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Classical Fermi Fluid and Geometric Action for $c = 1$

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ABSTRACT

We formulate the $c = 1$ matrix model as a quantum fluid and discuss its classical limit in detail, emphasizing the \hbar corrections. We view the fermi fluid profiles as elements of w_∞ -coadjoint orbit and write down a geometric action for the classical phase space. In the specific representation of fluid profiles as ‘strings’ the action is written in a four-dimensional form in terms of gauge fields built out of the embedding of the ‘string’ in the phase plane. We show that the collective field action can be derived from the above action provided one restricts to quadratic fluid profiles and ignores the dynamics of their ‘turning points’.

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1. Introduction:

Two-dimensional string theory provides us with a good laboratory to discuss principles of string theory and gravity. One of its attractive features is the high degree of solvability both at the classical and quantum level. In particular for a flat background metric and a linear dilaton the theory is equivalent to $c = 1$ matter coupled to two-dimensional gravity [1-5] and has a non-perturbative formulation in terms of the double-scaled $c = 1$ matrix model [6-9]. It is known for a long time that this matrix model is equivalent to a gas of non-interacting fermions moving in an external potential[10]. The fermions allow us not only to solve the theory [11-14] but also to find a large symmetry group (W_∞) [15-19] that has the interpretation of performing generalized gauge transformation on the fermions (generalized because it involves both multiplying the fermion field by a phase and transporting it in space). This symmetry has also been found in two-dimensional string theory [20, 21], and in the collective field theory which approximates the fermion field theory [22]. In particular in [20] there is a realization of vertex operators in terms of a ring of functions in a two-dimensional phase space and associated vector fields. See also [23]. In [15] and [17] we explored the consequences of the W_∞ symmetry in the quantum theory. It turns out [15] that the “classical limit” of the above symmetry can be understood very naturally in terms of the classical limit of the fermi theory. In the semi-classical approximation the states of the fermi theory are represented by a “fermi fluid” of uniform density occupying a certain two-dimensional region (not necessarily connected or simply connected) in the single-particle phase plane. Excitations of this state correspond to deformations of the fermi fluid. Such a concept is well-known in condensed matter physics [24] and has been applied to the problem at hand in a limited sense in [14]. In the quantum theory the excitations must satisfy the condition that the total number of fermions is conserved– in the semiclassical approximation this means that the total area of the fermi fluid must be conserved. Hence the excitations of the quantum theory are semiclassically represented by area-preserving deformations of the fermi fluid. If we concentrate for the moment on those deformations that are differentiable, then we see that the

basic excitations of the theory are area-preserving diffeomorphisms, hence they are elements of w_∞ . We thus recover the classical limit of W_∞ . In [15] we used this observation to represent fluid profiles by w_∞ ‘angles’ and discussed the dynamics in terms of the latter.

In this paper we develop in more detail the classical limit of the one dimensional fermi gas and its correspondence with the quantum theory. We summarize the results below.

(1) Correspondence between classical and quantum theory: We show that the $W(p, q, t)$ operator introduced in [15-17] have the interpretation that the expectation value of its fourier transform in any state $|F\rangle$ gives the quantum distribution function $u_F(x, p)$, satisfying the property that

$$\langle F | \int dx \Psi^\dagger(x, t) \hat{f}(x, -i\partial_x) \Psi(x, t) | F \rangle = \int dx dp f(x, p) u_F(x, p) \quad (1)$$

We show that the equation of motion of $W(p, q, t)$ is simply a statement of Liouville’s theorem for the quantum distribution function. We discuss the classical limit of $u_F(x, p)$ which leads to density functions that are one inside a region R and zero outside. We derive the change of $u_F(x, p)$ as the state $|F\rangle$ is changed to $U|F\rangle$ (U belonging to the W_∞ -group). This gives a natural representation of W_∞ in the space of u_F ’s. We show that the classical limit of this representation corresponds to a representation of w_∞ on the classical distribution functions, which coincides with the representation we had introduced in [15] and will use below. We also present the operator algebra of the quantum fluid characterized by $W(p, q, t)$ and $u(x, p, t)$.

(2) In [15] we identified the classical phase space of the $c = 1$ matrix model with the space of shapes of an incompressible fermi fluid (fluid ‘profiles’) in the two-dimensional phase plane of a single fermion. As suggested by correspondence with the quantum theory, we represent the fluid profiles by density functions that are characteristic functions. The w_∞ -action on these fluid profiles corresponds to thinking of the area-preserving diffeomorphism as a hamiltonian flow and evolving

each fermion constituting the fluid under that flow. We show that there is a natural notion of scalar product between fluid profiles and canonical transformations which enables us to identify the w_∞ -action on the fermi fluid as a co-adjoint action. This allows us to write down a geometric action on the classical phase space a la Kirillov. We find that this action is the classical limit of the action we wrote down in [16] involving elements of the W_∞ -group.

(3) We show that there is a rather interesting ‘string representation’ of the above classical dynamics. One parametrizes the boundary of the fermi fluid in the form of a (closed or open) string. Under this parametrization the dynamics of the fluid profile resembles a ‘string theory’ in 2 space dimensions with a built-in reparametrization invariance that follows from the indistinguishability of fermions. In order to write down the geometric action mentioned in (2) in these variables, one needs to introduce a time t and an additional variable s , so that the basic variables of the theory are $x^i(\sigma, \tau, t, s)$, $i = 1, 2$; $x^1 = x$, $x^2 = p$ which, at any given t, s , describes two-dimensional region (we consider connected simply connected regions) occupied by the fermi fluid which is thought of as the image of a map (x^i) from a two-dimensional parameter space σ, τ . The most interesting feature of the string representation is that the geometric action is given by

$$S_0 = \int_M dt ds d\sigma d\tau F_{\sigma\tau} F_{st} \quad (2)$$

where $F_{\mu\nu}$ is the field strength of an abelian gauge potential $A_\mu = \epsilon_{ik} x^i \partial_\mu x^k$, $i, k = 1, 2$; $\mu = 1, 2, 3, 4$. We are using the notation $\xi^\mu = (\sigma, \tau, t, s)$ for coordinates of the four-dimensional space M (the topology of M is discussed in section 5 below).

