POISSON CONVERGENCE OF EIGENVALUES OF CIRCULANT TYPE MATRICES

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Abstract. We consider the point processes based on the eigenvalues of the reverse circulant, symmetric circulant and k-circulant matrices with i.i.d entries and show that they converge to a Poisson random measures in vague topology. The joint convergence of upper ordered eigenvalues and their spacings follow from this. We extend these results partially to the situation where the entries are come from a two sided moving average process.

Keywords Large dimensional random matrix, eigenvalues, circulant matrix, symmetric circulant matrix, reverse circulant matrix, k-circulant matrix, point process, Poisson random measure, moving average process, spectral density, normal approximation.

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1. Introduction

There appears to have been only limited studies on the weak convergence of point process based on the eigenvalues of random matrices. Soshnikov (2004) considers the point process based on the positive eigenvalues of appropriately scaled Wigner matrix with heavy tailed entries $\{x_{ij}\}$ satisfying $\mathrm{P}(|x_{ij}| > x) = h(x)x^{-\alpha}$ where h is slowly varying function at infinity and $0 < \alpha < 2$. He showed that it converges to an inhomogeneous Poisson random point process. A similar result was proved for sample covariance matrices in Soshnikov (2006). These results were extended in Auffinger, Ben Arous and Peche (2008) to $2 \leq \alpha < 4$.

On the other hand, Bose, Hazra and Saha (2009) establish the distributional convergence of the maximum of the eigenvalues of circulant, reverse circulant and symmetric circulant matrices. Same result for k circulant matrix for $n = k^2 + 1$ was derived in Bose, Mitra and Sen (2008). The main tool for proving such a result was the strong approximation theorem by Einmahl (1989) for i.i.d random vectors.

In this article, we deal with circulant type matrices with light tailed entries and consider the point process based on the points $(\omega_k, \frac{\lambda_k - b_q}{a_q})$ where λ_k is k-th eigenvalue and $\omega_k = \frac{2\pi k}{n}$ is the Fourier frequency and a_q, b_q are appropriate scaling and centering constants appearing in the weak convergence of the maximum. We show that the limit measure is Poisson. In particular this yields the distributional convergence of any k-upper ordered eigenvalues of these matrices and also yields the joint distributional convergence of any k spacings of the upper ordered eigenvalues. Then we extend these results partially to two sided moving average process entries under certain restriction on the process.

2. Results for i.i.d. inputs

2.1. **Reverse circulant** (RC_n)**.** This is a symmetric matrix where the (i, j)-th element of the matrix is $x_{(i+j-2) mod\ n}$. Let $\lambda_{n,x}(\omega_0), \lambda_{n,x}(\omega_1), \ldots, \lambda_{n,x}(\omega_{n-1})$ be the eigenvalues of $n^{-1/2}RC_n$.

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These eigenvalues are given by (see Bose and Mitra (2002)):

$$(1)\qquad \begin{cases} \lambda_{n,x}(\omega_0) & = n^{-1/2}\sum_{t=0}^{n-1} x_t \\ \lambda_{n,x}(\omega_{n/2}) & = n^{-1/2}\sum_{t=0}^{n-1}(-1)^t x_t, \quad \text{if } n \quad \text{is even} \\ \lambda_{n,x}(\omega_k) = -\lambda_{n,x}(\omega_{n-k}) & = \sqrt{I_{n,x}(\omega_k)}, \ 1 \le k \le [\frac{n-1}{2}]. \end{cases}$$

where

$$I_{n,x}(\omega_k) = \frac{1}{n}|\sum_{t=0}^{n-1} x_t e^{-it\omega_k}|^2 \text{ and } \omega_k = \frac{2\pi k}{n}.$$

Note that $\{|\lambda_{n,x}(\omega_k)|^2;\ 1 \le k < n/2\}$ is the periodogram of $\{x_i\}$ at the frequencies $\{\omega_k = \frac{2\pi k}{n};\ 1 \le k < n/2\}$. This explains our notation of using ω_k as an argument of the eigenvalues $\lambda_{n,x}$. Since the eigenvalues occur in pairs with opposite signs (except for perhaps one eigenvalue), it suffices for our purposes to define our point process based on the points $(\omega_k, \frac{\lambda_{n,x}(\omega_k)-b_q}{a_q})$ for $k = 0, 1, 2, \ldots, [n/2]$. Let $\epsilon_x(\cdot)$ denote the point measure which gives unit mass to any set containing x. With $q = [\frac{n}{2}]$, $a_q = \frac{1}{2\sqrt{\ln q}}$ and $b_q = \sqrt{\ln q}$, define

$$(2)\qquad \eta_n(\cdot) = \sum_{j=0}^{q} \epsilon_{\left(\omega_j, \frac{\lambda_{n,x}(\omega_j)-b_q}{a_q}\right)}(\cdot).$$

Let $M_p([0,\pi]\times(-\infty,\infty])$ denote the set of all point measures on the set $[0,\pi]\times(-\infty,\infty]$ endowed with topology of vague convergence. We then have the following Theorem.

Theorem 1. *Let $\{x_t\}$ be i.i.d random variables with* $\mathrm{E}[x_0] = 0$, $\mathrm{E}[x_0]^2 = 1$ *and* $\mathrm{E}\,|x_0|^s < \infty$ *for some* $s > 2$. *Then for the sequence of point processes η_n defined in (2), we have $\eta_n \xrightarrow{\mathcal{D}} \eta$, where η is a Poisson process on $[0,\pi]\times(-\infty,\infty]$ with intensity measure $\pi^{-1}dt \times e^{-x}dx$ and $\xrightarrow{\mathcal{D}}$ denotes the convergence in distribution on the space $M_p([0,\pi]\times(-\infty,\infty])$ relative to the vague topology.*

The relation $\eta_n \xrightarrow{\mathcal{D}} \eta$ immediately yields the joint weak convergence of a finite vector of k upper ordered eigenvalues. To be precise, we introduce for every n the ordered version of the sample $\lambda_{n,x}(\omega_j), j = 0, 1, \ldots, n-1$,

$$\lambda_{n,(q)} \le \cdots \le \lambda_{n,(2)} \le \lambda_{n,(1)}.$$

Let $x_k < \cdots < x_1$ be any real numbers, and write $N_{i,n} = \eta_n([0,\pi]\times(x_i,\infty))$ for the number of exceedances of x_i by $\frac{\lambda_{n,x}(\omega_j)-b_q}{a_q}$, $j = 1, \ldots, q$. Then

$$\{\frac{\lambda_{n,(1)}-b_q}{a_q} \le x_1, \ldots, \frac{\lambda_{n,(k)}-b_q}{a_q} \le x_k\} = \{N_{1,n} = 0, N_{2,n} \le 1, \ldots, N_{k,n} \le k-1\}.$$

Thus the joint limit distribution of the vector of the k upper ordered eigenvalues $\lambda_{n,x}(\omega_j)$ as well as their spacings can be derived from Theorem 1.

Corollary 0.1. *Under the assumption of Theorem 1,*

(i) for any real numbers $x_k < \cdots < x_2 < x_1$,

$$\mathrm{P}\left(\frac{\lambda_{n,(1)}-b_q}{a_q} \le x_1, \cdots, \frac{\lambda_{n,(k)}-b_q}{a_q} \le x_k\right) \to \mathrm{P}(Y_{(1)} \le x_1, \cdots, Y_{(k)} \le x_k),$$

where $(Y_{(1)}, \cdots, Y_{(k)})$ has the density $\exp(-\exp(-x_k) - (x_1 + \cdots + x_{k-1}))$.
(ii) $\left(\frac{\lambda_{n,(i)}-\lambda_{n,(i-1)}}{a_q}\right)_{i=1,\ldots,k} \xrightarrow{\mathcal{D}} (i^{-1}E_i)_{i=1,\ldots,k}$ *where $\{E_i\}$ is an i.i.d standard exponential sequence.*

2.2. **Symmetric circulant** (SC_n)**.** It is symmetric version of the usual circulant matrix and (i,j)-th element of the matrix is given by $x_{n/2-|n/2-|i-j||}$.

Let $\lambda_{n,x}(\omega_0), \lambda_{n,x}(\omega_1), \ldots, \lambda_{n,x}(\omega_{n-1})$ be the eigenvalues of $n^{-1/2}SC_n$. These eigenvalues are given by: (i) for n odd:

$$\left\{\begin{array}{ll} \lambda_{n,x}(\omega_0) & = \frac{1}{\sqrt{n}}\big[x_0 + 2\sum_{j=1}^{[n/2]} x_j\big] \\ \lambda_{n,x}(\omega_k) & = \frac{1}{\sqrt{n}}\big[x_0 + 2\sum_{j=1}^{[n/2]} x_j \cos(\omega_k j)\big], \quad 1 \le k \le [n/2] \end{array}\right. \tag{3}$$

(ii) for n even:

$$\left\{\begin{array}{ll} \lambda_{n,x}(\omega_0) & = \frac{1}{\sqrt{n}}\big[x_0 + 2\sum_{j=1}^{\frac{n}{2}-1} x_j + x_{n/2}\big] \\ \lambda_{n,x}(\omega_k) & = \frac{1}{\sqrt{n}}\big[x_0 + 2\sum_{j=1}^{\frac{n}{2}-1} x_j \cos(\omega_k j) + (-1)^k x_{n/2}\big], \quad 1 \le k \le \frac{n}{2} \end{array}\right. \tag{4}$$

with $\lambda_{n,x}(\omega_{n-k}) = \lambda_{n,x}(\omega_k)$ in both cases.

