Nambu-Jona-Lasinio Model: A Phenomenological Lagrangian for the Strong Interactions

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We present qualitative arguments for the emergence of the Nambu-Jona-Lasinio model within Quantum Chromodynamics. Then within this model we sketch the derivation of the chiral meson model including the Wess-Zumino term. We also discuss the perturbative and non-perturbative spectrum.

Introduction and the Nambu-Jona-Lasinio model

In this article I will discuss the Nambu-Jona-Lasinio (NJL) model[1] in the context of some recent developments in strong interactions physics which addressed themselves to the question of the effective low energy lagrangian of quantum chromodynamics (QCD). This is the subject of the nonlinear chiral lagrangian, the Wess-Zumino term and topological solitons.[2),3)] Here we present qualitative arguments for the emergence of the NJL model as an effective field theory at an intermediate length scale. Then within this model we will describe the derivation of the chiral meson model and the Wess-Zumino term. The resulting nonlinear lagrangian generalizes Skyrme's model and supports soliton solutions.[4]

At present there is no first principles calculation that derives the chiral model from QCD. The issue is as difficult and similar to a derivation of hydrodynamics from the principles of atomic physics. Here we shall recourse to a plausible qualitative scenario. The theoretical framework to discuss phenomenological lagrangians is provided by the Kadanoff-Wilson renormalization group. We begin with 4-dimensional QCD on a hypercubical lattice. When the lattice spacing is small (much less than a fermi) we assume that QCD is described by Wilson's lattice action for quarks and gauge fields. This short distance lagrangian has no chiral couplings. To obtain an effective lagrangian at a larger lattice spacing we have to integrate out the high momentum fluctuations of the gauge and fermion fields. Now suppose we do that and after a few iterations we reach a lattice spacing of the size of the correlation length of the gauge fields: ξ_G. (Non-abelian gauge fields in 4-dimensions have a finite correlation length and this is one of their most important properties.) The lattice action at this length scale will contain higher order operators and possibly non-local couplings. These couplings can be chiral.

Now suppose at this scale we focus on Green's functions of local gauge invariant fermionic operators, e.g., $O_a{}^{ij}(x)=\sum\bar{\psi}\,{}^{\alpha}_{xi}\,\Gamma_a\psi\,{}^{\alpha}_{xj}$ where $\Gamma_a=\{1,\gamma_5,\gamma_\mu,\gamma_\mu\gamma_5\cdots\}$. α is a $SU(N)$ colour index and i, j are $U(n)\times U(n)$ flavour indices. Restricting attention to properties of these Green's functions for distances much larger than ξ_G (which is now one lattice spacing) we are effectively dealing with a nonlinear theory of fermions with contact interactions. This is basically the NJL model.

The main point in the above reasoning is that non-abelian gauge fields develop a finite correlation length and the nonlinear fermion theory only evaluates correlations of gauge invariant fermionic operators. With this understanding of the model there is no conflict with the local gauge invariance of the underlying QCD and with the question of quark confinement. A similar though far simpler situation where a massive gauge field leads to a theory with contact interactions, is the familiar Glashow-Salam-Weinberg model where the gauge symmetry is realized in the Higgs mode. For wavelengths much larger than the compton wavelength of the W boson the gauge theory is well approximated by the Fermi theory without violating gauge invariance.

With these qualitative remarks we write down the effective NJL model which evaluates gauge invariant correlations of fermionic operators for distances longer than the correlation length of the gauge field

$$-iL=\bar{\psi}_{L\alpha j}i\partial\!\!\!/\psi_{L\alpha j}+\bar{\psi}_{R\alpha j}i\partial\!\!\!/\psi_{R\alpha j}+g_1{}^2(\bar{\psi}_{L\alpha j}\psi_{R\alpha k})(\bar{\psi}_{R\beta k}\psi_{L\beta j})$$
$$-\frac{ig_2{}^2}{4}[(\bar{\psi}_{L\alpha j}\gamma_\mu\psi_{L\alpha k})(\bar{\psi}_{L\beta k}\gamma_\mu\psi_{L\beta j})+L\langle-\rangle R+\cdots], \tag{1}$$