We also show that under a certain ‘gauge choice’ which is valid only in some restrictive class of configurations (namely that the fluid profile is quadratic, that is, the fluid boundary is given by $0 = F(x, p)$ where F is at most quadratic in p and that the turning point of the fluid boundary on the x -axis is static), we recover the collective field theory action [25, 26].

The preceding couple of paragraphs deal with fermi fluids that occupy one filled region. The situations with multiple filled regions or regions with holes (multiply connected regions) correspond, in the string-interpretation, to splitting and joining of strings, and hence to an interacting string theory. We should mention that by using the action in terms of the U variable, it is possible in principle to calculate the action for a “pants” diagram, by invoking U -variables which are perhaps singular and which take a one-string configuration to two-string configurations. This is similar in spirit to the calculation of a three-string vertex using the Polyakov path integral.

The plan of the paper is as follows. In section 2 we discuss the correspondence between classical and quantum theory. In section 3 we review the Kirillov method briefly to set up the notation. In section 4, we apply this method to the case of the fermi fluid and derive the geometric action. In section 5, we discuss the string representation, derive the action (2) and show how to recover collective field theory from this action under the restricted circumstance mentioned above.

2. Correspondence between the classical and the quantum theory:

In [15-17] we introduced the fermion bilinear $\Phi(x, y, t) = \Psi^\dagger(x, t)\Psi(y, t)$ or its relative

$$W(p, q, t) \equiv \int dx \Psi^\dagger(x, t) \hat{g}(p, q) \Psi(x, t), \quad \hat{g}(p, q) \equiv \exp(ip\hat{X} + iq\hat{P}) \quad (3)$$

as the basic dynamical variable of the theory^{*}. Since all states of the fermi theory are obtained by multiple application of the $W(p, q, t)$ ’s to the ground state (filled fermi sea)[†], it is clear that they provide a complete set of physical observables.

* The definition of $W(p, q, t)$ given here differs from the one in our earlier papers [15-17] by a factor of two.

† To see this, note that elementary particle-hole excitations can be written as linear combination of $W(p, q, t)$ ’s (cf. [15]).

Let us recall the commutation relation and equation of motion of the $W(p, q, t)$'s [15]

$$[W(p, q, t), W(p', q', t)] = 2i \sin\left[\frac{\hbar}{2}(pq' - qp')\right] W(p + p', q + q', t) \quad (4)$$

$$(\partial_t + p\partial_q + q\partial_p)W(p, q, t) = 0 \quad (5)$$

In writing the last equation we have used a specific hamiltonian $h(x, -i\partial_x) = (-\partial_x^2 - x^2)/2$.

We shall see now that $W(p, q, t)$'s have a rather interesting interpretation in the classical limit. Let us compute the expectation value in a state $|F\rangle$ of an observable

$$O_f = \int dx \Psi^\dagger(x, t) \hat{f} \Psi(x, t) \quad (6)$$

where $\hat{f}(x, -i\partial_x)$ is an operator corresponding to a classical function $f(x, p)$ in the single-particle phase space. We shall fix the operator ordering in \hat{f} by defining

$$\hat{f} \equiv \int d\alpha d\beta \tilde{f}(\alpha, \beta) \hat{g}(\alpha, \beta), \quad \tilde{f}(\alpha, \beta) \equiv \int \frac{dx}{2\pi} \frac{dp}{2\pi} f(x, p) \exp(-i\alpha x - i\beta p) \quad (7)$$

\hat{f} , defined in this fashion is called a Weyl-ordered operator. To understand (7), note that it can be obtained from the classical relation

$$f(x, p) = \int d\alpha d\beta \tilde{f}(\alpha, \beta) g(\alpha, \beta; x, p), \quad g(\alpha, \beta; x, p) \equiv \exp(i\alpha x + i\beta p) \quad (8)$$

by replacing the symbol x by \hat{X} and p by \hat{P} on the right hand side so that $g(\alpha, \beta; x, p)$ becomes $\hat{g}(\alpha, \beta)$ (recall the definition of \hat{g} in (3)). As an example, if $f(x, p) = xp^2$ then $\tilde{f}(\alpha, \beta) = (-i\partial_\alpha)\delta(\alpha)(-i\partial_\beta)^2\delta(\beta)$, so that $\hat{f} = \int d\alpha d\beta (-i\partial_\alpha)\delta(\alpha)(-i\partial_\beta)^2\delta(\beta) \exp(i\alpha \hat{X} - i\beta \hat{P}) = \hat{X}(\hat{P})^2 - i\hat{P} = (1/3)[\hat{X}(\hat{P})^2 + \hat{P}\hat{X}\hat{P} + (\hat{P})^2\hat{X}]$ which is perhaps a more familiar statement of Weyl ordering.

The expectation value of O_f in a state $|F\rangle$ is then

$$\begin{aligned}
\langle F|O_f|F\rangle &= \int d\alpha d\beta \langle F| \int dx \Psi^\dagger(x, t) \hat{g}(\alpha, \beta) \Psi(x, t) |F\rangle \tilde{f}(\alpha, \beta) \\
&= \int_{\alpha, \beta} \langle F|W(\alpha, \beta)|F\rangle \tilde{f}(\alpha, \beta) \\
&= \int_{x, p} \langle F|u(x, p)|F\rangle f(x, p)
\end{aligned} \tag{9}$$

where $u(x, p, t)$ is the Fourier transform of $W(\alpha, \beta, t)$:

$$u(x, p, t) = \int_{\alpha, \beta} \exp(-i\alpha x - i\beta p) \left[\int dy \Psi^\dagger(y, t) \hat{g}(\alpha, \beta) \Psi(y, t) \right] \tag{10}$$

Equation (9) is rather interesting. It tells us that the expectation value of the operator O_f can be exactly expressed by a phase space integral of $f(x, p)$ with a density function

$$u_F(x, p, t) \equiv \langle F|u(x, p, t)|F\rangle \tag{11}$$

Since this statement is valid at the quantum level, we call $u_F(x, p)$ the quantum distribution function in the state $|F\rangle$.