Now define a sequence of point processes based on the points $(\omega_j, \frac{\lambda_{n,x}(\omega_j)-b_q}{a_q})$ for $k = 0, 1, \ldots, q(= [n/2])$, where $\lambda_{n,x}$ are same as in (3). Note, we have not considered the eigenvalues λ_{n-k} for $k = 1, \ldots, [\frac{n}{2}]$ to define the point process since $\lambda_{n,x}(\omega_{n-k}) = \lambda_{n,x}(\omega_k)$ for $k = 1, \ldots, [\frac{n}{2}]$ and it does not effect our goal of finding the limit distribution of upper order eigenvalues. Define

$$\eta_n(\cdot) = \sum_{j=0}^{q} \epsilon_{\left(\omega_j, \frac{\lambda_{n,x}(\omega_j)-b_q}{a_q}\right)}(\cdot) \tag{5}$$

where

$$b_n = c_n + a_n \ln 2,\ a_n = (2\log n)^{-1/2} \text{ and } c_n = (2\log n)^{1/2} - \frac{\log\log n + \log 4\pi}{2(2\log n)^{1/2}}. \tag{6}$$

Theorem 2. *Let $\{x_t\}$ be i.i.d random variables with $\mathrm{E}[x_0] = 0$, $\mathrm{E}[x_0]^2 = 1$ and $\mathrm{E}[x_0]^s < \infty$ for some $s > 2$. Then for the sequence of point processes η_n defined in (5), we have $\eta_n \stackrel{\mathcal{D}}{\to} \eta$, where η is a Poisson process on $[0,\pi] \times (-\infty, \infty]$ with intensity measure $\pi^{-1}dt \times e^{-x}dx$.*

Note that a similar result as Corollary 0.1 holds in this case too.

2.3. k**-circulant.** For positive integers k and n, define the $n \times n$ square matrix

$$A_{k,n} = \begin{bmatrix} x_0 & x_1 & x_2 & \ldots & x_{n-2} & x_{n-1} \\ x_{n-k} & x_{n-k+1} & x_1 & \ldots & x_{n-k-2} & x_{n-k-1} \\ x_{n-2k} & x_{n-2k+1} & x_0 & \ldots & x_{n-2k-2} & x_{n-2k-1} \\ & & & \vdots & & \end{bmatrix}_{n\times n}.$$

The first row of $A_{k,n}$ is $(x_0, x_1, x_2, ..., x_{n-1})$ and for $1 \le j < n-1$, its $(j+1)$-th row is obtained by giving its j-th row a right circular shift by k positions (equivalently, k mod n positions).
We consider k-circulant matrix only for $n = k^2+1$. First we describe the eigenvalues of k-circulant matrix. Let

$$\nu = \nu_n := \cos(2\pi/n) + i\sin(2\pi/n),\ i^2 = -1 \text{ and } \lambda_k(x) = \sum_{l=0}^{n-1} x_l \nu^{kl}, \quad 0 \le j < n. \tag{7}$$

For any positive integers k, n, let $p_1 < p_2 < \ldots < p_c$ be all their common prime factors so that,

$$n = n' \prod_{q=1}^{c} p_q^{\beta_q} \quad \text{and} \quad k = k' \prod_{q=1}^{c} p_q^{\alpha_q}.$$

Here α_q, $\beta_q \ge 1$ and n', k', p_q are pairwise relatively prime. For any positive integer s, let $\mathbb{Z}_s = \{0, 1, 2, \ldots, s-1\}$. Define the following sets

$$S(x) = \{xk^b \bmod n' : b \ge 0\},\ 0 \le x < n'. \tag{8}$$

Let $g_x = |S(x)|$. Define

$$\upsilon_{k,n'} := |\{x \in \mathbb{Z}_{n'} : g_x < g_1\}| . \tag{9}$$

We observe the following about the sets $S(x)$.

(1) $S(x) = \{xk^b \bmod n' : 0 \le b < |S(x)|\}$.
(2) For $x \neq u$, either $S(x) = S(u)$ or $S(x) \cap S(u) = \phi$. As a consequence, the distinct sets from the collection $\{S(x) : 0 \le x < n'\}$ forms a partition of $\mathbb{Z}_{n'}$.

We shall call $\{S(x)\}$ the *eigenvalue partition* of $\{0, 1, 2, \ldots, n-1\}$ and we will denote the partitioning sets and their sizes by

$$\{\mathcal{P}_0, \mathcal{P}_1, \ldots, \mathcal{P}_{l-1}\}, \text{ and } \quad n_i = |\mathcal{P}_i|, \ 0 \le i < l. \tag{10}$$

Define

$$y_j := \prod_{t \in \mathcal{P}_j} \lambda_{ty}, \quad j = 0, 1, \ldots, l-1 \quad \text{where} \quad y = n/n'.$$

Then the characteristic polynomial of $A_{k,n}$ (whence its eigenvalues follow) is given by

$$\chi(A_{k,n}) = \lambda^{n-n'} \prod_{j=0}^{\ell-1} (\lambda^{n_j} - y_j) . \tag{11}$$

In the present case, by Lemma 7 of Bose, Mitra and Sen (2008), the eigenvalue partition of $\{0, 1, 2, \ldots, n-1\}$ contains exactly $q = [\frac{n}{4}]$ sets of size 4 and each set is self-conjugate. Moreover, if k is even then there is only one more partition set containing only 0, and if k is odd then there are two more partition sets containing only 0 and only $n/2$ respectively.
Now for the development of the point process we need a clear picture of the eigenvalue partition of $\{0, 1, 2, \ldots, n-1\}$. For this we represent the set $\mathbb{Z}_n = \{0, 1, 2, \ldots, n-1\}$ in the following form

$$\mathbb{Z}_n = \{ak + b; 0 \le a \le k-1, 1 \le b \le k\} \cup \{0\}. \tag{12}$$

Then we can write $S(x)$ defined in (8) as follows

$$S(ak+b) = \{ak+b, bk-a, n-ak-b, n-bk+a\}; \ 0 \le a \le k-1, \ 1 \le b \le k.$$

Lemma 1. *For $n = k^2 + 1$,*

$$\mathbb{Z}_n = \bigcup_{0 \le a \le [\frac{k-2}{2}], a+1 \le b \le k-a-1} S(ak+b) \bigcup S(0), \text{ if } k \text{ is even} \tag{13}$$

and

$$\mathbb{Z}_n = \bigcup_{0 \le a \le [\frac{k-2}{2}], a+1 \le b \le k-a-1} S(ak+b) \bigcup S(0) \bigcup S(n/2), \text{ if } k \text{ is odd} \tag{14}$$

where all $S(ak+b)$ are mutually disjoint and hence form a (eigenvalue) partition of $\mathbb{Z}_n$.

Now we are ready to define our point process based on the eigenvalues of the k-circulant matrix. For our purpose we neglect $\{0, n/2\}$ if n is even and $\{0\}$ if n is odd. Denote

$$S = \mathbb{Z}_n - \{0, n/2\}, \ T_n = \{(a,b) : 0 \le a \le [\frac{k-2}{2}], a+1 \le b \le k-(a+1)\},$$

$$\beta_{x,n}(a,b) = \prod_{t \in S(ak+b)} \lambda_t(x) \text{ and } \lambda_x(a,b) = (\beta_{x,n}(a,b))^{1/4}.$$

Now define a sequence of point process based on points $\{(\frac{a}{\sqrt{n}}, \frac{b}{\sqrt{n}}, \frac{\lambda_x(a,b)-d_q}{c_q}) : (a,b) \in T_n\}$. Define

$$\eta_n(\cdot) = \sum_{(a,b) \in T_n} \epsilon_{\left(\frac{a}{\sqrt{n}}, \frac{b}{\sqrt{n}}, \frac{\lambda_x(a,b)-d_q}{c_q}\right)}(\cdot) \tag{15}$$

where $q = q(n) = \lfloor \frac{n}{4} \rfloor$ and

$$c_n = (8 \ln n)^{-1/2} \quad \text{and} \quad d_n = \frac{(\ln n)^{1/2}}{\sqrt{2}} \left(1 + \frac{1}{4} \frac{\ln \ln n}{\ln n}\right) + \frac{1}{2(8 \ln n)^{1/2}} \ln \frac{\pi}{2}. \tag{16}$$

Theorem 3. *Let $\{x_t\}$ be i.i.d random variables with $\mathrm{E}[x_0] = 0$, $\mathrm{E}[x_0]^2 = 1$ and $\mathrm{E}\,|x_0|^s < \infty$ for some $s > 2$. Then for the sequence of point processes η_n defined in (15), we have $\eta_n \xrightarrow{\mathcal{D}} \eta$, where η is a Poisson process on $[0, 1/2] \times [0, 1] \times [0, \infty]$ with intensity measure $4\mathbb{I}_{\{s \le t \le 1-s\}} e^{-x} ds dt dx$.*

A similar result as Corollary 0.1 holds in this case too.

3. Result for dependent input

Let $\{x_n; n \geq 0\}$ be a two sided moving average process,

$$x_n = \sum_{i=-\infty}^{\infty} a_i \epsilon_{n-i} \tag{17}$$

where $\{a_n; n \in \mathbb{Z}\} \in l_1$, that is $\sum_n |a_n| < \infty$, are nonrandom and $\{\epsilon_i; i \in \mathbb{Z}\}$ are i.i.d. random variables with $E(\epsilon_i) = 0$ and $V(\epsilon_i) = 1$. Let $f(\omega)$, $\omega \in [0, 2\pi]$ be the spectral density of $\{x_n\}$. Note that if $\{x_n\}$ is i.i.d. with mean 0 and variance σ^2, then $f \equiv \frac{\sigma^2}{2\pi}$.

3.1. **Reverse circulant** (RC_n)**.** Now define a sequence of point processes based on the points $\left(\omega_k, \frac{\frac{\lambda_{n,x}(\omega_k)}{\sqrt{2\pi f(\omega_k)}} - b_q}{a_q}\right)$ for $k = 1, 2, \ldots, q$, where $\lambda_{n,x}(\omega_k)$ are the eigenvalues of $n^{-1/2} RC_n$ defined in (1). Define $\tilde{\lambda}_{n,x}(\omega_k) = \frac{\lambda_{n,x}(\omega_k)}{\sqrt{2\pi f(\omega_k)}}$ and

$$\tilde{\eta}_n(\cdot) = \sum_{j=1}^{q} \epsilon_{\left(\omega_j, \frac{\tilde{\lambda}_{n,x}(\omega_k) - c_q}{a_q}\right)}(\cdot) \tag{18}$$

where $a_q = \frac{1}{2\sqrt{\ln q}}$ and $b_q = \sqrt{\ln q}$.