Ψ_L and Ψ_R refer to the left and right chiral projections. g_1 and g_2 are constants of mass dim -1. There are phenomenological parameters. We denote the cutoff implicit in (1) by Λ: $\Lambda^{-1}\gg\xi_G$. Further specifications will be made as we go on. The dots in (1) refer to higher tensor couplings which we do not consider here. The above lagrangian can be rewritten using colour gauge invariant collective fields:[5)]

$$-iL=\bar{\psi}_L(i\partial\!\!\!/+iL\!\!\!/)\psi_L+\bar{\psi}_R(i\partial\!\!\!/+iR\!\!\!/)\psi_R+\bar{\psi}_LM\psi_R+\bar{\psi}_RM^\dagger\psi_L$$
$$+\frac{i}{g_1{}^2}\mathrm{tr}\,M^\dagger M+\frac{i}{g_2{}^2}\mathrm{tr}\,(L_\mu{}^2+R_\mu{}^2)+\cdots. \tag{2}$$

M is a complex scalar and L_μ and R_μ are vector fields which couple to the left and right chirality fermions.

Integrating over the fermions in the path integral corresponding to (2), we get the path integral entirely over the collective fields

$$Z=\int_{M,M^\dagger,L,R}\exp(NS_{\mathrm{eff}}),$$

$$S_{\mathrm{eff}}=\ln\det D-\frac{1}{Ng_1{}^2}\int d^4x\,\mathrm{tr}\,M^\dagger M-\frac{1}{Ng_2{}^2}\int d^4x\,\mathrm{tr}\,(L_\mu{}^2+R_\mu{}^2), \tag{3}$$

$$D_{x,y}=(i\partial\!\!\!/-L\!\!\!/P_L-R\!\!\!/P_R+iMP_L+iM^\dagger P_R)\delta^4(x-y),$$

$$P_{L,R}=\frac{(1\pm\gamma_5)}{2}. \tag{4}$$

The central object of our investigation will be the determinant the chiral Dirac operator D. In (4) we can parametrize $M(x)=H(x)U(x)$, where H is hermitian and U is unitary.

Spontaneous symmetry breaking

In the large N limit, we can seek for configurations which minimize the effective action (3). A Lorentz invariant configuration that does it is $L_\mu = R_\mu = 0$ and $M = \bar{H}1$. Substituting in (3) and minimizing it leads to the gap equation (we have used a sharp momentum cutoff for illustration)

$$\bar{H}\left[1-\frac{\bar{H}^2}{\Lambda^2}\ln\left(1+\frac{\Lambda^2}{\bar{H}^2}\right)-\frac{8\pi^2}{N\Lambda g_1{}^2}\right]=0\,. \tag{5}$$

There are 2 solutions $\bar{H}=0$ and $\bar{H}\neq 0$. The latter exists provided $\Lambda g_1{}^2 > 8\pi^2/N$. The solution $\bar{H}=0$ is unstable to small perturbations, whilst the non-trivial solution which is stable, corresponds to the true minimum of the effective action. This is the famous result of Nambu and Jona-Lasinio:[1] Chiral symmetry is spontaneously broken. In this case from $U(n)\times U(n)$ to diag $U(n)$.

Determinant of chiral Dirac operator and Wess-Zumino term

We now discuss the determinant of the chiral Dirac operator (4). For the present we do not assume spontaneous symmetry breaking. $\det D$ is formally invariant under local $U(n)\times U(n)$ gauge transformations: $M\to\Omega M$, $R\to R^\Omega=\Omega R\Omega^{-1}+i\partial\Omega\Omega^{-1}$, and similarly for the left-handed fields. This invariance is formal because the currents corresponding to this symmetry are not conserved due to the presence of anomalies.[6] We shall see that a chiral rotation on the collective fields translates the phase of $\det D$. To study these matters and in fact to define the phenomenological model more precisely we turn to defining $\det D$.