Thus, we find that $W(p, q, t)$ is simply the Fourier transform of the quantum distribution function operator $u(x, p, t)$

$$u(x, p, t) = \int d\alpha d\beta \exp(i\alpha x + i\beta p) W(\alpha, \beta, t) \tag{12}$$

whose expectation value gives us the quantum distribution function $u_F(x, p, t)$.

Note that the equation of motion (5) for the W -operator implies the following equation for u (using (12))

$$(\partial_t + p\partial_x + x\partial_p)u(x, p, t) = 0 \tag{13}$$

which is simply the statement of Liouville theorem for the quantum distribution (for the hamiltonian $h(x, p) = (p^2 - x^2)/2$).

Relation to the “first-order density matrix”:

Given a many-body wave-function $|F\rangle = \sum F(x_1, \dots, x_N) |x_1, \dots, x_N\rangle$ the first-order density matrix is defined as

$$\begin{aligned}\phi_F(x, y) &= \int dx_2 dx_3 \cdots dx_N F^*(x, x_2, x_3, \dots, x_N) F(y, x_2, x_3, \dots, x_N) \\ &= \langle F | \Phi(x, y) | F \rangle, \quad \Phi(x, y) = \Psi^\dagger(x) \Psi(y)\end{aligned}\quad (14)$$

The usefulness of this quantity is that a single-particle operator like $O_f = \sum_i \hat{f}(x_i, -i\partial/\partial x_i) = \int dx \Psi^\dagger(x, t) \hat{f} \Psi(x, t)$ is given by

$$\langle F | O_f | F \rangle = \sum_{x,y} \langle x | \hat{f} | y \rangle \phi_F(x, y) \quad (15)$$

By using (7) in (15) we get

$$\langle F | O_f | F \rangle = \int_{x',p'} f(x', p') \left[\int_{x,y} \phi_F(x, y) \langle x | \tilde{g}(x', p') | y \rangle \right] \quad (16)$$

where $\tilde{g}(x, p) \equiv \int_{\alpha,\beta} \hat{g}(\alpha, \beta) \exp(-i\alpha x - i\beta p)$.

Comparing with the previous expression of $\langle F | O_f | F \rangle$ we find that the quantum distribution function is a simple transform of the first-order density matrix:

$$u_F(x, p) \equiv \langle F | u(x, p) | F \rangle = \int_{x_1, y_1} K(x, p; x_1, y_1) \phi_F(x_1, y_1) \quad (17)$$

where

$$K(x, p; x_1, y_1) \equiv \langle x_1 | \tilde{g}(x, p) | y_1 \rangle = \exp(-ip(x_1 - y_1)) \delta\left(\frac{x_1 + y_1}{2} - x\right) \quad (18)$$

This kernel actually expresses the transformation between the two different forms of the fermion bilinear, $\Phi(x_1, y_1, t)$ and $u(x, p, t)$.

The semi-classical limit:

Let us try to calculate $u_F(x, p)$ using (17), when $|F\rangle$ is the ground state. For simplicity, let us consider a system of free particles first (potential= 0). It is easy to calculate $\phi_F(x_1, y_1)$ using plane waves for the single-particle wavefunctions:

$$\phi_F(x_1, y_1) = \sum_{k < k_F} \exp[ik(x_1 - y_1)] \quad (19)$$

This gives

$$u_F(x, p) = \text{const. } \theta(p_F - p) \quad (20)$$

The constant ensures that the integral $\int dx dp u_F(x, p)$ which measures the total fermion number, is N (in the case of double scaling where $N \rightarrow \infty$ the normalization has to be appropriately redefined).

Equation (20) is the first example of recovering a classical result from the quantum distribution function. We recall that semiclassically the ground state of a fermi system corresponds to filling up the region in the phase space corresponding to $h(x, p) < e_F$ where $h(x, p)$ is the energy function. In other words, the classical density function is

$$u_F^C(x, p) = \text{const. } \theta(e_F - h(x, p)) \quad (21)$$

where the constant is again determined by the condition that the integral of u_F^C should reproduce the total fermion number. We can see that for the free particle case ($h(x, p) = p^2$) (21) coincides with (20). In other words, the quantum distribution function coincides with the classical distribution function.

In general (for a more general potential and more general states) the classical distribution function is given by

$$u_F^C(x, p) = \chi_R(x, p) \quad (22)$$

where $\chi_R(x, p)$ is the characteristic function for some region R in the phase space ($\chi_R(x, p) = 1$ if $(x, p) \in R$, $= 0$ otherwise). R describes the region occupied by the

fermions in the state $|F\rangle$ — in general this is only an approximate concept, and hence the quantum distribution function has an \hbar -expansion:

$$u_F(x, p) = \chi_R(x, p) + O(\hbar) \quad (23)$$

corresponding to a quantum softening of the step function. Of course at finite temperature there is a softening of the step function even at the classical level. In the above we have ignored the constant multiplying $\chi_R(x, p)$.

In the case of $h = (p^2 - x^2)/2$ the quantum distribution function is indeed not exactly equal to the characteristic function, and has non-trivial \hbar corrections as in (23), but we skip the details here.