Theorem 4. *Let $\{x_n\}$ be the two sided moving average process defined in (17) with $\mathrm{E}(\epsilon_0) = 0$, $\mathrm{E}(\epsilon_0^2) = 1$ and $\mathrm{E}\,|\epsilon_0|^s < \infty$ for some $s > 2$ and*

$$\sum_{j=-\infty}^{\infty} |a_j| |j|^{1/2} < \infty \text{ and } f(\omega) > 0 \text{ for all } \omega \in [0, 2\pi]. \tag{19}$$

Then for the sequence of point processes $\tilde{\eta}_n$ defined in (18), we have $\tilde{\eta}_n \xrightarrow{\mathcal{D}} \eta$, where η is a Poisson process on $[0, \pi] \times (-\infty, \infty]$ with intensity measure $\pi^{-1} dt \times e^{-x} dx$.

3.2. **Symmetric circulant** (SC_n)**.** Here we consider two sided moving average process defined in (17) with an extra assumption that $a_j = a_{-j}$ for all $j \in \mathbb{N}$. Define

$$\tilde{\eta}_n(\cdot) = \sum_{j=0}^{q} \epsilon_{\left(\omega_j, \frac{\tilde{\lambda}_{n,x}(\omega_j) - b_q}{a_q}\right)}(\cdot) \tag{20}$$

where $q = q(n) \sim \frac{n}{2}$, $\tilde{\lambda}_{n,x}(\omega_j) = \frac{\lambda_{n,x}(\omega_j)}{\sqrt{2\pi f(\omega_j)}}$ and $\lambda_{n,x}(\omega_j)$ are the eigenvalues of symmetric circulant matrix given in (3) and a_q, b_q are as in (6).

Theorem 5. *Let $\{x_n\}$ be the two sided moving average process defined in (17) with $a_j = a_{-j}$, $\mathrm{E}(\epsilon_0) = 0$, $\mathrm{E}(\epsilon_0^2) = 1$ and $\mathrm{E}\,|\epsilon_0|^s < \infty$ for some $s > 2$ and*

$$\sum_{j=-\infty}^{\infty} |a_j| |j|^{1/2} < \infty \text{ and } f(\omega) > 0 \text{ for all } \omega \in [0, 2\pi]. \tag{21}$$

Then for the sequence of point processes $\tilde{\eta}_n$ defined in (20), we have $\tilde{\eta}_n \xrightarrow{\mathcal{D}} \eta$, where η is a Poisson process on $[0, \pi] \times (-\infty, \infty]$ with intensity measure $\pi^{-1} dt \times e^{-x} dx$.

3.3. **k-circulant.** First recall the eigenvalues of k-circulant matrix for $n = k^2 + 1$ given in Section (2.3) and define following notation based on that

$$\beta_{\epsilon,n}(a,b) = \prod_{t \in S(ak+b)} \lambda_t(\epsilon), \quad \lambda_\epsilon(a,b) = (\beta_{\epsilon,n}(a,b))^{1/4},$$

$$\tilde{\beta}_{x,n}(a,b) = \frac{\prod_{t \in S(ak+b)} \lambda_t(x)}{4\pi^2 f(\omega_{ak+b}) f(\omega_{bk-a})} \text{ and } \tilde{\lambda}_x(a,b) = (\tilde{\beta}_{x,n}(a,b))^{1/4}.$$

Now with $d_q,\ c_q$ as in (16), define our point process based on points $\{(\frac{a}{\sqrt{n}}, \frac{b}{\sqrt{n}}, \frac{\tilde{\lambda}_x(a,b)-d_q}{c_q}) : (a,b) \in T_n\}$ as:

$$\tilde{\eta}_n(\cdot) = \sum_{j=0}^{q} \epsilon_{\left(\frac{a}{\sqrt{n}}, \frac{b}{\sqrt{n}}, \frac{\tilde{\lambda}_x(a,b)-d_q}{c_q}\right)}(\cdot). \tag{22}$$

Theorem 6. *Let $\{x_n\}$ be the two sided moving average process defined in (17) with* $\mathrm{E}(\epsilon_0) = 0$, $\mathrm{E}(\epsilon_0^2) = 1$ *and* $\mathrm{E}|\epsilon_0|^s < \infty$ *for some $s > 2$ and*

$$\sum_{j=-\infty}^{\infty} |a_j||j|^{1/2} < \infty \text{ and } f(\omega) > 0 \text{ for all } \omega \in [0, 2\pi]. \tag{23}$$

Then for the sequence of point processes $\tilde{\eta}_n$ defined in (20), we have $\tilde{\eta}_n \xrightarrow{\mathcal{D}} \eta$, where η is a Poisson process on $[0, 1/2] \times [0, 1] \times [0, \infty]$ with intensity measure $4\mathbb{I}_{\{s \le t \le 1-s\}} e^{-x} ds dt dx$.

4. Proofs of results

4.1. **Proof of Theorem 1.** Before going into the proof of Theorem 1 we state two results which play a key role in the proof and which will also be used later. The proof of Lemma 2 is available in Kallenberg (1983), Resnick (1987) and Embrechts, P., Kluppelberg, C. and Mikosch, T. (1997).

Lemma 2. *Let (N_n) and N be point processes on a complete separable metric space E and N be a simple point process. Let $\mathcal{T}$ be a basis of relatively compact open sets such that $\mathcal{T}$ is closed under finite unions and intersections and for $I \in \mathcal{T}$,* $\mathrm{P}[N(\partial I) = 0] = 1$. *If* $\lim_{n\to\infty} \mathrm{P}[N_n(I) = 0] = \mathrm{P}[N(I) = 0]$ *and* $\lim_{n\to\infty} \mathrm{E}[N_n(I)] = \mathrm{E}[N(I)] < \infty$ *then $N_n \xrightarrow{\mathcal{D}} N$ in $M_p(E)$.*

The following Lemma is from Davis and Mikosch (1999) which follows from strong approximation results of Einmahl (1989). Denote the truncated and centered i.i.d random variables by

$$\overline{x}_t = x_t I(|x_t| < n^{1/s}) - \mathrm{E}[x_t I(|x_t| < n^{1/s}].$$

Lemma 3. *For $d \ge 1$, define*

$$v_d(t) = (\cos(\omega_{i_1} t), \sin(\omega_{i_2} t), ..., \cos(\omega_{i_d} t), \sin(\omega_{i_d} t))' \tag{24}$$

where $\omega_{i_1}, ..., \omega_{i_d}$ are any distinct Fourier frequencies. Let $\{x_t\}$ be i.i.d random variables with $\mathrm{E}[x_0] = 0$, $\mathrm{E}[x_0]^2 = 1$ *and* $\mathrm{E}[x_0]^s < \infty$ *for some $s > 2$. Let $\tilde{p}_n$ be the density function of*

$$2^{1/2} n^{-1/2} \sum_{t=1}^{n} (\overline{x}_t + \sigma_n N_t) v_d(t),$$

where $\{N_t\}$ is a sequence of i.i.d $N(0,1)$ random variables, independent of $\{x_t\}$ and $\sigma_n^2 = \mathrm{Var}(\overline{x}_t) s_n^2$. *If $n^{-2c_6} \ln n < s_n^2 \le 1$ with $c_6 = 1/2 - (1-\delta)/s$ for arbitrarily small $\delta > 0$, then uniformly for $|x|^3 = o_d(\min(n^{c_6}, n^{1/2-1/s}))$,*

$$\tilde{p}_n(x) = \phi_{(1+\sigma_n^2) I_{2d}}(x)(1 + o(1)).$$

Proof of Theorem 1. **Step 1:** We first show that $\eta_n^* \xrightarrow{\mathcal{D}} \eta$ where

$$\eta_n^*(\cdot) = \sum_{j=1}^{q} \epsilon_{\left(\omega_j, \frac{\lambda_{n,\bar{x}+\sigma_n N}(\omega_j)-b_q}{a_q}\right)}(\cdot)$$

and $\lambda_{n,\bar{x}+\sigma_n N}(\omega_k)$ are the eigenvalues of $n^{-1/2}RC_n$ with entries $\{\bar{x}_t + \sigma_n N_t\}$ with $\sigma_n^2 = n^{-c_6}$ and c_6 is as in the Lemma 3. First note that if we define the set

$$A_q^d = \{(x_1, y_1, ..., x_d, y_d)' : \sqrt{x_i^2 + y_i^2} > 2z_q\}$$

where $z_q = a_q x + b_q$, it easily follows that

$$\begin{aligned} \mathrm{P}\left(\lambda_{n,\bar{x}+\sigma_n N}(\omega_{i_1}) > z_q, ..., \lambda_{n,\bar{x}+\sigma_n N}(\omega_{i_d}) > z_q\right) &= \mathrm{P}\left(2^{1/2} n^{-1/2} \sum_{t=1}^{n} (\bar{x}_t + \sigma_n N_t) v_d(t) \in A_q^d\right) \\ &= \int_{A_q^d} \phi_{(1+\sigma_n^2) I_{2d}}(x)(1+o(1))dx \\ &= q^d \exp(-dx)(1+o(1)) \qquad (25) \end{aligned}$$