In euclidean space time the operator D is elliptic and non-self adjoint w.r.t. the scalar product $(\chi,\psi)=\int d^4x\chi^\dagger\psi$. The non-self adjointness is due to the presence of chiral couplings. We also assume that space time is compactified to S^4. Hence the eigenvalues of D are discrete and complex: $\varepsilon_n=e^{i\Delta n}|\varepsilon_n|$. We can formally define $\ln\det D=i\sum_n\Delta_n+\frac{1}{2}\sum_n\ln|\varepsilon_n|^2=i\ \mathrm{Im}(\ln\det D)+\frac{1}{2}\ln\det D^\dagger D$. Now in a cutoff field theory these sums must be cutoff at $n\sim\Lambda$. A smooth cutoff procedure maintaining certain symmetry principles is desirable. Since we are modelling the strong interactions we define $\det D$ to preserve vector symmetries. Further in our phenomenological theory we parametrize the definition of $\det D$ in terms of an arbitrary function $g(x)$ which vanishes for large x, together with all its derivatives. We also assume that $g(x)$ is monotonically decreasing and $g(0)=1$. An example is $g(x)=e^{-ax}$ where $a>0$ is a parameter. With this understanding we define $\ln\det D=i\Delta+\frac{1}{2}\,\mathrm{tr}\ln D^\dagger D$ by

$$\Delta=\mathrm{Im}(\ln\det D)=-\frac{1}{2}\mathrm{Im}\int_{1/\Lambda^2}^{\infty}\frac{ds}{s}\mathrm{tr}\,g(s\hat{D}^2), \tag{6}$$

$$\frac{1}{2}\mathrm{tr}\ln D^\dagger D=\frac{1}{2}\mathrm{tr}\ln\hat{D}^\dagger\hat{D}=-\frac{1}{2}\int_{1/\Lambda^2}^{\infty}\frac{ds}{s}\mathrm{tr}\,g(s\hat{D}^\dagger\hat{D}), \tag{7}$$

where $\hat{D}=i\gamma_5 D$.

Calculation of the phase Δ

We establish a differential equation for the determinant of the gauge rotated Dirac operator. $D^{[\Omega]}=i\partial\!\!\!/-L\!\!\!/P_L-R\!\!\!/^{\Omega}P_R+iH(\Omega U^{\dagger}P_L+U\Omega^{\dagger}P_R)$. Then under $\Omega\to\Omega+\delta\Omega$ we have $\delta D=\frac{1}{2}[\delta\Omega\Omega^{-1}, D^{[\Omega]}]-\frac{1}{2}\{\gamma_5\delta\Omega\Omega^{-1}, D^{[\Omega]}\}$. Since our definition of Δ preserves vector symmetries, the first variation does not contribute and we have

$$\delta\Delta(\Omega)=-\mathrm{Im\,tr}\left\{\gamma_5\delta\Omega\Omega^{-1}g\left(\left(\frac{D^{[\Omega]}}{\Lambda}\right)^2\right)\right\}. \tag{8}$$

The above anomalous variation of the phase can be calculated to give

$$\delta\Delta(\Omega)=\int d^4x\,\mathrm{tr}(i\,\delta\Omega\Omega^{-1}B(x))+o\left(\frac{1}{\Lambda^2}\right). \tag{9}$$

$B(x)$ is the non-abelian anomaly computed many years ago by Bardeen using different methods

$$B(x)=-\frac{1}{8\pi^2}\left[\frac{1}{4}F_v{}^2+\frac{1}{12}F_A{}^2-\frac{2}{3}i(F_vA^2+AF_vA+A^2F_v)-\frac{8}{3}A^4\right]. \tag{10}$$

$F_v=dV+iV^2+iA^2$ and $F_A=dA+iA\,V+iVA$ are the vector and axial vector 2-forms with $V=\frac{1}{2}(L+R^{\Omega})$ and $A=\frac{1}{2}(L-R^{\Omega})$. V and A are vector and axial vector 1-forms. Note that at the level of phenomenological lagrangians it is justified to drop terms $o(1/\Lambda^2)$ because we are assuming that the background fields are slowly varying on this scale. Also $B(x)$ is independent of the function $g(x)$ and hence universal.