We see therefore that the natural set of observables in the quantum theory, namely $u_F(x, p) = \langle F | u(x, p) | F \rangle$ correspond in the classical limit to the set of density functions $\chi_R(x, p)$ for all regions R in the phase plane. In section four we will define the classical phase space of the fermi system in terms of characteristic functions.

Before we end this section, let us show how the W_∞ -transformations in the quantum theory give rise to w_∞ -transformations on the fluid regions R in the classical theory modulo \hbar -corrections.

We have shown in [15] that the fermion fock space forms an irreducible representation of the W_∞ -algebra. This is a consequence of the fact that an arbitrary state in the theory can be reached from the fermi ground state by particle-hole pair excitations, which in turn can be expressed as linear combination of the W_∞ -generators. This implies that an arbitrary state $|F\rangle$ in the theory can be written as

$$|F\rangle = U|F_0\rangle \quad (24)$$

where U is a W_∞ -group element:

$$U = \exp[i \int_{p,q} \epsilon(p, q) W(p, q, t)] \quad (25)$$

Let us ask the following question: if a state $|F_0\rangle$ (not necessarily the ground state) is changed by a W_∞ -transformation according to (24), how does the corresponding quantum distribution function change? By definition

$$u_0 \rightarrow u_F(x, p) = \langle F_0 | U^\dagger u(x, p) U | F_0 \rangle \quad (26)$$

We have abbreviated u_{F_0} by u_0 .

Let us consider the case when $U = 1 + iH = 1 + i \int_{\alpha, \beta} \epsilon(\alpha, \beta) W(\alpha, \beta)$ with H infinitesimal. (26) now reads

$$\delta u_0(x, p) = -i \int_{\alpha, \beta} \epsilon(\alpha, \beta) \langle F_0 | [W(\alpha, \beta), u(x, p)] | F_0 \rangle \quad (27)$$

We can evaluate the commutator by using the structure constants (5) of the W_∞ -algebra. We have

$$\begin{aligned} [W(\alpha, \beta, t), u(x, p, t)] &= 2i \int_{\alpha', \beta'} \sin\left[\frac{\hbar}{2}(\alpha\beta' - \alpha'\beta)\right] \exp(-i\alpha'x - i\beta'p) W(\alpha + \alpha', \beta + \beta') \\ &= 2i \sin\left[\frac{i\hbar}{2}(\alpha\partial_p - \beta\partial_x)\right] [e^{i\alpha x + i\beta p} u(x, p, t)] \end{aligned} \quad (28)$$

Putting this in (27) one gets

$$\delta u_0(x, p) = i \int_{\alpha, \beta} \epsilon(\alpha, \beta) \delta_{\alpha, \beta} u_0(x, p) \quad (29)$$

where

$$\delta_{\alpha, \beta} u_0(x, p) = \int_{x', p'} M_{\alpha, \beta}(x, p; x', p') u_0(x', p') \quad (30)$$

with

$$\begin{aligned} M_{\alpha, \beta}(x, p; x', p') &= 2i \int_{\alpha', \beta'} \sin\left[\frac{\hbar}{2}(\alpha\beta' - \alpha'\beta)\right] \exp[-i\alpha'x - i\beta'p + i(\alpha + \alpha')x' + i(\beta + \beta')p'] \\ &= 2i \sin\left[\frac{i\hbar}{2}(\alpha\partial_p - \beta\partial_x)\right] [\delta(x - x') \delta(p - p') \exp(i\alpha x' + i\beta p')] \end{aligned} \quad (31)$$

Equations (30) and (31) define the representation of W_∞ -algebra on the quantum distribution functions.

To understand the classical limit of equations (30) and (31) we need to make an \hbar -expansion of these equations. The net result is that the sine function gets expanded in power series, the first term representing the classical limit. Using this, we get

$$M_{\alpha,\beta}(x, p; x', p') = -\hbar(\alpha\partial_p - \beta\partial_x)[\delta'(x - x')\delta(p - p')\exp(i\alpha x' + i\beta p')] + O(\hbar^2) \quad (32)$$

The last equation implies

$$\frac{1}{i\hbar}\delta_{\alpha,\beta}u_0 = \{\exp(i\alpha x + i\beta p), u_0\}_{PB} + O(\hbar) \quad (33)$$

In other words, if one performs a W_∞ -transformation on the quantum distribution function by a generator corresponding to the single-particle operator $\exp(i\alpha\hat{X} + i\beta\hat{P})$ the result, to leading order in \hbar , is to send $u_0(x, p) \rightarrow u_0(x', p')$ where the points x', p' are obtained from x, p by a canonical transformation under the corresponding classical function $\exp(i\alpha x + i\beta p)$. The division by $i\hbar$ is related to the difference between commutators and classical Poisson brackets. Note that the $\{\}_{PB}$ here denotes Poisson bracket in the single-particle phase space. Equation (33) is remarkable because (a) it relates the Poisson bracket structure in the many-body phase space to that in the single-particle phase space and (b) it sets up a correspondence between W_∞ -transformations on the left hand side to w_∞ -transformations on the right hand side.

Thus we proved that W_∞ -transformation on the quantum distribution imply w_∞ -transformation on the classical distribution. The coadjoint orbit construction in section four for the classical case will precisely use this transformation.

To end this section we mention that a detailed discussion of the operator algebra of the quantum fluid, considered briefly here, is under preparation [27]. We

summarize the operator algebra in the following:

$$\begin{aligned}
[W(p, q, t), W(p', q', t)] &= 2i \sin\left[\frac{\hbar}{2}(pq' - qp')\right] W(p + p', q + q', t) \\
[W(p, q, t), u(p', q', t)] &= 2i \sin\left[\frac{i\hbar}{2}(p\partial_{p'} - q\partial_{q'})\right] [e^{i(pq' + qp')} u(p', q', t)] \\
[u(p, q, t), u(p', q', t)] &= 2i \sin\left[\frac{\hbar}{2}(\partial_p \partial_{q'} - \partial_q \partial_{p'})\right] [\delta(p - p')\delta(q - q') u(p', q', t)]
\end{aligned} \tag{34}$$

where $u(p, q, t)$ in the last two lines denotes $u(x, p, t)$ with $x = q$.