Since the limit process η is simple, to show $\eta_n^* \xrightarrow{\mathcal{D}} \eta$ it suffices to show that

$$\mathrm{E}\,\eta_n^*((a,b] \times (x,y]) \to \mathrm{E}\,\eta((a,b] \times (x,y]) = \frac{b-a}{\pi}(e^{-x} - e^{-y}) \qquad (26)$$

for all $0 \le a < b \le \pi$ and $x < y$ and, for all $k \ge 1$,

(27)
$$\mathrm{P}(\eta_n^*((a_1,b_1]\times R_1) = 0, \ldots, \eta_n^*((a_k,b_k]\times R_k) = 0) \to \mathrm{P}(\eta((a_1,b_1]\times R_1) = 0, \ldots, \eta((a_k,b_k]\times R_k) = 0),$$

where $0 \le a_1 < b_1 < \cdots < a_k < b_k \le \pi$ and $R_1, \ldots, R_k$ are bounded Borel sets, each consisting of a finite union of intervals on $(-\infty, \infty]$. To prove (26), note that

$$\begin{aligned} \mathrm{E}\,\eta_n^*((a,b] \times (x,y]) &= \sum_{\omega_j \in (a,b]} \mathrm{P}(a_q x + b_q < \lambda_{n,\bar{x}+\sigma_n N}(\omega_j) \le a_q y + b_q) \\ (\text{by (25)}) \quad &\sim \frac{(b-a)n}{2\pi} q^{-1}(e^{-x} - e^{-y}) \to \frac{(b-a)}{\pi}(e^{-x} - e^{-y}). \end{aligned}$$

Now to prove (27), set $n_j := \#\{i : \omega_i \in (a_j, b_j]\} \sim n(b_j - a_j)$. Then the complement of the event in (27) is the union of $m = n_1 + \ldots + n_k$ events, that is,

(28)
$$1 - \mathrm{P}(\eta_n^*((a_1,b_1]\times R_1) = 0, \ldots, \eta_n^*((a_k,b_k]\times R_k) = 0) = \mathrm{P}\left(\cup_{j=1}^k \cup_{\omega_i \in (a_j,b_j]} \{\frac{\lambda_{n,\bar{x}+\sigma_n N}(\omega_i) - b_q}{a_q} \in R_j\}\right).$$

Now for any choice of d distinct integers $i_1, \ldots, i_d \in \{1, \ldots, q\}$ and integers $j_1, \ldots, j_d \in \{1, \ldots, k\}$ we have from (25) that

$$\mathrm{P}\left(\cap_{r=1}^d \{\frac{\lambda_{n,\bar{x}+\sigma_n N}(\omega_{i_r}) - b_q}{a_q} \in R_{j_r}\}\right) = q^{-d} \prod_{r=1}^{d} \lambda(R_{j_r})(1+o(1)), \qquad (29)$$

where λ is the measure on $(-\infty, \infty]$ given by $e^{-x}dx$ and the relation is uniform over all d-tuples $i_1, \ldots, i_d$. Using elementary counting argument and (29), the sum of the probabilities of all collections of d distinct sets from the m sets that comprise the union in (28) is given by

$$\begin{aligned} S_d &= \sum_{(u_1,\ldots,u_k); u_1+\cdots+u_k=d} \binom{n_1}{u_1} \cdots \binom{n_k}{u_k} q^{-u_1} \lambda^{u_1}(R_1) \cdots q^{-u_k} \lambda^{u_k}(R_k)(1+o(1)) \\ &= \sum_{(u_1,\ldots,u_k); u_1+\cdots+u_k=d} \frac{1}{u_1! u_2! \cdots u_k! \pi^d} ((b_1-a_1)\lambda(R_1))^{u_1} \cdots ((b_k-a_k)\lambda(R_k))^{u_k}(1+o(1)) \\ &\to (d!)^{-1} \pi^{-1} ((b_1-a_1)\lambda(R_1) + \cdots + (b_k-a_k)\lambda(R_k))^d. \end{aligned}$$

Now it follows that,

$$\sum_{j=1}^{2s}(-1)^{j-1}S_j \overset{n\to\infty}{\longrightarrow} \sum_{j=1}^{2s}\frac{(-1)^{j-1}}{j!\pi^j}((b_1-a_1)\lambda(R_1)+\ldots+(b_k-a_k)\lambda(R_k))^j$$
$$\overset{s\to\infty}{\longrightarrow} 1-\exp\big(-\sum_{j=1}^{k}(b_j-a_j)\pi^{-1}\lambda(R_j)\big),$$

which by Bonferroni inequality and (28), proves (27).

Step 2: It remains to transfer the convergence of η_n^* onto η_n. First define the point process

$$\bar{\eta}_n(\cdot)=\sum_{j=1}^{q}\epsilon_{\left(\omega_j,\frac{\lambda_{n,\bar{x}}(\omega_j)-b_q}{a_q}\right)}(\cdot) \text{ and } \eta_n'(\cdot)=\sum_{j=1}^{q}\epsilon_{\left(\omega_j,\frac{\lambda_{n,x}(\omega_j)-b_q}{a_q}\right)}(\cdot).$$

It then suffices to show that (see Theorem 4.2 of Kallenberg)

$$\bar{\eta}_n-\eta_n^* \overset{\mathcal{P}}{\longrightarrow} 0, \tag{30}$$

$$\bar{\eta}_n-\eta_n' \overset{\mathcal{P}}{\longrightarrow} 0 \tag{31}$$

and

$$\eta_n'-\eta_n \overset{\mathcal{P}}{\longrightarrow} 0 \tag{32}$$

Equivalently, that for any continuous function f on $[0,\pi]\times(-\infty,\infty]$ with compact support,

$$\bar{\eta}_n(f)-\eta_n^*(f) \overset{\mathcal{P}}{\longrightarrow} 0\ ,\ \bar{\eta}_n(f)-\eta_n'(f) \overset{\mathcal{P}}{\longrightarrow} 0, \text{ and } \eta_n'(f)-\eta_n(f) \overset{\mathcal{P}}{\longrightarrow} 0$$

where the notation $\eta(f)$ denotes $\int f d\eta$. Suppose the compact support of f is contained in the set $[0,\pi]\times[K+\gamma_0,\infty)$ for some $\gamma_0>0$ and $K\in\mathbb{R}$. Since f is uniformly continuous , $\omega(\gamma):=\sup\{|f(t,x)-f(t,y)|;\ t\in[0,\pi],\ |x-y|\le\gamma\}\to 0$ as $\gamma\to 0$.
On the set $A_n=\{\max_{j=1,\ldots,q}|\frac{\lambda_{n,\bar{x}+\sigma N}(\omega_j)}{a_q}-\frac{\lambda_{n,\bar{x}}(\omega_j)}{a_q}|\le\gamma\}$, we have for $\gamma<\gamma_0$,

$$\big|f(\omega_j,\frac{\lambda_{n,\bar{x}+\sigma N}(\omega_j)-b_q}{a_q})-f(\omega_j,\frac{\lambda_{n,\bar{x}}(\omega_j)-b_q}{a_q})\big|\le\begin{cases}\omega(\gamma) & \text{if } \frac{\lambda_{n,\bar{x}+\sigma N}(\omega_j)-b_q}{a_q}>K\\ 0 & \text{if } \frac{\lambda_{n,\bar{x}+\sigma N}(\omega_j)-b_q}{a_q}\le K.\end{cases} \tag{33}$$

Also note

$$\begin{aligned}\frac{1}{a_q}\max_{1\le j\le q}|\lambda_{n,\bar{x}+\sigma N}(\omega_j)-\lambda_{n,\bar{x}}(\omega_j)| &\le \frac{1}{a_q}\max_{1\le j\le q}|\frac{\sigma_n}{\sqrt{n}}\sum_{t=1}^{n}N_te^{i\omega_jt}|\\ &\le \frac{\sigma_n}{a_q}\max_{1\le j\le q}\sqrt{\frac{1}{n}(\sum_{t=1}^{n}N_t\cos\frac{2\pi kt}{n})^2+\frac{1}{n}(\sum_{t=1}^{n}N_t\sin\frac{2\pi kt}{n})^2}\\ &\le \frac{\sigma_n}{a_q}\max_{1\le j\le q}\sqrt{X_{1j}^2+X_{2j}^2}\end{aligned}$$

where $\{X_{1j},X_{2j};1\le j\le q\}$ are i.i.d. $N(0,1)$. Now $\frac{\sigma_n}{a_q}\max_{1\le j\le q}\sqrt{X_{1j}^2+X_{2j}^2}=O_P(\sigma_n\ln n)$. Therefore $\lim_{n\to\infty}\mathrm{P}(A_n^c)=0$. Now, for any $\epsilon>0$, choose γ sufficiently small that $\gamma<\gamma_0$. Define $B_n=\{|\bar{\eta}_n(f)-\eta_n^*(f)|>\epsilon\}$. Then

$$\begin{aligned}\limsup_{n\to\infty}\mathrm{P}(B_n) &\le \limsup_{n\to\infty}(\mathrm{P}(B_n\cap A_n)+\mathrm{P}(A_n^c))\\ &\le \limsup_{n\to\infty}\mathrm{P}(\omega(\gamma)\eta_n^*([0,\pi]\times[K,\infty))>\epsilon)+\limsup_{n\to\infty}\mathrm{P}(A_n^c)\\ &\le \limsup_{n\to\infty}\mathrm{E}\,\eta_n^*([0,\pi]\times[K,\infty))\omega(\gamma)/\epsilon\le e^{-K}\omega(\gamma)/\epsilon.\end{aligned}$$

Since $\omega(\gamma) \to 0$ as $\gamma \to 0$, (30) follows.

