq b$.

$$x_1 \geq \max_{l \in I^-}\{(-A_{l1})^{-1}(\sum_{j=2}^{n} A_{lj}x_j \; - \; b_l)\}\}$$

$$x_1 \leq \min_{k \in I^+}\{(A_{k1})^{-1}(-\sum_{j=2}^{n} A_{kj}x_j \; + \; b_k)\}\}$$

$$\sum_{j=2}^{n} A_{ij}x_j \leq b_i \;\; \forall \; i \in I^0$$

Now the elimination of x_1 is achieved by setting the maximum above to be no larger than the minimum. Thus for any $\tilde{x}$ solving $\tilde{A}\tilde{x} \leq \tilde{b}$, it is ensured that we will obtain a non-empty range for x_1 (since the greatest lower bound on x_1 is forced to be no larger than the least upper bound in $\tilde{A}\tilde{x} \leq \tilde{b}$). $\Box$

Note that the proof of Proposition 2.1 gives us a simple way of actually constructing a solution to the original system $Ax \leq b$ by tracing back through the elimination method as demonstrated in the example below. Also, the proof indicates that elimination as an algebraic technique implements the geometric operation of projection, a theme that we expand on in Section 4.

Example 2.2 *Let S denote the system of linear inequalities*

$$\{-x_1 + 2x_2 \leq 3, \; 2x_1 - 7x_2 \leq -15, \; -x_1 \leq 0, \; -x_2 \leq 0\}$$

Let us eliminate variable x_1. This results in the system

$$\{-7x_2 \leq -15, \; -3x_2 \leq -9, \; -x_2 \leq 0\}$$

If we take $x_2 = 5$ (which satisfies all three inequalities) and back substitute this value in S we obtain $\max\{0, 7\} \leq x_1 \leq \min\{10\}$ *as the corresponding range of feasible values for x_1.*

In the worst case, the Fourier elimination method can be quite inefficient. Let k be any positive integer and n the number of variables be $2^k + k + 2$.

If the input inequalities have lefthand sides of the form $\pm x_r \pm x_s \pm x_t$ for all possible $1 \leq r < s < t \leq n$ it is easy to prove by induction that after k variables are eliminated, by Fourier's method, we would have at least $2^{\frac{n}{2}}$ inequalities. This explosion in the number of inequalities can be observed on a wide variety of problems in practice and is a manifestation of the problem of *intermediate swell* that plagues many algorithms in symbolic computation.

Let us re-examine the stopping conditions of Fourier Elimination in terms of the final $\tilde{b}$.

- $(\tilde{b} \not\geq 0)$: The system $Ax \leq b$ is unsolvable.
- (`Undefined` $\tilde{b}$) : The system $Ax \leq b$ is strongly solvable.
- $(\min_i \tilde{b}_i > 0)$: The system $Ax \leq b$ is (simply) solvable and the polyhedron $\{x \in \Re^n : Ax \leq b\}$ is full-dimensional.
- $(\min_i \tilde{b}_i = 0)$: The system $Ax \leq b$ is solvable and contains implicit equations.

We will now take a closer look at these stopping conditions and see that they each provide useful insights on the structure of the underlying polyhedron $\{x \in \Re^n : Ax \leq b\}$.

2.2 Unsolvability and Farkas Lemma

The celebrated Farkas Lemma [3] is a simple consequence of analysing the stopping condition of Fourier Elimination when presented with an unsolvable system of linear inequalities. The lemma states the remarkable property that the unsolvability of a system of linear inequalities is directly related to the solvability of a "dual" system.

Farkas Lemma Exactly one of the alternatives

$$I.\ \ \exists\, x \in \Re^n \,:\, Ax \leq b \qquad II.\ \ \exists\, y \in \Re^m_+ \,:\, y^t A = 0,\, y^t b < 0$$

is true for any given real matrices A, b (where $\Re^m_+$ denotes the non-negative orthant of $\Re^m$).

Proof: Let us analyse the case when Fourier Elimination provides a proof of the unsolvability of a given linear inequality system $Ax \leq b$. The method clearly converts the given system into $RAx \leq Rb$ where RA is zero and Rb has atleast one negative component. Therefore there is some row of the non-negative matrix R, say $r \in \Re^m_+$, such that $rA = 0$ and $rb < 0$. Thus $\neg I$

implies II. It is easy to check that I and II cannot both be simultaneously true for fixed A, b. $\square$

Remark 2.3 *An alternate form of Farkas Lemma that we will have occasion to use in Section 3 states that:*

Exactly one of the alternatives

$$I. \ \exists \, x \in \Re^n_+ \, : \, Ax \le b \qquad II. \ \exists \, y \in \Re^m_+ \, : \, y^t A \ge 0, \, y^t b < 0$$

is true for any given real matrices A, b.