3. Review of the Kirillov method[28]:

Traditionally, the Kirillov method is a method of inventing physical systems that form representations of a group G . The steps are the following:

- (a) Find a dual Γ to the Lie algebra \mathcal{G} , that is, invent a linear space Γ so that there is a scalar product $\langle x|u \rangle$ between elements $x \in \Gamma$ and $u \in \mathcal{G}$.
- (b) Define an action of G (and consequently of \mathcal{G}) on Γ by the rule $G \ni U : x \rightarrow x^U \equiv \tilde{A}(U).x \in \Gamma$ where $\tilde{A}(U).x$ is defined by

$$\langle \tilde{A}(U).x | u \rangle = \langle x | U u U^{-1} \rangle \text{ for all } u \in \mathcal{G} \tag{35}$$

The group action $\tilde{A}(U)$ satisfying (35) is called a co-adjoint action and the property (35) is called the co-adjoint property (for obvious reasons, since the right hand side of (35) uses the adjoint action on the Lie algebra \mathcal{G}).

Note that \tilde{A} has the property $\tilde{A}(UV) = \tilde{A}(U)\tilde{A}(V)$.

We shall also use the infinitesimal version of \tilde{A} , namely if $U = \exp(tv)$ then $\tilde{a}(v).x \equiv \lim_{t \rightarrow 0} (d/dt) \tilde{A}(U).x$. In this limit, the coadjoint property reads as

$$\langle \tilde{a}(v).x | u \rangle = \langle x | [v, u] \rangle \quad u, v \in G, x \in \Gamma \tag{36}$$

Coadjoint orbit: for any given $x_0 \in \Gamma$ the set of points $\tilde{A}(U).x_0$ obtained by applying all group elements $U \in G$ to x_0 is called the coadjoint orbit of x_0 (denoted $C(x_0)$).

(c) Parametrize points of $C(x_0)$ by group elements, as follows. Suppose $x \in C(x_0)$. Clearly, by definition of $C(x_0)$, there is at least one group element U that has brought us from x_0 to x , i.e.

$$\tilde{A}(U).x_0 = x \quad (37)$$

Actually there would usually be an ambiguity in the definition of U , because if some U applied to x_0 gives x , then so would UV where

$$\tilde{A}(V)x_0 = x_0 \quad (38)$$

The set of such V 's forms a subgroup $H \subset G$, called the stability subgroup of x_0 . This means that points of $C(x_0)$ are characterized by an equivalence class UH ; in other words, $C(x_0) = G/H$.

(d) Let us assume that we have fixed the ambiguity by choosing a particular element out of each equivalence class. Then each point $x \in C(x_0)$ has an image $U \in G$. Similarly, curves $x(t) \in C(x_0)$ have images $U(t) \in G$. It is easy to see that the tangent vector dx/dt corresponds to the tangent vector dU/dt . By the well-known isomorphism between tangent vectors in a group manifold and elements of its Lie algebra we know that a tangent vector dU/dt at the point $U(t)$ corresponds to the Lie algebra element $(dU/dt)U^{-1} \equiv u_t$. Note that $dx/dt = \tilde{a}(u_t).x$.

Kirillov's prescription for the symplectic form is the following. Suppose we have two tangent vectors at the point $x \in \Gamma$, given by $t_1 = \tilde{a}(u_1).x$ and $t_2 = \tilde{a}(u_2).x$. Then the symplectic form $\Omega(x)$ is defined by

$$\Omega(x).(t_1, t_2) \equiv \langle x | [u_1, u_2] \rangle \quad (39)$$

One can check that Ω is antisymmetric and closed.

(e) The way one arrives at the classical action for a path $x(t) \in \Gamma$ is by extending the path $x(t)$ to a two-dimensional region $x(t, s)$ whose boundary is

$x(t)$. The classical action is simply the integral of the symplectic form over this two dimensional region*. Consider a little two-dimensional region formed by the two tangent vectors at x :

$$dx/dt = \tilde{a}(\partial_t UU^{-1}).x, \quad dx/ds = \tilde{a}(\partial_s UU^{-1}).x \quad (40)$$

By the above definition for the symplectic form, the action is therefore

$$S_0 = \int dt ds \langle x | [\partial_t UU^{-1}, \partial_s UU^{-1}] \rangle \quad (41)$$

The above action seems to depend on the extension $x(t, s)$ of the original path in the s direction. The fact that Ω is closed ensures that the action does not change under small deformations of the map $x(t, s)$ which keep the boundary $x(t)$ invariant.

One can formally integrate the Lagrangian with respect to s (in the absence of any topological obstruction). Thus

$$\begin{aligned} S_0 &= \int dt ds \langle x_0 | [U^{-1} \partial_t U, U^{-1} \partial_s U] \rangle \\ &= \int dt \langle x_0 | U^{-1} \partial_t U \rangle \\ &= \int dt \langle x | \partial_t UU^{-1} \rangle \end{aligned} \quad (42)$$

where in writing the first and the third lines line one has used the co-adjoint property and the relation between $x(t, s)$ and x_0 .

The question of well-definedness of the action: note that we have managed to write an action (41) or (42) in terms of the U -variable. However, the action must have an invariance with respect to the change $U \rightarrow UV$ where V satisfies

* Actually, this corresponds to the $p\dot{q}$ part of the Lagrangian, we will include the hamiltonian (Hdt) part shortly.