We now integrate (9). To write it as a differential equation we parametrize $\Omega=e^{it_a\eta_a}$. Then $i\delta\Omega\Omega^{-1}=-t_a u_a{}^a\delta\eta^a$, where $u_a{}^a$ is the vierbein on the group. The generator of group motions is the operator $\mathcal{D}_a=iu_a{}^a(\delta/\delta\eta^a)$ and (9) reads $\mathcal{D}_a\Delta=\mathrm{tr}\,t_aB$. It is a technical matter to recast this equation in the form

$$\mathcal{D}_a(\Delta+C_1+C_2)=-\frac{1}{48\pi^2}\mathrm{tr}\,t_a\omega^4,\qquad \omega=id\Omega\Omega^{-1}\,, \tag{11}$$

$\omega^4=\varepsilon_{\mu\nu\rho\sigma}\partial_\mu\Omega\Omega^{-1}\partial_\nu\Omega\Omega^{-1}\partial_\rho\Omega\Omega^{-1}\partial_\sigma\Omega\Omega^{-1}$ is a 4-form and C_1 and C_2 are actions of local densities given by

$$C_1(L, R^{\Omega})=\frac{1}{48\pi^2}\int d^4x\,\mathrm{tr}\left\{i[R^{\Omega}, L](dL+dR^{\Omega})+R^{\Omega}L^3+(R^{\Omega})^3L-\frac{1}{2}LR^{\Omega}LR^{\Omega}\right\},$$

$$C_2(R^{\Omega}, \omega)=\frac{1}{48\pi^2}\int d^4x\,\mathrm{tr}\left\{(R^{\Omega})^3\omega+\omega^3R^{\Omega}+\frac{1}{2}\omega R^{\Omega}\omega R^{\Omega}+i\omega(R^{\Omega}dR^{\Omega}+dR^{\Omega}R^{\Omega})\right\}. \tag{12}$$

Now the 4-form ω^4 in (11) cannot be written as the variation of a local action hence to solve (11) we integrate it along a path in the space of the chiral field $\Omega(x)$. The path can be specified by a parameter x_5: $\Omega(x;x_5)$, $0\le x_5\le 1$. x_5 is a function of the path length. Along the path $\mathcal{D}_a=i\,\mathrm{tr}\,t_a(\partial\Omega\Omega^{-1}/\partial x_5)(d/dx_5)$ and denoting $\Delta+C_1+C_2=\Gamma$, (11) becomes

$$\frac{d}{dx_5}\Gamma=\frac{1}{48\pi^2}\int d^4x\,\mathrm{tr}\frac{\partial\Omega\Omega^{-1}}{\partial x_5}(d\Omega\Omega^{-1})^4\,. \tag{13}$$

We choose the boundary conditions $\Omega(x; x_5=0)=1$ and $\Omega(x; x_5=1)=U(x)$ and integrate (13) to obtain

$$\Gamma(1)-\Gamma(0)=\frac{1}{48\pi^2}\int_0^1 dx_5\int d^4x\,\mathrm{tr}\frac{\partial\Omega\Omega^{-1}}{\partial x_5}(d\Omega\Omega^{-1})^4\,. \tag{14}$$

Since the 5-form in (14) is closed, the solution is path independent and is only a function of the end point configuration $U(x)$.

Hence we have our basic result:[7)]

$$\begin{aligned}\Delta(U)-\Delta(1)&=\mathrm{Im}\ln\det(i\partial\!\!\!/-\not{L}P_L-\not{R}P_R+iH(UP_L+U^\dagger P_R))\\&\quad-\mathrm{Im}\ln\det(i\partial\!\!\!/-\not{L}P_L-\not{R}^UP_R+iH)\\&=\frac{1}{48\pi^2}\int_0^1 dx_5d^4x\,\mathrm{tr}\frac{\partial\Omega\Omega^{-1}}{\partial x_5}(d\Omega\Omega^{-1})^4\\&\quad+i[C_1(L;R^U)-C_1(L;R)+C_2(R^U;idUU^{-1})].\end{aligned} \tag{15}$$

In the context of the NJL model and QCD, where chiral symmetry spontaneously broken, to leading order in large N we have $H(x)=\bar{H}$. Then in the long wavelength expansion in powers of $1/\bar{H}$, we find $\mathrm{Im}\ln\det(i\partial\!\!\!/-\not{L}P_L-R^UP_R+i\bar{H})=0$[4)] and so (15) evaluates Δ the phase of the Dirac operator (4). On the other hand in case $H(x)=0$ and for simplicity $L_\mu=R_\mu=0$, (15) tells us that

$$\mathrm{Im}(\ln\det i\partial\!\!\!/)-\mathrm{Im}(\ln\det(i\partial\!\!\!/-i\partial\!\!\!/UU^{-1}))=\frac{1}{48\pi^2}\int_0^1 dx_5\int d^4x\,\mathrm{tr}\frac{\partial\Omega\Omega^{-1}}{\partial x_5}(d\Omega\Omega^{-1})^4\,. \tag{16}$$