2.3 Strong Solvability

A solvable system of inequalities is said to be *strongly solvable* if any modification of the right-hand side constants of the inequalities results in a solvable set. This is a valuable property to detect since in one fell swoop we would have the answer to an entire family of solvability problems.

In the Fourier elimination method, the final $\tilde{b}$ may not be defined if all the inequalities are deleted by the monotone sign condition of step 1 while trying to eliminate some variable. In such a situation we declare the system $Ax \le b$ *strongly solvable*. This is a valid conclusion since, regardless of what b in $\Re^m$ is chosen, the last variable, say x_j, to be eliminated has only upper (lower) bounds depending on whether it appears with a positive (negative) coefficient in all the remaining inequalities. Thus for any values assigned to the other uneliminated variables, we can always choose a small (large) enough value for x_j that solves the system.

A geometric characterisation of strong solvability states that a system of linear inequalities $S = \{Ax \le b\}$ is strongly solvable if and only if the polyhedron $P = \{x : Ax \le b\}$ contains spheres of unbounded radii. The proof of this characterisation is left to the reader as an exercise.

Example 2.4 *In Example 2.2 if we try to carry out one more iteration of the Fourier method we see that x_2 is monotone negative in all three inequalities. Hence the system is actually strongly solvable. It is not difficult to see that the original system defines a two-dimensional cone that admits circular disks of arbitrary radius.*

2.4 Implicit Equations

An important problem in symbolic computation with linear constraints is the identification of implicit equations in a system of linear inequalities. In geometric terms, we need the affine hull of the convex polyhedron represented by the linear inequalities. Jean-Louis Lassez and Michael Maher [6] made the interesting observation that the detection of implicit equations by Fourier elimination is easily accomplished. Run the Fourier method until all variables have been eliminated. The implicit equalities are exactly those original constraints used in producing $\tilde{b}_i = 0$. This is best illustrated through an example.

Example 2.5 *Consider the system S of linear inequalities*

$$\{2x_1 - x_2 + x_3 \leq 2,\ x_1 - 3x_2 \leq -2,\ -x_1 + x_2 \leq 0,\ x_2 \leq 1\}$$

While eliminating x_1 the second and third inequalities generate the fifth inequality $(-2x_2 \leq -2)$. While eliminating x_2, the fourth and the fifth inequality generate $0 \leq 0$ which results in $\min_i\{\tilde{b}_i\} = 0$. Thus the second, third and fourth inequalities are actually implicit equations and the affine hull of P, the polyhedron corresponding to S, is given by $\{x \in \Re^3 : x_1 = x_2 = 1\}$ as seen in Figure 1.

Figure 1. Implicit Equations of a Polyhedron

3 The Linear Programming Duality Theorem

Building on polarity in cones and polyhedra, duality in linear programming is a fundamental concept which is related to both the complexity of linear programming and to the design of algorithms for solvability and optimisation. If we take the *primal* linear programme to be

$$(P) \quad \min_{x \in \Re^n} \{cx : Ax \geq b\}$$

there is an associated *dual* linear programme

$$(D) \quad \max_{y \in \Re^m} \{b^T y : A^T y = c^T,\ y \geq 0\}$$

and the two problems are related through several properties that we now recount.

Proposition 3.1 `(Weak Duality):` *For any $\hat{x}$ and $\hat{y}$ feasible in (P) and (D) (i.e. they satisfy the respective constraints), we have $c\hat{x} \geq b^T\hat{y}$.*

Proof: $c\hat{x} = \hat{y}^T A\hat{x} \geq \hat{y}^T b$ where the first (equality) relation holds from (D)-feasibility of $\hat{y}$ and the second (inequality) follows from (P)-feasibility of $\hat{x}$ and non-negativity of $\hat{y}$. □

The weak duality condition gives us a technique for obtaining lower bounds for minimization problems and upper bounds for maximization problems. Hence,

Corollary 3.2 *The linear programme (P) has a finite optimal solution if and only if its dual (D) does.*

If the linear programmes have finite optima, the inequality of the weak duality relation can be strengthened to an equality.