(38), because both U and UV correspond to the same point x in the configuration space, and the action should depend only on the path drawn in the configuration space (and not in G). In other words, in case of (41), we demand that

$$S_0[U(s, t)] = S_0[U(s, t)V(s, t)] \text{ where } \tilde{A}(V(s, t)).x_0 = x_0 \quad (43)$$

and in case of (42) we demand that

$$S_0[U(t)] = S_0[U(t)V(t)] \text{ where } \tilde{A}(V(t)).x_0 = x_0 \quad (44)$$

We call these criteria the criteria for well-definedness of the action. Equation (43) is easily satisfied by using the coadjoint property of the scalar product. (44) is more tricky [29] and instead of the Lagrangian remaining invariant it changes by a total derivative in time.

Adding the Hamiltonian piece:

So far we dealt with only the ‘symplectic form’ or the $p\dot{q}$ part of the action. In general, we will have a hamiltonian piece, where the hamiltonian h corresponds to a given element of the Lie algebra, $h \in \mathcal{G}$. We want to add $-dtH$ term in the action so that we get an equation of motion

$$\frac{dx}{dt} = \tilde{a}(h).x \quad (45)$$

The additional piece is

$$S_h = - \int dt \langle x | h \rangle \quad (46)$$

and the total action

$$S = S_0 + S_h \quad (47)$$

It is easy to check that the equation of motion following from the above action is indeed (45).

This is the standard story.

Our approach will be slightly different, mainly in emphasis. We will consider the physical system, say Q , as already given to us, with a specified action of the relevant group G . We will try to see if we can invent a notion of scalar product between elements of Q (to be suitably identified as a subset of some linear space Γ) so that under that scalar product the group-action satisfies the co-adjoint property. Then we can use the Kirillov device to write down a classical action on Q (or more specifically on coadjoint orbits in Q). In the case of the fermi fluid, Q is going to be the space of fluid profiles (= the space of characteristic functions), and the cojoint orbits $\subset Q$ going to be fermi fluids of a given area. We shall see that there is an obvious scalar product between Q and the elements of w_∞ algebra, indeed if we use the identification of the space of fluid profiles with the space of characteristic functions then Q is naturally imbedded in the space Γ of distributions and the above scalar product is then inherited from Γ .

A Toy Example: single classical spin as co-adjoint orbit of $SU(2)$

Let us explain a toy example. This is the problem of a single classical spin, characterised by a “spin” vector $x^i \in R^3$ satisfying the the constraint of unit norm $\vec{x} \cdot \vec{x} = 1$. The configuration space of the spin is $Q = S^2$. How does one write a natural action for this spin?

First remark that there is an obvious action of $SU(2)$ on this spin, namely rotation of the spin. To be precise, if we denote an element of $su(2)$ (the algebra) as $u = u_i \sigma^i$ where σ are the Pauli matrices, then

$$\tilde{a}(u) \cdot \vec{x} = \vec{u} \times \vec{x} \tag{48}$$

One way of quickly arriving at this equation is to think of R^3 as identified with the Lie algebra $su(2)$ where \vec{x} is identified with a matrix $X = x^i \sigma_i$. In this case, the rotation by u of X is given by $\tilde{a}(u) \cdot X = [u, X]$ using the natural action of Lie algebra on itself. (48) is obtained by using the fact that $[u, X] = \vec{u} \times \vec{x} \cdot \vec{\sigma}$.

The above identification of R^3 with $su(2)$ also suggests a natural scalar product between elements of R^3 (and hence of $Q \subset R^3$) and the elements of $su(2)$, namely

$$\langle x|u \rangle = \text{tr}(Xu) = \vec{x} \cdot \vec{u} \quad (49)$$

Clearly the above action (48) of $su(2)$ satisfies the co-adjoint property under the scalar product (49). The co-adjoint orbits are spheres in R^3 of constant radii. By choice, our config. space Q is the coadjoint orbit characterised by unit radius.

Classical action: Let us try to construct the classical action for a periodic path $\vec{x}(t)$, $\vec{x}(0) = \vec{x}(1)$ which in our case is a closed path drawn on the sphere Q . The classical action is the integral of the symplectic form over the “filled circle”*. We denote the filled circle by $\vec{x}(t, s)$, where $s \in [0, 1]$, $s = 1$ describing the boundary.

Consider now the infinitesimal region formed by the two tangent vectors $d\vec{x}/dt$ and $d\vec{x}/ds$ drawn at the point $\vec{x}(t, s)$. The integral of the symplectic form over this infinitesimal region is

$$ds dt \langle x|[u_t, u_s] \rangle \quad (50)$$

where u_t, u_s are elements of the $su(2)$ algebra defined by the equations

$$\begin{aligned} \tilde{a}(u_t) \cdot \vec{x} &= \vec{u}_t \times \vec{x} = \frac{d\vec{x}}{dt} \\ \tilde{a}(u_s) \cdot \vec{x} &= \vec{u}_s \times \vec{x} = \frac{d\vec{x}}{ds} \end{aligned} \quad (51)$$

It is easy to solve (51), giving

$$\begin{aligned} \vec{u}_t &= \vec{x} \times \frac{d\vec{x}}{dt} \\ \vec{u}_s &= \vec{x} \times \frac{d\vec{x}}{ds} \end{aligned} \quad (52)$$

We have used the notation $u_t = \vec{u}_t \cdot \vec{\sigma}$, etc. and the fact that the tangent vectors

* There is an ambiguity here regarding whether we want to fill the ‘inside’ or the ‘outside’ of the circle, but we’ll see that the symplectic form is an integral form which means that the results will differ only by an integral multiple of 2π , hence in computing $\exp(iS)$ we would not see any difference

$d\vec{x}/dt$ and $d\vec{x}/ds$ are perpendicular to x (also that $\vec{x} \cdot \vec{x} = 1$).

Using these one arrives at a classical action

$$S_0 = \int dt ds \vec{x} \cdot \frac{d\vec{x}}{dt} \times \frac{dx}{ds} \quad (53)$$

which is the famous solid angle action for a classical spin.