The proof of (31) is essentially identical to the argument given for (30). For completeness we give the details. Now define $C_n = \{\max_{1\le j\le q} |\frac{\lambda_{n,x}(\omega_j)}{a_q} - \frac{\lambda_{n,\bar{x}}(\omega_j)}{a_q}| < \gamma\}$. Again on the set C_n, we have for $\gamma < \gamma_0$

$$\big|f(\omega_j, \frac{\lambda_{n,x}(\omega_j) - b_q}{a_q}) - f(\omega_j, \frac{\lambda_{n,\bar{x}}(\omega_j) - b_q}{a_q})\big| \le \begin{cases} \omega(\gamma) & \text{if } \frac{\lambda_{n,\bar{x}}(\omega_j)-b_q}{a_q} > K \\ 0 & \text{if } \frac{\lambda_{n,\bar{x}}(\omega_j)-b_q}{a_q} \le K. \end{cases} \tag{34}$$

Now

$$\begin{aligned}
\frac{1}{a_q}\,\mathrm{E}\{\max_{1\le j\le q} |\lambda_{n,x}(\omega_j) - \lambda_{n,\bar{x}}(\omega_j)|\} &\le \frac{1}{a_q}\,\mathrm{E}\{\max_{1\le j\le q} |\frac{1}{\sqrt{n}}\sum_{t=1}^{n} x_t \mathbb{I}(|x_t| > n^{1/s}) e^{i\omega_j t}|\} \\
&\le \frac{\sqrt{\ln n}}{\sqrt{n}}\,\mathrm{E}\{\sum_{t=1}^{n} |x_t| \mathbb{I}(|x_t| > n^{1/s})\} \\
&\le \sqrt{n \ln n}\,\mathrm{E}\,|x_1| \mathbb{I}(|x_t| > n^{1/s}) \\
&= \sqrt{n\ln n}\big[n^{1/s}\,\mathrm{P}(|x_1| > n^{1/s}) + \int_{n^{1/s}}^{\infty} \mathrm{P}(X_1 > x)dx\big] \\
&\le \sqrt{n\ln n}\big[n^{1/s}\frac{\mathrm{E}\,|x_1|^s}{n} + \frac{\mathrm{E}\,|x_1|^s}{n^{1-1/s}}\big] \\
&\le 2\frac{\ln n}{n^{1/2-1/s}}\,\mathrm{E}\,|x_1|^s \to 0, \text{ as } n \to \infty.
\end{aligned}$$

Therefore $\mathrm{P}(C_n^c) \to 0$. Now for any $\epsilon > 0$, choose γ sufficiently small that $\gamma < \gamma_0$. Then by intersecting the event $\{|\eta_n(f) - \bar{\eta}_n(f)| > \epsilon\}$ with A_n and A_n^c, respectively and using (34) and (26), (30) we obtain

$$\begin{aligned}
\limsup_{n\to\infty} \mathrm{P}(|\bar{\eta}_n(f) - \eta(f)| > \epsilon) &\le \limsup_{n\to\infty} (\mathrm{P}(\omega(\gamma)\bar{\eta}_n([0,\pi]\times[K,\infty)) > \epsilon) + \mathrm{P}(A_n^c)) \\
&\le \limsup_{n\to\infty} \mathrm{E}\,\bar{\eta}_n([0,\pi]\times[K,\infty))\omega(\gamma)/\epsilon \le e^{-K}\omega(\gamma)/\epsilon.
\end{aligned}$$

Since $\omega(\gamma) \to 0$ as $\gamma \to 0$, (31) follows.

Finally for any $\epsilon > 0$

$$\begin{aligned}
\mathrm{P}(|\eta_n'(f) - \eta_n(f)| > \epsilon) &= \mathrm{P}(|f(0, \frac{\lambda_{n,x}(\omega_0) - b_q}{a_q})| > \epsilon) \\
&\le \mathrm{P}(\frac{\lambda_n(\omega_0) - b_q}{a_q} \ge K) = \mathrm{P}(\frac{1}{\sqrt{n}}\sum_{l=0}^{n-1} x_l > K a_q + b_q) \to 0, \text{ as } n \to \infty.
\end{aligned}$$

Therefore $\eta_n - \eta_n' \xrightarrow{\mathcal{P}} 0$. □

Proof of Corollary 0.1. The proof is similar to the proof of Theorem 4.2.8 of Embrechts, P., Kluppelberg, C. and Mikosch, T. (1997). We just briefly sketch the steps. We have already seen that for finite k,

$$\begin{aligned}
\mathrm{P}(\frac{\lambda_{n,(1)} - b_q}{a_q} \le x_1, \ldots, \frac{\lambda_{n,(k)} - b_q}{a_q} \le x_k) &= \mathrm{P}(N_{1,n} = 0, N_{2,n} \le 1, \ldots, N_{k,n} \le k-1) \\
&\to \mathrm{P}(N_1 = 0, N_2 \le 1, \cdots, N_k \le k-1),
\end{aligned}$$

where $N_i = \eta([0,\pi]\times(x_i,\infty])$. Let us denote $Z_i = \eta([0,\pi]\times(x_i, x_{i-1}])$ with $x_0 = \infty$. Now observe that to calculate $\mathrm{P}(N_1 = 0, N_2 \le 1, \cdots, N_k \le k-1)$, it is enough to consider $\mathrm{P}(N_1 = a_1, N_2 = a_1 + a_2, \cdots, N_k = a_1 + \cdots + a_k)$, where $a_i \ge 0$ and

$$\begin{aligned}
&\mathrm{P}(N_1 = a_1, N_2 = a_1 + a_2, \cdots, N_k = a_1 + \cdots + a_k) \\
&= \mathrm{P}(Z_1 = a_1, Z_2 = a_2, \cdots, Z_k = a_k)
\end{aligned}$$

$$= \frac{(e^{-x_1})^{a_1}}{a_1!}\frac{(e^{-x_2}-e^{-x_1})^{a_2}}{a_2!}\cdots\frac{(e^{-x_k}-e^{-x_{k-1}})^{a_k}}{a_k!}e^{-e^{-x_k}}.$$

This proves Part (i). Part (ii) is an easy consequence of Part (i). □

4.2. **Proof of Theorem 2.** The following lemma, similar to Lemma 3 is a strong approximation result which we shall use in the proof of Theorem 2. Suppose $v_d(0) = \sqrt{2}(1,1,...,1)$ and for $1 \le t \le j$, $v_d(t)$ are same as (24).

Lemma 4. *Let $n = 1+2j$ and $\sigma_j^2 = (1+2j)^{-c}$ for some $c > 0$ and let $\{x_t\}$ be i.i.d mean zero with $Ex_0^2 = 1$ and $E|x_0|^s < \infty$ for some $s > 2$. Suppose N_t's are i.i.d. $N(0,1)$ random variables independent of $\{x_t\}$ and $\tilde{p}_j(x)$ is the density of*

$$\frac{1}{\sqrt{1+2j}}\sum_{t=0}^{j}(\bar{x}_t+\sigma_j N_t)v_d(t).$$

Then for any measurable subset E of $\mathbb{R}^d$,

$$\Big|\int_E \tilde{p}_j(x)dx - \int_E \phi_{(1+\sigma_j^2)I_d}(x)dx\Big| \;\le\; \epsilon_j\int_E \phi_{(1+\sigma_j^2)I_d}(x)dx + O(\exp(-(1+2j)^{\eta}))$$

where $\epsilon_j \to 0$, $\eta > 0$ and the above holds uniformly over d-tupules $1 \le i_1 < i_2 < ... < i_d < j$.

Proof of Theorem 2. The idea of the proof is similar to proof of Theorem 1. So we mention only the main steps and a few technical details. We first establish convergence in distribution for the point process based on the points $(\omega_j, \frac{\lambda'_{n,\bar{x}+\sigma_n N}(\omega_j)-b_q}{a_q})$ for $j = 1,2,\ldots,q$, where

$$\lambda'_{n,\bar{x}+\sigma_n N}(\omega_j) = \frac{1}{\sqrt{n}}\big[\sqrt{2}(\bar{x}_0+\sigma_n N_0) + 2\sum_{t=1}^{[n/2]}(\bar{x}_t+\sigma_n N_t)\cos\frac{2\pi jt}{n}\big],\quad 0 \le j \le [n/2].$$

Define

$$\eta_n^*(\cdot) = \sum_{j=1}^{q}\epsilon_{\left(\omega_j, \frac{\lambda'_{n,\bar{x}+\sigma_n N}(\omega_j)-b_q}{a_q}\right)}(\cdot).$$

Since the limit process η is simple, it suffices to show (26) and (27) for above η_n^*. We can establish them following arguments similar to those given in the proof of Theorem 1 and using Lemma 4. Now define the following point processes

$$\bar{\eta}'_n(\cdot) = \sum_{j=1}^{q}\epsilon_{\left(\omega_j, \frac{\lambda'_{n,\bar{x}}(\omega_j)-b_q}{a_q}\right)}(\cdot),\quad \bar{\eta}_n(\cdot) = \sum_{j=1}^{q}\epsilon_{\left(\omega_j, \frac{\lambda_{n,\bar{x}}(\omega_j)-b_q}{a_q}\right)}(\cdot) \text{ and } \eta'_n(\cdot) = \sum_{j=1}^{q}\epsilon_{\left(\omega_j, \frac{\lambda_{n,x}(\omega_j)-b_q}{a_q}\right)}(\cdot)$$

where

$$\lambda'_{n,\bar{x}}(\omega_j) = \frac{1}{\sqrt{n}}\big[\sqrt{2}\bar{x}_0 + 2\sum_{t=1}^{[n/2]}\bar{x}_t\cos\frac{2\pi jt}{n}\big],\quad 0 \le j \le [n/2],$$

and $\{\lambda_{n,\bar{x}}(\omega_j)\}$ are given in (3) with x_t replaced by $\bar{x}_t$. As before it now suffices to show that (see Theorem 4.2 of Kallenberg)

$$\bar{\eta}'_n - \eta_n^* \xrightarrow{\mathcal{P}} 0,\quad \bar{\eta}_n - \bar{\eta}'_n \xrightarrow{\mathcal{P}} 0,\quad \bar{\eta}_n - \eta'_n \xrightarrow{\mathcal{P}} 0 \text{ and } \eta'_n - \eta_n \xrightarrow{\mathcal{P}} 0. \tag{35}$$

For the first relation in (35) define $A_n = \{\max_{1\le j\le q}|\lambda'_{n,\bar{x}}(\omega_j) - \lambda_{n,\bar{x}+\sigma N}(\omega_j)| \le \gamma\}$ and observe that

$$\max_{1\le j\le q}|\lambda'_{n,\bar{x}}(\omega_j) - \lambda_{n,\bar{x}+\sigma N}(\omega_j)| \;=\; \frac{\sigma_n}{\sqrt{n}}\max_{1\le j\le q}|\sqrt{2}N_0 + 2\sum_{t=1}^{q}N_t\cos\frac{2\pi jt}{n}| = O_p(\sigma_n\ln n).$$

Hence $\mathrm{P}(A_n^c) \to 0$. The remaining argument is similar to the proof of (30). For the second relation note that

$$\mathrm{P}\big(\max_{1\le j\le q}|\lambda_{n,\bar{x}}(\omega_j) - \lambda'_{n,\bar{x}}(\omega_j)| > \epsilon\big) \le \mathrm{P}\big(\frac{(\sqrt{2}-1)|x_0|}{\sqrt{n}} > \epsilon\big) \to 0.$$

Proof of the third and fourth relations are similar to the proofs of (31) and (32) in Theorem 1. □

4.3. **Proof of Theorem 3.** We begin by proving the Lemma.