Theorem 3.3 `(Strong Duality):` *x^* and y^* are a pair of optimal solutions for (P) and (D) respectively, if and only if x^* and y^* are feasible in (P) and (D) (i.e. they satisfy the respective constraints)* `and` *$cx^* = b^T y^*$.*

Proof: Let x^* and y^* be a pair of feasible solutions to (P) and (D) respectively that satisfy $cx^* = b^T y^*$. It follows from the weak duality proposition that x^* and y^* are a pair of optimal solutions for (P) and (D) respectively.

To prove the "only if" direction of the theorem we assume that (P) and (D) are optimised by x^* and y^* respectively. Consider the two inequality systems.

$$(I)\ \{Ax \leq b,\ -A^T y \leq -c^T,\ b^T y - cx \leq 0,\ -x \leq 0,\ -y \leq 0\}$$

$$(II)\ \{A^T\alpha - c^T\lambda \geq 0,\ -A\beta + b\lambda \geq 0,\ b^T\alpha - c\beta < 0,\ \alpha \geq 0,\ \beta \geq 0,\ \lambda \geq 0\}$$

From Farkas Lemma we know that (I) or (II) must be solvable but not both. We demonstrate below that (I) must be solvable since (II) is unsolvable. But then it follows that (x^*, y^*) must solve (I) for if it does not and $(\bar{x}, \bar{y})$ does then $c\bar{x}$ would get above cx^* or $b^T\bar{y}$ would get below $b^T y^*$. Since $\bar{x}$ and $\bar{y}$ are (P) and (D) feasible, this would contradict the optimality of x^* and y^*.

It therefore remains to show that the inequality system (II) is unsolvable. Suppose otherwise, i.e. (α, β, λ) is a solution to (II) and let us consider two cases.

- $(\lambda = 0)$: In this case $A^T\alpha \geq 0,\ -A\beta \geq 0,\ b^T\alpha - c\beta < 0,\ \alpha \geq 0,\ \beta \geq 0$. If $b^T\alpha \geq 0$ then $c\beta > 0$ and the maximum of (P) is unbounded above which is impossible. Conversely if $b^T\alpha < 0$ then the minimum of (D) is unbounded below which is also impossible. This rules out this case.

- $(\lambda > 0)$: Since system (II) is homogeneous (right hand side constants are all 0) we may as well take $\lambda = 1$. But then (β, α) are feasible in (P),(D) with $b^T\alpha - c\beta < 0$ which is impossible since it violates weak duality. Hence this case is also ruled out.□

Remark 3.4 *From the above proof it is evident that we can simultaneously optimise (P) and (D) by solving the system of inequalities (I). Therefore the solvability of linear inequalities subsumes linear optimisation.*

Remark 3.5 *The strong duality condition above gives us a good stopping criterion for optimisation algorithms. It would be useful to have constructions for moving from dual to primal solutions and vice-versa. The necessary and sufficient conditions for optimality (which follow from Theorem 3.3) as given below, provide just that.*

`(Complementary Slackness):` *x^* and y^* are a pair of optimal solutions for (P) and (D) respectively, if and only if x^* and y^* are feasible in (P) and (D) (i.e. they satisfy the respective constraints)* and $(Ax^* - b)^T y^* = 0$.

Note that the properties above have been stated for linear programmes in a particular form. The reader should be able to check, that if for example the primal is of the form

$$(P') \quad \min_{x \in \Re^n} \{cx : Ax = b,\ x \geq 0\}$$

then the corresponding dual will have the form

$$(D') \quad \max_{y \in \Re^m} \{b^T y : A^T y \leq c^T\}$$

The tricks needed for seeing this is that any equation can be written as two inequalities, an unrestricted variable can be substituted by the difference of two non-negatively constrained variables and an inequality can be treated as an equality by adding a non-negatively constrained variable to the lesser side. Using these tricks, the reader could also check that dual construction in linear programming is involutory (i.e. the dual of the dual is the primal).

4 Projection: The Geometry of Elimination

We saw earlier that Fourier elimination of a variable in a linear inequality system actually constructs the projection or shadow of the convex polyhedron in the space that is diminished in dimension by one. Not surprisingly, the projection of a convex polyhedron is another convex polyhedron as described by the system of linear inequalities produced by the Fourier construction.

Figure 2. Variable Elimination and Projection

It is natural to wonder if elimination of a block of variables can be exceuted simultaneously - rather than one variable at a time. Indeed this is possible and in fact leads to a technique that is a much improved elimination method.