A typical hamiltonian term corresponds to inclusion of a magnetic field which may be viewed as either a vector \vec{B} or equivalently as a matrix $B = \vec{B} \cdot \vec{\sigma}$. The corresponding term in the action is

$$S_h = \int dt \langle x | B \rangle = \int dt \vec{x} \cdot \vec{B} \quad (54)$$

4. The fermi fluid as co-adjoint orbit of w_∞ :

We consider the space Q of fermi fluid profiles as the classical phase space. In [15] we derived the transformation of the fluid profile under area-preserving diffomorphisms (equivalently, canonical transformations). In section two we re-derived this result by looking at the classical limit of the quantum distribution function.

Following the discussion in section 2, we shall parametrize the fermi fluid as follows. If the fluid occupies a region R in the phase plane we shall characterize it by a density function $\chi_R(x, p)$ defined by

$$\begin{aligned} \chi_R(p, x) &= 1, \text{ if } (x, p) \in R \\ &= 0, \text{ otherwise} \end{aligned} \quad (55)$$

Thus, our classical phase space is the space of density functions $Q = \{\chi_R, R \subset \mathbb{R}^2\}$ where this R can be any two dimensional subset of the phase plane.

The group we are concerned with is the group of area-preserving diffeomorphisms. The Lie algebra consists of hamiltonian flows under arbitrary functions of the phase plane. Thus we shall parametrize the Lie algebra elements u by functions $f(x, p)$ or vector fields $X_f = \partial_p f \partial_x - \partial_x f \partial_p$. When we think of the functions as Lie algebra elements, we define the Lie bracket as identical to the Poisson bracket. Note that

$$[X_f, X_g] = X_{\{f, g\}_{PB}}$$

The action on fluid elements is specified by considering the action on the corresponding density functions:

$$\tilde{a}(f) \cdot \chi_R(x, p) = -(\partial_p f \partial_x - \partial_x f \partial_p) \chi_R(x, p) \equiv -X_f \chi_R(x) \quad (56)$$

Let us justify the above definition. Note that for infinitesimal f , the above can be re-stated as

$$\chi_R(x, p) + \tilde{a}(f) \chi_R(x, p) = \chi_R(x', p') \quad (57)$$

where (x', p') is the point obtained by evolving (x, p) for unit time under the hamiltonian $-f$: $x' = x - \partial_p f, p' = p + \partial_x f$. Now instead of evolving the coordinates x, p we might alternatively evolve the region R . We know that if we evolve both R to R' and the coordinate x, p to x', p' then $\chi_R(x, p) = \chi_{R'}(x', p')$. This also shows that $\chi_R(x', p') = \chi_{R'}(x, p)$ where R' is now obtained by evolving R under the hamiltonian f for unit time. Thus we get the result that, for infinitesimal f ,

$$\chi_R(x, p) + \tilde{a}(f) \chi_R(x, p) = \chi_{R'}(x, p) \quad (58)$$

In other words, under the w_∞ transformation f , the region R evolves to a new region R' as if each fermion inside evolves under the hamiltonian f . In the next sections we shall use (58) instead of (56) to find the action of infinitesimal w_∞ transformations in the different parametrizations.

Note that the above representation of w_∞ is precisely the one obtained in section two as a limiting case of W_∞ -transformation on the quantum distribution function.

The action of finite group elements, like $U(t) = \exp(-iX_f t)$ is given by

$$\tilde{A}(U).\chi_R(x, p) = \exp(it(\partial_p f \partial_x - \partial_x f \partial_p))\chi_R(x, p) \quad (59)$$

The scalar product:

We define now a notion of a scalar product between the elements X_f of w_∞ and fluid elements $\chi_R(x, p)$. A natural scalar product is:

$$\langle \chi_R(x, p) | X_f \rangle = \int_{\mathbb{R}^2} \chi_R(x, p) f(x, p) = \int_R f(x, p) \quad (60)$$

In other words the scalar product counts the total amount of $f(x, p)$ contained in the fermi fluid.

Note that if we think of elements of w_∞ as simply functions $f(x, p)$ on the phase plane, the obvious dual Γ is the space of ‘generalized functions’ or distributions (indeed distributions are defined that way). The scalar product of a distribution D with a function f is by definition the integral of the distribution with f as the ‘test’ function. Our density function χ_R is defined as the characteristic function corresponding to the region R , hence the space of fluid profiles Q (defined as the space of density functions) is naturally embedded in the linear space Γ of distributions. Indeed one can extend the definition of w_∞ action on Γ in the obvious way. To be precise, the action of X_f on a distribution D would be given by

$$\langle \tilde{a}(f).D | g \rangle = \langle D | X_f.g \rangle \equiv \langle D | \{f, g\}_{PB} \rangle \quad (61)$$

(61) makes it clear that the above scalar product satisfies the co-adjoint property. The coadjoint orbits are fluids of the same area. From the point of view of the fermion theory, this is a natural consequence of fermion number conservation.

Classical action:

We want to compute the classical action for a fluid trajectory $R(t)$ (which we assume periodic for the moment), given by density functions $\chi_{R(t)}$. We assume that $R(t)$ has the same area for all t , so that it is lying on a single coadjoint orbit. We “fill in” the one-dimensional trajectory to a two-dimensional one as mentioned in the last section: the fluid profiles are called $R(s, t)$ and the corresponding density functions $\chi_{R(s,t)}$. The action is given by the following two dimensional integral

$$S_0 = \int dt ds \langle \chi_{R(s,t)} | [\partial_t UU^{-1}, \partial_s UU^{-1}] \rangle \quad (62)$$

The group element $U(s, t)$ is defined to be the one which brings us from a certain “base fluid profile” R_0 to the current one $R(s, t)$:

$$\tilde{A}(U(s, t)) \cdot \chi_{R_0} = \chi_{R(s,t)} \quad (63)$$

As explained earlier, $U(s, t)$ as defined by (63) is ambiguous upto right multiplication by elements $V(s, t)$ which do not move R_0 :

$$\tilde{A}(V(s, t)) \cdot \chi_{R_0} = \chi_{R_0} \quad (64)$$

This means that our configuration space (space of fluid profiles of a given area that form an orbit of w_∞) is actually a coset G/H where H is the set of V ’s satisfying (64). One can check that the action written above satisfies the criterion of well-definedness expressed in (43).