Back to the NJL model we mention that (15) is the Wess-Zumino term and it contains all the anomalous vertices involving pions and vector mesons. It also serves to fix the quantum numbers of the soliton solution.[8)]

Calculation of |det D|

In this part of the calculation there is no universal piece like the anomaly (15). We will present $\ln|\det D|$ as defined in (7) as an expansion in powers of $1/\bar{H}$ upto dim 4 operators. We simply write down the final answer. For technical details we refer to Ref. 4).

$$\begin{aligned}N\ln|\det D|=-N\int d^4x\,\mathrm{tr}\Big\{&d_1\Big[\frac{1}{3}(F^v_{\mu\nu})^2+\frac{1}{3}(F^A_{\mu\nu})^2+4\bar{H}^2A_\mu^2\\&+(\partial_\mu H)^2+(H^2-\bar{H}^2)^2\Big]+d_2i[A_\mu,A_\nu]F^v_{\mu\nu}+d_3A_\mu{}^2(H^2-\bar{H}^2)\\&+d_4(\partial_\mu A_\nu+i[V_\mu,A_\nu])^2+d_5(A_\mu{}^2)^2+d_6[A_\mu,A_\nu]^2\Big\}+o\Big(\frac{1}{\bar{H}^2}\Big).\end{aligned} \tag{17}$$

$F^v_{\mu\nu}$ and $F^A_{\mu\nu}$ are the vector and axial vector field strengths same as defined in (10). $V=\frac{1}{2}(L+R^\Omega)$ and $A=\frac{1}{2}(L-R^\Omega)$. The coefficients d_i are given in terms of the regulator function $g(x)$, $d_1=F_{-1}$, $d_2=-4F_0$, $d_3=4F_{-1}-8F_0$, $d_4=-(4/3)F_0$, $d_5=-(8/3)(2F_0-F_1)$, $d_6=-(4/3)(F_0+F_1)$ where $F_n=\int_x^\infty ds\, s^{1-n}g^{(n)}(s)$, $x=\bar{H}^2/\Lambda^2$ and $g^{(n)}(s)$ is the nth derivative of $g(s)$. A reasonable theory has $x \lesssim 1$. As expected (17) is invariant under vector gauge transformations. Axial vector transformations are not a symmetry since the expansion has been done in the broken symmetry phase.

Perturbative spectrum

Collecting the expressions (3), (15) and (17) the effective action of the NJL model is

$$\frac{S_{\rm eff}}{N}=\ln|\det D|+i\Delta(U)-\frac{1}{Ng_1{}^2}\int d^4x\,{\rm tr}\,H^2-\frac{1}{Ng_2{}^2}\int d^4x\,{\rm tr}(L_\mu{}^2+R_\mu{}^2). \qquad (18)$$

The perturbative spectrum around the large N classical ground state $L_\mu=R_\mu=0$, $M=\bar{H}$ is easily calculated by looking at the quadratic part of (18). It consists of massive vector and axial vector mesons, the neutral σ-meson and the massless pions. The masses and $F_\pi{}^2$ are given by

$$m_v{}^2=\frac{3}{d_1Ng_2{}^2},\quad m_A{}^2=\frac{3}{d_1Ng_2{}^2}+6\bar{H}^2,\quad m_\sigma{}^2=4\bar{H}^2,$$

$$F_\pi{}^2=\frac{N\bar{H}^2d_1}{1+2\bar{H}^2d_1Ng_2{}^2}. \qquad (19)$$

Note the relation $m_A{}^2-m_v{}^2=(3/2)m_\sigma{}^2$.