Proof of Lemma 1. First observe that $S(0) = \{0\}$ and $S(n/2) = \{n/2\}$ if k is odd and

$$\#\{x: \ x \in S(ak+b); 0 \le a \le [\frac{k-2}{2}], a+1 \le b \le k-a-1\} = \begin{cases} n-1 & \text{if } \ k \text{ even} \\ n-2 & \text{if } \ k \text{ odd.} \end{cases}$$

So if we can show that $S(ak+b); 0 \le a \le [\frac{k-2}{2}], a+1 \le b \le k-a-1$ are mutually disjoint then we are done. We shall show $S(a_1k+b_1) \cap S(a_2k+b_2) = \phi$ for $a_1 \ne a_2$ or $b_1 \ne b_2$. We divide the proof into four different cases.

Case (i) $(a_1 < a_2, b_1 > b_2)$ Note that

$$a_1 + 1 < a_2 + 1 \le b_2 < b_1 \le k - (a_1+1).$$

Since $\{S(x); 0 \le x \le n-1\}$ forms a partition of $\mathbb{Z}_n$, it is enough to show that $a_1k+b_1 \notin S(a_2k+b_2)$. As $(a_2-a_1)k > k$ and $(b_1-b_2) < k$, we have $a_1k+b_1 \ne a_2k+b_2$. Also $(b_2-a_1)k \ge 2k$ and $a_2+b_1 \le [\frac{k-2}{2}] + k - (a_1+1) \le \frac{3k}{2}$, therefore $a_1k+b_1 \ne b_2k-a_2$. Note that

$$\begin{aligned} a_1k+b_1+a_2k+b_2 &\le (a_1+a_2)k + 2k - 2(a_1+1) \\ &\le 2[\frac{k-2}{2}]k + 2k - 2(a_1+1) \\ &\le k^2 - 2k + 2k - 2(a_1+1) \\ &< k^2+1 = n. \end{aligned}$$

Therefore $a_1k+b_1 \ne n-(a_2k+b_2)$. Similarly,

$$a_1k+b_1+b_2k-a_2 \ \le \ a_1k + k - (a_1+1) + (k-(a_2+1))k - a_2 < k^2+1 = n$$

and therefore $a_1k+b_1 \ne n-(b_2k-a_2)$. Hence in this case $S(a_1k+b_1) \cap S(a_2k+b_2) = \phi$.

Case (ii) $(a_1 < a_2, b_1 < b_2)$ In this case it is very easy to see that $a_1k+b_1 \notin S(a_2k+b_2)$ and hence $S(a_1k+b_1) \cap S(a_2k+b_2) = \phi$.

Case (iii) $(a_1 = a_2, b_1 < b_2)$ Let $a_1 = a_2 = a$. Obviously $ak+b_1 \ne ak+b_2$. Since $0 \le a \le [\frac{k-2}{2}]$ and $a+1 \le b_1 < b_2 \le k-(a+1)$, we have $(b_2-a)k \ge 2k > (a+b_1)$. Hence $ak+b_1 \ne b_2k-a$. Also $2ak+b_1+b_2 \le k(k-2)+2k = k^2 < n$, so $ak+b_1 \ne n-(ak+b_2)$. Finally,

$$b_1+b_2k+ak-a \le [k-(a+1)](k+1)+ak-a = k^2-2a-1 < k^2+1 = n,$$

implies $ak+b_1 \ne n-(b_2k-a)$. Hence $ak+b_1 \notin S(ak+b_2)$ and $S(a_1k+b_1) \cap S(a_2k+b_2) = \phi$.

Case (iv) $(a_1 < a_2, b_1 = b_2)$ In this case also it is very easy to show that $S(a_1k+b_1) \cap S(a_2k+b_2) = \phi$. This completes the proof. □

Proof of Theorem 3. Though the main idea of the proof is similar to the proof of Theorem 1, the details are more complicated. We prove it in two steps.

Step 1: We first establish convergence in distribution for the point process based on the points $\{(\frac{a}{\sqrt{n}}, \frac{b}{\sqrt{n}}, \frac{\lambda_{\bar{x}+\sigma_n N}(a,b)-d_q}{c_q}) : (a,b) \in T_n\}$ where $\lambda_{\bar{x}+\sigma_n N}(a,b)$ is obtained from $\lambda_x(a,b)$ replacing $\{x_i\}$ by $\{\bar{x}_i + \sigma_n N_i\}$. Define

$$\eta_n^*(\cdot) = \sum_{(a,b)\in T_n} \epsilon_{\left(\frac{a}{\sqrt{n}}, \frac{b}{\sqrt{n}}, \frac{\lambda_{\bar{x}+\sigma_n N}(a,b)-d_q}{c_q}\right)}(\cdot).$$

Observe that first two components of the limit is uniformly distributed over a triangle whose vertices are $(0,0), (1/2,1/2), (0,1)$. Denote this triangle by $\triangle$. Since the limit process is simple it suffices to show that

$$\text{(36)} \qquad \mathrm{E}\,\eta_n^*((a_1,b_1] \times (a_2,b_2] \times (x,y]) \to \mathrm{E}\,\eta((a_1,b_1] \times (a_2,b_2] \times (x,y])$$

for all $0 \le a_1 < b_1 \le 1/2$, $0 \le a_2 < b_2 \le 1$ and $x < y$, and for all $l \ge 1$,

$$\text{(37)} \qquad \mathrm{P}(\eta_n^*((a_1,b_1]\times(c_1,d_1]\times R_1)=0,\ldots,\eta_n^*((a_l,b_l]\times(c_l,d_l]\times R_l)=0)$$
$$\longrightarrow \mathrm{P}(\eta((a_1,b_1]\times(c_1,d_1]\times R_1)=0,\ldots,\eta((a_l,b_l]\times(c_l,d_l]\times R_l)=0),$$

where $\cap_{i=1}^{l}(a_i,b_i]\times(c_i,d_i]=\phi$ and $R_1,\ldots,R_l$ are bounded Borel sets, each consisting of a finite union of intervals on $[0,\infty]$.

Proof of (36): We shall first prove condition (36) for the following type of sets:
(i) $(a_1,b_1]\times(a_2,b_2]$ lies entirely inside the trangle $\triangle$.
(ii) $(a_1,b_1]\times(a_1,b_1]$ where $0 \le a_1 < b_1 \le 1/2$.
(iii) $(a_1,b_1]\times(1-b_1,1-a_1]$ where $0 \le a_1 < b_1 \le 1/2$.
(iv) $(a_1,b_1]\times(a_2,b_2]$ lies entirely outside of the triangle $\triangle$.

Graphically the mentioned boxes are as in Figure 1.

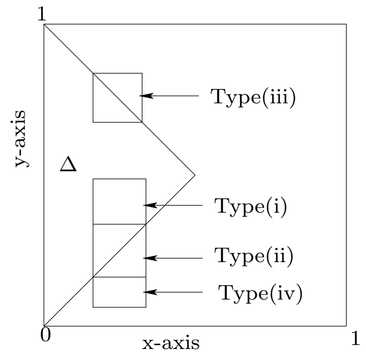

Figure 1

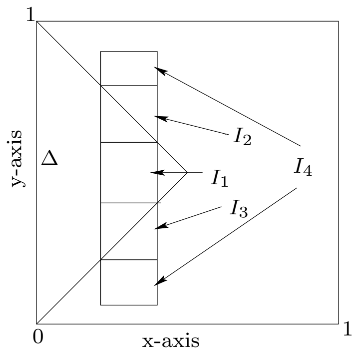

Figure 2

Figure 1 shows four types of basic sets and Figure 2 shows the decomposition of a rectangle in four types of sets.