First let us identify the set of variables to be eliminated. Let the input system be of the form

$$P \;=\; \{\, (x, u) \in \Re^{n_1+n_2} \;\mid\; Ax + Bu \;\leq\; b \,\}$$

where u is the set to be eliminated. The projection of P onto x or equivalently the effect of eliminating the u variables is

$$P_x \;=\; \{\, x \in \Re^{n_1} \;\mid\; \exists\, u \in \Re^{n_2} \text{ such that } Ax + Bu \;\leq\; b \,\}$$

Now W, the *projection cone* of P, is given by

$$W \;=\; \{\, w \in \Re^{m} \;\mid\; wB = 0,\; w \geq 0\}.$$

A simple application of Farkas Lemma yields a description of P_x in terms of W.

Projection Lemma Let G be any set of generators (eg. the set of extreme rays) of the cone W. Then $P_x \;=\; \{\, x \in \Re^{n_1} \mid (gA)x \leq gb \;\;\forall\, g \in G \,\}$.

The lemma, sometimes attributed to Černikov [1], reduces the computation of P_x to enumerating the extreme rays of the cone W or equivalently the extreme points of the polytope $W \cap \{\, w \in \Re^m \mid \sum_{i=1}^{m} w_i = 1 \,\}$. We will see in the next section that Fourier Elimination can be used to solve this problem.

5 Implicit and Parametric Representations

A system of linear inequalities of the form $Ax \leq b$ represents a convex polyhedron $\mathcal{K}$, *implicitly*. It is implicit in that we are given the bounding halfspaces and the representation does not directly provide a scheme for generating points in $\mathcal{K}$. An explicit or *parametric* representation of $\mathcal{K}$ would require the lists of extreme points $\{p^1, p^2, \cdots, p^K\}$ and extreme rays $\{r^1, r^2, \cdots, r^L\}$ of $\mathcal{K}$. And then the convex multipliers $\{\alpha_1, \alpha_2, \cdots, \alpha_K\}$ and the positive cone multipliers $\{\mu_1, \mu_2, \cdots, \mu_L\}$ are the *parameters* that give us the representation:

$$\begin{aligned} \mathcal{K} \;=\; \{x \in \Re^n \;:\quad x \;&=\; \textstyle\sum_{i=1}^{K} \alpha_i p^i \;+\; \sum_{j=1}^{L} \mu_j r^j \\ \textstyle\sum_{i=1}^{K} \alpha_i \;&=\; 1, \\ \alpha_i \;\geq\; 0 \;\forall i,\;\; \mu_j \;&\geq\; 0 \;\forall j \;\} \end{aligned}$$

An organic role of Fourier Elimination is in obtaining an implicit representation of a convex polyhedron from a parametric representation. The parametric representation above is a system of linear equations and inequalities in x, α and μ. If we eliminate the α and μ variables from this system, we would obtain an implicit representation.

The converse problem of generating a parametric representation from a given implicit representation of a polyhedron can also be attacked with Fourier Elimination. In an intriguing paper, Paul Williams [8] shows that a dual interpretation of Fourier Elimination yields a scheme for enumerating the extreme rays and extreme points of a polyhedron defined by a system of linear constraints $\{Ax \leq b,\; x \geq 0\}$.

Picking an arbitrary $c^T \in \Re^n$, we start with the dual pair of linear programmes

$$(P) \qquad \max\{cx \;:\; Ax \leq b,\; x \geq 0\}$$

$$(D) \qquad \min\{b^T y \;:\; A^T y \geq c^T,\; y \geq 0\}$$

Now introduce a new variable z in (D) to get

$$(\mathrm{D}') \qquad \min\{\mathrm{z} \;:\; \mathrm{z} - \mathrm{A}^{\mathrm{T}}\mathrm{y} \geq \mathrm{c}^{\mathrm{T}},\; \mathrm{y} \geq 0\}$$

The linear programme dual to (D') is given by

$$(\mathrm{P}') \qquad \max\{\mathrm{cx} \;:\; \mathrm{Ax} - \mathrm{b}\alpha + \beta \;=\; 0,\; \alpha = 1,\; \mathrm{x} \geq 0,\; \alpha \geq 0,\; \beta \geq 0\}$$

Using Fourier Elimination on the constraints of (D') we eliminate all y variables and are left with bounds (lower and upper) on z. Fourier Elimination on (D') takes positive combinations of the constraints to eliminate y variables but at the cost, in general, of many new constraints. In (P') this corresponds to column operations to eliminate rows (constraints) at the cost of generating many new columns (variables).