Non-perturbative spectrum: Topological solitons of the NJL model

We now discuss the large N solitons of the effective action (18). For simplicity we set the vector fields $L_\mu=R_\mu=0$, in (18). In principle we should consider the coupled system of $H(x)$ and $U(x)$. However our investigation indicates that the classical value of $H(x)$ deviates negligibly from $\bar{H}$ and is never zero. Further if we look for static solutions the Wess-Zumino term $\Delta(U)$ does not contribute and the static energy function to be minimized is

$$\frac{E}{N}=\int d^3x\,{\rm tr}\left\{4\bar{H}^2d_1(l_i)^2+3d_3(\partial_il_i)^2+\frac{d_5}{16}(l_i{}^2)^2+\frac{d_4-d_6}{6}[l_i,l_j][l_j,l_i]\right\},$$

$$l_i=i\partial_iUU^{-1}. \qquad (20)$$

If we look for configurations which are slowly varying over the length $1/\bar{H}$, the second term is negligible and we have

$$\frac{E}{N}=\int d^3x\,{\rm tr}\left\{4\bar{H}^2d_1(l_i)^2+\frac{d_5}{16}(l_i{}^2)^2+\frac{d_4-d_6}{16}[l_i,l_j][l_j,l_i]\right\}. \qquad (21)$$

The coefficients $d_1=\int_x^\infty(ds/s)g(s)$ and $d_4-d_6=(1/12\pi^2)[g(x)-xg'(x)]$ are positive for all x, in particular for $x\lesssim 1$. However the coefficient $d_5=(1/16\pi^2)[g(x)+(x-2)\times(-g'(x))]$ is not positive for $x\lesssim 1$, for all functions $g(x)$ in the class of admissible regulators, e.g., if we choose $g(x)=e^{-xa}$ then $d_5=(1/16\pi^2)e^{-ax}[1+a(x-2)]$, which is positive for $x\lesssim 1$ only when $a<(2-x)^{-1}$. The point we are making is that the NJL model is parametrized by the function $g(x)$ and the choice of this function must be made on physical grounds and cannot be fixed a priori. In practice one will have to parametrize

$g(x)$ by several parameters and fix them by matching observed quantities. With these remarks, when all coefficients in (21) are positive we do find a soliton solution. The skyrme model comprises of the first and third term in (21).[9] The numerical work is in progress and will be reported later.

The soliton we have discussed is a baryon. To see this let us compute the baryon current $B_\mu=(1/N)\langle\bar{\psi}\gamma_\mu\psi\rangle$. The expectation value is evaluated in the presence of the 'external' fields L_μ, R_μ and M. It is clear that $B_\mu=\mathrm{tr}\{(\delta/\delta L_\mu)+(\delta/\delta R_\mu)\}\ln\det D$. Now since the real part of ln det D, is invariant under vector gauge transformations only the phase contributes and $B_\mu=\mathrm{tr}\{(\delta/\delta L_\mu)+(\delta/\delta R_\mu)\}\varDelta$. In general this is a complicated function of the various fields in $\varDelta$. However in the limit of very long wavelengths when massive vector and axial vector particles decouple from the effective lagrangian, we can consider L_μ and R_μ as infinitesimal sources and in the formula for the baryon current only the '5-dimensional term' in the phase (15) contributes and we get the topological formula for the baryon current

$$B_\mu=-\frac{1}{24\pi^2}\varepsilon_{\mu\nu\rho\sigma}\mathrm{tr}(\partial_\nu UU^{-1}\partial_\rho UU^{-1}\partial_\sigma UU^{-1}). \tag{22}$$

The baryon charge $B_0=(1/24\pi^2)\int d^3x\varepsilon_{ijk}(\partial_i UU^{-1}\partial_j UU^{-1}\partial_k UU^{-1})$ is a topological invariant in the broken symmetry phase where $U(x\to\infty)=1$.

Clearly much work of a numerical nature remains to be done to fit various aspects of hadron phenomenology in the scheme we have outlined. We hope that progress in this direction will be possible. The freedom offered by the regulator function in the definition of the determinant, in a phenomenological approach, is bound to play an important role in this endeavour.

Epilogue

Our work on the Nambu-Jona-Lasinio model emerged from a desire to understand the 'topological' Wess-Zumino term from a structural point of view: An understanding of this peculiar 5-dim term must be contained in the fact that 2 quarks make up a meson. It is this question that led us to establish a connection between the nonlinear fermion theory and the nonlinear σ-model. The understanding of the NJL model in the context of QCD only came towards the end. Professor Nambu has often emphasized to me the importance of understanding and elucidating phenomena in terms of structure and constituents. It is my privilege to dedicate this article to him.

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