Since any boxes in $[0,1/2]\times[0,1]$ can be expressed as disjoint union of these four kinds of sets (see Figure 2), it is sufficient to prove (36) and (37) for the above four kind of boxes only. Let I_i denote i-th type of set. Enough to prove that for each i, as $n\to\infty$, $\mathrm{E}\,\eta_n^*(I_i\times(x,y])\to\mathrm{E}\,\eta(I_i\times(x,y])$.
Proof of (36) for Type (i) sets:

$$\begin{aligned}
\mathrm{E}\,\eta_n^*((a_1,b_1]\times(a_2,b_2]\times(x,y]) &= \mathrm{E}\Big(\sum_{(a,b)\in T_n}\epsilon_{\big(\frac{a}{\sqrt{n}},\frac{b}{\sqrt{n}},\frac{\lambda_{\bar{x}+\sigma_n N(a,b)}-d_q}{c_q}\big)}((a_1,b_1]\times(a_2,b_2]\times(x,y])\Big)\\
&= \sum_{(\frac{a}{\sqrt{n}},\frac{b}{\sqrt{n}})\in(a_1,b_1]\times(a_2,b_2]} \mathrm{P}\Big(\frac{\lambda_{\bar{x}+\sigma_n N}(a,b)-d_q}{c_q}\Big)\in(x,y]\Big)\\
&\sim (b_1-a_1)(b_2-a_2)n\frac{1}{q}(e^{-x}-e^{-y})(1+o(1))\\
&\to 4(b_1-a_1)(b_2-a_2)(e^{-x}-e^{-y})\\
&= \mathrm{E}\,\eta((a_1,b_1]\times(a_2,b_2]\times(x,y]).
\end{aligned}$$

Proof of (36) for Type (ii) sets:

$$\begin{aligned}
\mathrm{E}\,\eta_n^*((a_1,b_1]\times(a_1,b_1]\times(x,y]) &= \mathrm{E}\Big(\sum_{(a,b)\in T_n}\epsilon_{\big(\frac{a}{\sqrt{n}},\frac{b}{\sqrt{n}},\frac{\lambda_{\bar{x}+\sigma_n N(a,b)}-d_q}{c_q}\big)}((a_1,b_1]\times(a_1,b_1]\times(x,y])\Big)\\
&= \sum_{(\frac{a}{\sqrt{n}},\frac{b}{\sqrt{n}})\in(a_1,b_1]\times(a_1,b_1]} \mathrm{P}\Big(\frac{\lambda_{\bar{x}+\sigma_n N}(a,b)-d_q}{c_q}\Big)\in(x,y]\Big)\\
&\sim \frac{1}{2}(b_1-a_1)(b_1-a_1)n\frac{1}{q}(e^{-x}-e^{-y})(1+o(1))
\end{aligned}$$

$$\begin{aligned} &\to \frac{1}{2}(b_1-a_1)^2 4(e^{-x}-e^{-y}) \\ &= \mathrm{E}\,\eta((a_1,b_1]\times(a_1,b_1]\times(x,y]). \end{aligned}$$

Proof of (36) for Type (iii) sets is exactly similar as Type (ii) sets.

Proof of (36) for Type (iv) sets:

$$\begin{aligned} \mathrm{E}\,\eta_n^*((a_1,b_1]\times(a_2,b_2]\times(x,y]) &= \mathrm{E}\Big(\sum_{(a,b)\in T_n}\epsilon_{\big(\frac{a}{\sqrt{n}},\frac{b}{\sqrt{n}},\frac{\lambda_{\bar{x}+\sigma_n N(a,b)}-d_q}{c_q}\big)}((a_1,b_1]\times(a_2,b_2]\times(x,y])\Big) \\ &= \sum_{(\frac{a}{\sqrt{n}},\frac{b}{\sqrt{n}})\in(a_1,b_1]\times(a_2,b_2]} \mathrm{P}\Big(\frac{\lambda_{\bar{x}+\sigma_n N}(a,b)-d_q}{c_q}\Big)\in(x,y]\Big) \\ &= 0 = \mathrm{E}\,\eta((a_1,b_1]\times(a_2,b_2]\times(x,y]), \end{aligned}$$

since $\{(a,b)\in T_n : (\frac{a}{\sqrt{n}},\frac{b}{\sqrt{n}})\in(a_1,b_1]\times(a_2,b_2]\}=\phi$. This completes the proof of (36).

Proof of (37): We prove (37) for four types of sets separately.

Proof of (37) for Type (i) sets: $(a_i,b_i]\times(c_i,d_i]$ lies completely inside the triangle $\triangle$ for all $i=1,2,\ldots,l$. Let

$$\begin{aligned} n_j &= \#\{(a,b) : (\frac{a}{\sqrt{n}},\frac{b}{\sqrt{n}})\in(a_j,b_j]\times(c_j,d_j]\} \\ &\sim \sqrt{n}(b_j-a_j)\sqrt{n}(d_j-c_j) = n(b_j-a_j)(d_j-c_j). \end{aligned}$$

Then the complement of the event in (37) is the union of $m=n_1+\ldots+n_l$ events, that is

$$\begin{aligned} &1-\mathrm{P}\big(\eta_n^*((a_1,b_1]\times(c_1,d_1]\times R_1)=0,\ldots,\eta_n^*((a_l,b_l]\times(c_l,d_l]\times R_l)=0\big) \\ &= \mathrm{P}\big(\cup_{j=1}^{l}\cup_{(\frac{a}{\sqrt{n}},\frac{b}{\sqrt{n}})\in(a_j,b_j]\times(c_j,d_j]}\{\frac{\lambda_{x+\sigma_n N-d_q}}{c_q}\in R_j\}\big). \end{aligned}$$

Now following the argument to prove (27) given in Theorem 1, we get

$$\begin{aligned} &\mathrm{P}\big(\eta_n^*((a_1,b_1]\times(c_1,d_1]\times R_1)=0,\ldots,\eta_n^*((a_l,b_l]\times(c_l,d_l]\times R_l)=0\big) \\ &\stackrel{n\to\infty}{\longrightarrow} \exp\{-\sum_{j=1}^{l}(b_j-a_j)(d_j-c_j)4\lambda(R_j)\} \\ &= \mathrm{P}(\eta((a_1,b_1]\times(c_1,d_1]\times R_1)=0,\ldots,\eta((a_l,b_l]\times(c_l,d_l]\times R_l)=0). \end{aligned}$$

This proves (37) for Type (i) sets.

Proof of (37) for Type (ii) sets: Here $c_i=a_i$, $d_i=b_i$ and

$$\begin{aligned} n_j &= \#\{(a,b) : (\frac{a}{\sqrt{n}},\frac{b}{\sqrt{n}})\in(a_j,b_j]\times(a_j,b_j]\} \\ &\sim \frac{1}{2}\sqrt{n}(b_j-a_j)\sqrt{n}(b_j-a_j) = \frac{n}{2}(b_j-a_j)^2. \end{aligned}$$

Remaining part of the proof is as in the previous case. Finally we get

$$\begin{aligned} &\mathrm{P}\big(\eta_n^*((a_1,b_1]\times(a_1,b_1]\times R_1)=0,\ldots,\eta_n^*((a_l,b_l]\times(a_l,b_l]\times R_l)=0\big) \\ &\stackrel{n\to\infty}{\longrightarrow} \exp\{-\sum_{j=1}^{l}\frac{1}{2}(b_j-a_j)^2 4\lambda(R_j)\} \\ &= \mathrm{P}(\eta((a_1,b_1]\times(a_1,b_1]\times R_1)=0,\ldots,\eta((a_l,b_l]\times(a_l,b_l]\times R_l)=0). \end{aligned}$$

Proof of (37) for Type (iii) is same as Type (ii) sets.
Finally we prove it for Type (iv) sets. In this case $(a_i, b_i] \times (c_i, d_i] \bigcap \triangle = \phi$ for all $i = 1, \ldots, l$. Note that for all i, $\#\{(a,b) \in T_n : (\frac{a}{\sqrt{n}}, \frac{b}{\sqrt{n}}) \in (a_i, b_i] \times (c_i, d_i]\} = 0$ and therefore

$$\mathrm{P}\left(\eta_n^*((a_1, b_1] \times (c_1, d_1] \times R_1) = 0, \ldots, \eta_n^*((a_l, b_l] \times (c_l, d_l] \times R_l) = 0\right) = 1.$$

Also from intensity measure of η,

$$\mathrm{P}(\eta((a_1, b_1] \times (c_1, d_1] \times R_1) = 0, \ldots, \eta((a_l, b_l] \times (c_l, d_l] \times R_l) = 0) = 1.$$

Hence (37) is proved for all four types of sets separately.

Step 2: It remains to transfer the convergence of η_n^* onto η_n. First define the following process

$$\bar{\eta}_n(\cdot) = \sum_{(a,b)\in T_n} \epsilon_{\left(\frac{a}{\sqrt{n}}, \frac{b}{\sqrt{n}}, \frac{\lambda_{\bar{x}}(a,b) - d_q}{c_q}\right)}(\cdot).$$

Then it suffices to show that for any continuous function f on $[0, 1/2] \times [0, 1] \times [0, \infty)$ with compact support,

$$\bar{\eta}_n(f) - \eta_n^*(f) \xrightarrow{\mathcal{P}} 0 \text{ and } \bar{\eta}_n(f) - \eta(f) \xrightarrow{\mathcal{P}} 0. \tag{38}$$

Suppose the compact support of f is contained in the set $[0, 1/2] \times [0, 1] \times [K + \gamma_0, \infty)$ for some $\gamma_0 > 0$ and $K \in \mathbb{R}_{\geq 0}$. Since f is uniformly continuous, $\omega(\gamma) := \sup\{|f(s,t,x) - f(s,t,y)|;\ s \in [0, 1/2],\ t \in [0,1],\ |x - y| \leq \gamma\} \to 0$ as $\gamma \to 0$. On the set $A_n = \{\max_{(a,b)\in T_n} |\frac{\lambda_{\bar{x}+\sigma_n N}(a,b)}{c_q} - \frac{\lambda_{\bar{x}}(a,b)}{c_q}| \leq \gamma\}$, we have for $\gamma < \gamma_0$,
(39)

$$\left|f\left(\frac{a}{\sqrt{n}}, \frac{b}{\sqrt{n}}, \frac{\lambda_{\bar{x}+\sigma_n N}(a,b) - d_q}{c_q}\right) - f\left(\frac{a}{\sqrt{n}}, \frac{b}{\sqrt{n}}, \frac{\lambda_{\bar{x}}(a,b) - d_q}{c_q}\right)\right| \leq \begin{cases} \omega(\gamma) & \text{if } \frac{\lambda_{n,\bar{x}+\sigma N}(\omega_j) - b_q}{a_q} > K \\ 0 & \text{if } \frac{\lambda_{n,\bar{x}+\sigma N}(\omega_j) - b_q}{a_q} \leq K. \end{cases}$$