At the completion of the elimination of y variables in (D'), we have all constraints of (P') eliminated except for the transformed equation representing the original normalizing constraint $\alpha = 1$ and nonegativity restrictions on the transformed variables. The extreme points and rays of the polyhedron defined by a single equation on nonnegative variables can be simply read off. If we revert the column operations to interpret these extreme points and rays in terms of the original x variables we will obtain the extreme points and rays of the polyhedron defined by the constraints of (P). Unfortunately, as seen in the example below, we may also obtain some non-extreme solutions which need to be recognized and discarded. (*Exercise:* Devise a test for recognizing non-extreme solutions).

Example 5.1 *To get the extreme points of the polyhedron*

$$\mathcal{K} \;=\; \{(x_1, x_2) \,:\, -x_1 + 2x_2 \leq 3,\; 2x_1 - 7x_2 \leq -15,\; x_1 \geq 0,\; x_2 \geq 0\}$$

we define the pair of linear programs (P') and (D') as described above. Using Fourier's method to eliminate the y variables in (D') we end up with the transformed version of (P') whose constraints have the form

$$\{2v_1 + v_2 + 0v_3 + 0v_4 = 1,\; v_j \geq 0,\; j = 1, 2, 3, 4\}$$

We read off the two extreme points

$$(\frac{1}{2}, 0, 0, 0) \;\; and \;\; (0, 1, 0, 0)$$

and two extreme rays

$$(0, 0, 1, 0) \;\; and \;\; (0, 0, 0, 1)$$

Inverting the column operations (transformations) we obtain $(3, 3)$ *&* $(10, 5)$ *as the candidate extreme points of* $\mathcal{K}$*, and* $(7, 2)$ *&* $(2, 1)$ *as the candidate extreme rays of* $\mathcal{K}$*. From Figure 3, it is evident that the candidate extreme point* $(10, 5)$ *is the only spurious candidate since it is not a corner point of* $\mathcal{K}$*.*

Figure 3. Extreme Points and Rays

Working with the extreme points of the feasible region of a linear programme is important for optimisation since we know that if an optimal solution exists then it does so at an extreme point. Searching through all possible extreme points to pick the best one is too laborious. We want to be able to execute a partial search to zero in on an optimal extreme point. This, in essence, is what the Simplex Method does for linear optimisation as we shall see in the next article in this series.

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A VERY BRIEF HISTORY OF LINEAR PROGRAMMING [7]

Linear programming has been a fundamental topic in the development of the computational sciences. The subject has its origins in the early work of L.B.J. Fourier on solving systems of linear inequalities, dating back to the 1820's. The revival of interest in the subject in the 1940's was spearheaded by G.B.Dantzig in USA and L.V.Kantorovich in the erstwhile USSR. They were both motivated by the use of linear optimisation for optimal resource utilization and economic planning.

The 1950's and 1960's marked the period when linear programming fundamentals (duality, decomposition theorems, network flow theory, matrix factorizations) were worked out in conjunction with the advancing capabilities of computing machinery.

The 1970's saw the realization of the commercial benefits of this huge investment of intellectual effort. Many large-scale linear programmes were formulated and solved on mainframe computers to support applications in industry (for example: Oil, Airlines) and for the state (for example: Energy Planning, Military Logistics).

The 1980's were an exciting period for linear programmers. The polynomial time-complexity of linear programming had just been established. A healthy competition between the simplex and Karmarkar's interior methods ensued which ultimately led to rapid improvements in both technologies. This combined with remarkable advances in computing hardware and software have brought powerful linear programming tools to the electronic desktop of the 1990's.

A Glossary

- `Convex Polyhedron:` The set of solutions to a finite system of linear inequalities on real-valued variables. Equivalently, the intersection of a finite number of linear half-spaces in $\Re^n$.
- `Polyhedral (Convex) Cone:` A special convex polyhedron which is the set of solutions to a finite system of *homogeneous* linear inequalities on real-valued variables.
- `Extreme Ray:` Any direction vector in which we can move and still remain in the polyhedron is called a *ray*. A ray is extreme if it cannot be expressed as a strict positive combination of two or more rays of the polyhedron.
- `Extreme Point:` A point in the polyhedron is extreme if it cannot expressed as a strict convex combination of two or more points of the polyhedron.