Now if $\mathrm{P}(A_n^c) \to 0$, then using (39)

$$\limsup_{n\to\infty} \mathrm{P}\left(|\eta_n^*(f) - \bar{\eta}_n(f)| > \epsilon\right) \leq \frac{\omega(\gamma)}{\epsilon} 4e^{-K} \to 0, \quad \text{as } \gamma \to 0.$$

Now we show $\mathrm{P}(A_n^c) \to 0$. For any sequence of random variables $(X_i)_{0 \leq i < n}$, define

$$M_n(X) = \max_{1\leq t\leq n} \left|n^{-1/2} \sum_{l=0}^{n-1} X_l \exp(i2\pi t l/n)\right|.$$

We can use the basic inequalities

$$\big||z_1 z_2| - |w_1 w_2|\big| \leq (|z_1| + |w_2|) \max\{|z_1 - w_1|, |z_2 - w_2|\},$$

and

$$||w_1|^{1/2} - |w_2|^{1/2}| \leq |w_1 - w_2|^{1/2}, \quad z_i, w_i \in \mathbb{C},\ 1 \leq i \leq 2,$$

to obtain

$$\begin{aligned} \max_{a,b} \left|\lambda_{\bar{x}+\sigma_n N}(a,b) - \lambda_{\bar{x}}(a,b)\right| &\leq \left[(M_n(\bar{x} + \sigma_n N))^{1/2} + (M_n(\bar{x}))^{1/2}\right](M_n(\sigma_n N))^{1/2} \\ &\leq \left[2(M_n(\bar{x} + \sigma_n N))^{1/2} + (M_n(\sigma_n N))^{1/2}\right](M_n(\sigma_n N))^{1/2}. \end{aligned}$$

By Davis and Mikosch (1999), we have

$$M_n^2(\sigma_n N) = O_p(\sigma_n^2 \log n) \text{ and } M_n^2(\bar{x} + \sigma_n N) = O_p(\log n),$$

with $\sigma_n^2 = n^{-c}$. Therefore

$$\max_{a,b} \frac{1}{c_q} \left|\lambda_{\bar{x}+\sigma_n N}^2(a,b) - \lambda_{\bar{x}}^2(a,b)\right| = O_p((\log n) n^{-c/4}).$$

Hence

$$\mathrm{P}(A_n^c) = \mathrm{P}\left(\max_{a,b} \left|\frac{\lambda_{\bar{x}+\sigma_n N}(a,b)}{c_q} - \frac{\lambda_{\bar{x}}(a,b)}{c_q}\right| > \epsilon\right)$$

$$= \mathrm{P}\big(\frac{n^{c/4}}{\log n}\max_{a,b}\frac{1}{c_q}\big|\lambda_{\bar{x}+\sigma_n N}(a,b)-\lambda_{\bar{x}}(a,b)\big| > \frac{\epsilon n^{c/4}}{\log n}\big) \to 0 \text{ as } n\to\infty.$$

The proof of the other part of (38) is essentially identical with the conclusion of Lemma 8 of Bose, Mitra and Sen (2008) playing the key role. □

4.4. **Proofs for dependent inputs.**

Proof of Theorem 4. First observe that $\min_{\omega\in[0,2\pi]} f(\omega) > \alpha > 0$. We define another sequence of point process based on the points $\big(\omega_k, \frac{\lambda_{n,\epsilon}(\omega_k)-b_q}{a_q}\big)$ for $k=1,2,\ldots,q$ where $\lambda_{n,x}(\omega_k)$ are the eigenvalues of $n^{-1/2}RC_n$ with x_i replaced by ϵ_i. Define

$$\eta_n(\cdot) = \sum_{j=1}^{q} \epsilon_{\big(\omega_j, \frac{\lambda_{n,\epsilon}(\omega_j)-b_q}{a_q}\big)}(\cdot). \tag{40}$$

In Theorem 1, we have shown that $\eta_n \xrightarrow{\mathcal{D}} \eta$, where η is a Poisson process on $[0,\pi]\times(-\infty,\infty]$ with intensity measure $\pi^{-1}dt\times e^{-x}dx$. Now if we can show that $\tilde{\eta}_n - \eta_n \xrightarrow{\mathcal{P}} 0$, then we will be through. Equivalently, we have to show that for any continuous function g on E with compact support,

$$\tilde{\eta}_n(g) - \eta_n(g) \xrightarrow{\mathcal{P}} 0$$

as $n\to\infty$. Suppose the compact support of g is contained in the set $[0,\pi]\times[K+\gamma_0,\infty)$ for some $\gamma_0>0$ and $K\in\mathbb{R}$. Since g is uniformly continuous , $\omega(\gamma) := \sup\{|g(t,x)-g(t,y)|;\ t\in[0,1],\ |x-y|\le\gamma\} \to 0$ as $\gamma\to 0$. On the set $A_n = \{\max_{j=1,\ldots,q}|\frac{\lambda_{n,x}(\omega_j)}{a_q\sqrt{2\pi f(\omega_j)}} - \frac{\lambda_{n,\epsilon}(\omega_j)}{a_q}| \le \gamma\}$, we have for $\gamma<\gamma_0$,

$$\big|g(\omega_j, \frac{\tilde{\lambda}_{n,x}(\omega_k)-b_q}{a_q}) - g(\omega_j, \frac{\lambda_{n,\epsilon}(\omega_j)-b_q}{a_q})\big| \le \begin{cases} \omega(\gamma) & \text{if } \frac{\lambda_{n,\epsilon}(\omega_j)-b_q}{a_q} > K \\ 0 & \text{if } \frac{\lambda_{n,\epsilon}(\omega_j)-b_q}{a_q} \le K. \end{cases} \tag{41}$$

Observe

$$\begin{aligned} \frac{1}{a_q}\max_{1\le j\le q}|\frac{\lambda_{n,x}(\omega_j)}{\sqrt{2\pi f(\omega_j)}} - \lambda_{n,\epsilon}(\omega_j)| &\le \frac{1}{\alpha a_q}\max_{1\le j\le q}|\lambda_{n,x}(\omega_j) - \sqrt{2\pi f(\omega_j)}\lambda_{n,\epsilon}(\omega_j)| \\ &\le \frac{1}{\alpha a_q}\max_{1\le j\le q}|\frac{1}{\sqrt{n}}\sum_{l=0}^{n-1}x_l e^{i\omega_j l} - (\sum_{t=-\infty}^{\infty} a_t e^{i\omega_j t})\frac{1}{\sqrt{n}}\sum_{l=0}^{n-1}\epsilon_l e^{i\omega_j l}| \end{aligned}$$

and following the argument of Theorem 3 of Walker (1965) we can show that

$$\max_{1\le j\le q}|\frac{1}{\sqrt{n}}\sum_{l=0}^{n-1}x_l e^{i\omega_j l} - (\sum_{t=-\infty}^{\infty} a_t e^{i\omega_j t})\frac{1}{\sqrt{n}}\sum_{l=0}^{n-1}\epsilon_l e^{i\omega_j l}| = o_P(n^{-1/4}).$$

Therefore $\lim_{n\to\infty}\mathrm{P}(A_n^c)=0$. Now, for any $\delta>0$, choose γ sufficiently small that $\gamma<\gamma_0$. Then, by intersecting the event $\{|\tilde{\eta}_n(g)-\eta_n(g)|>\delta\}$ with A_n and A_n^c and using (41), we obtain

$$\begin{aligned} \limsup_{n\to\infty}\mathrm{P}(|\tilde{\eta}_n(g)-\eta_n(g)|>\delta) &\le \limsup_{n\to\infty}(\mathrm{P}(\{|\tilde{\eta}_n(g)-\eta_n(g)|>\delta\}\cap A_n) + \mathrm{P}(A_n^c)) \\ &\le \limsup_{n\to\infty}\mathrm{P}(\omega(\gamma)\eta_n([0,\pi]\times[K,\infty))>\epsilon) + \limsup_{n\to\infty}\mathrm{P}(A_n^c) \\ &\le \limsup_{n\to\infty}\mathrm{E}\,\eta_n([0,\pi]\times[K,\infty))\omega(\gamma)/\epsilon \le e^{-K}\omega(\gamma)/\epsilon. \end{aligned}$$

Since $\omega(\gamma)\to 0$ as $\gamma\to 0$, $\tilde{\eta}_n - \eta_n \xrightarrow{\mathcal{P}} 0$. □

Proof of Theorem 5. The line of argument is similar as in Theorem 4. We omit the details but mention that to show $\lim_{n\to\infty}\mathrm{P}(A_n^c)=0$, we use the following fact from (3.8) of Bose, Hazra and Saha (2009)

$$\max_{1\le k\le[n/2]}\Big|\frac{\lambda_k}{\sqrt{2\pi f(\omega_k)}} - \lambda_{k,\epsilon}\Big| = o_p(n^{-1/4}).$$

□

Proof of Theorem 6. First define a point process based on $\{(\frac{a}{\sqrt{n}}, \frac{b}{\sqrt{n}}, \frac{\lambda_\epsilon(a,b)-d_q}{c_q}) : (a,b) \in T_n\}$,

$$\eta_n(\cdot) = \sum_{j=0}^{q} \epsilon_{\left(\frac{a}{\sqrt{n}}, \frac{b}{\sqrt{n}}, \frac{\lambda_\epsilon(a,b)-d_q}{c_q}\right)}(\cdot).$$

First note that in Theorem 3, we have shown that $\eta_n \xrightarrow{\mathcal{D}} \eta$, where η is a Poisson process on $[0,1/2] \times [0,1] \times (-\infty,\infty]$ with intensity measure $4\mathbb{I}_{\{s \le t \le 1-s\}} e^{-x} dsdtdx$. Rest of the argument is similar to the proof of Theorem 4. The additional point that needs to be noted is that $\mathrm{P}(\max_{(a,b)\in T_n} \left|\tilde{\lambda}_x(a,b) - \lambda_\epsilon(a,b)\right| > \gamma) \to 0$ follows from the proof of Theorem 11 of Bose, Hazra and Saha (2009). □

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