As mentioned in the general outline in the previous section, we can transpose the U -action to rewrite (62) in terms of χ_{R_0} . The result, after one partial integration with respect to s (in the absence of a topological obstruction), is

$$\begin{aligned} S_0 &= \int dt \langle \chi_{R_0} | U^{-1} \partial_t U \rangle \\ &= \int \langle \chi_R | \partial_t UU^{-1} \rangle \end{aligned} \quad (65)$$

The hamiltonian piece: if the fluid profile is evolved by a hamiltonian function

$h(x, p)$ in the single-particle phase space (for instance, $h = (p^2 - x^2)/2$), then (46) becomes

$$S_h = - \int dt \langle \chi_R | X_h \rangle \quad (66)$$

By using the coadjoint property of the scalar product we can rewrite this as

$$S_h = - \int dt \langle \chi_{R_0} | U^{-1} X_h U \rangle \quad (67)$$

The right way to interpret the above expression is: first think of U as made of exponential of differential operators (like $U = \exp(X_f)$ etc.) so that $U^{-1} X_h U$ is also a differential operator, of the form X_g for some g . Then $\langle \chi_{R_0} | U^{-1} X_h U \rangle$ is actually defined to be $\langle \chi_{R_0} | g \rangle$.

The total action is given by

$$S = S_0 + S_h = \int dt \langle \chi_{R_0} | U^{-1} \partial_t U + U^{-1} X_h U \rangle \quad (68)$$

Correspondence with the action in [16]:

Let us make a brief remark to make correspondence with the W_∞ -action that we wrote down in [16]. The latter action was

$$\begin{aligned} S = S_0 + S_h, \quad S_0 &= \int dt \text{tr}(\Lambda U^{-1} \partial_t U), \\ S_h &= \int dt \text{tr}(\Lambda U^{-1} \bar{A} U) \end{aligned} \quad (69)$$

where $U = \exp(i \int \epsilon(p, q) \hat{g}(p, q))$ is an element of W_∞ -group, viewed as an operator in the single-particle Hilbert space or equivalently as an infinite dimensional matrix $U_{xy} = \langle x | U | y \rangle$. Λ is a fixed matrix, and \bar{A} is simply the hamiltonian operator h appearing in the fermion theory.

It is clear that the above action is the Kirillov action for the co-adjoint orbit of Λ under the group W_∞ . The coadjoint action is defined by identifying the dual of the algebra with itself (the scalar product being defined as the *trace*).

We are not going to prove the equivalence between (69) and (68) in great detail here; we indicate the steps instead. Basically, we found in section two that the scalar product $\langle \chi_R | X_f \rangle$ is merely the classical limit of the quantity $\langle u_F | X_f \rangle \equiv \int_{x,p} f(x,p) u_F(x,p)$ where $u_F(x,p)$ is the quantum distribution function in the state $|F\rangle$ (the state $|F\rangle$ here is *defined* by the property that $u_F(x,p) = \chi_R(x,p) + O(\hbar)$). Now the latter quantity is expressible as $\int_{x,y} f_{xy} \phi_F(x,y)$ where $f_{xy} = \langle x | \hat{f} | y \rangle$ (see equation (15)). Now consider the hamiltonian term $\langle \chi_{R_0} | U^{-1} h U \rangle = \langle \chi_R | h \rangle$ in (68). By the above remarks, this is the classical limit of $\text{tr}(h\phi_F)$ where we have suppressed the symbols x, y , treating them as matrix indices. If we now use the fact that an arbitrary state $|F\rangle$ state can be written as $|F\rangle = U|F_0\rangle$, then it is easy to deduce that $\phi_F = U^{-1}\phi_{F_0}U$. Thus, if we identify $\Lambda(x,y) = \phi_{F_0}(x,y)$, then the hamiltonian piece in (69) equals that in (68) plus order (\hbar) terms. Similar remarks hold for the kinetic term, though the proof is a little more lengthy*.

5. The String Representation:

In this section we shall employ a different, in a sense more intrinsic, representation of the fermi fluid, which we shall call the “string representation”. Let us consider for the moment a fermi fluid which occupies one single filled region (that is, a connected simply connected two-dimensional region of the phase plane). One example is the ground state distribution. If the hamiltonian is $h = (p^2 + x^2)/2$ then the ground state distribution is a fermi fluid filling the region $p^2 + x^2 \leq 2\mu$ where μ is the fermi energy. If the hamiltonian is $h = (p^2 - x^2)/2$ (and we restrict to $x < 0$, cf. [15]) then the ground state distribution is $x \leq -\sqrt{p^2 - 2\mu}$. We point out that the choice of signature (Euclidean or Minkowski) in the 2-dimensional target space dictated in [15] the choice of the hamiltonian (actually, in the Euclidian case the hamiltonian was $h = -(p^2 + x^2)/2$ so that the fermi fluid filled the outside of the circle mentioned above—this difference will not matter for most part of our

* Note that we have used the same notation U above for the single-particle operator $U = \exp[i \int dp dq \epsilon(p,q) \hat{g}(p,q)]$ and for the many-body operator $U = \exp[i \int dp dq \epsilon(p,q) W(p,q)]$. ■

discussion below and we'll choose to ignore it). As is clear from the example of the ground state distributions, the boundary of the fermi fluid can either be closed or open.

In the “string representation” we describe the fermi fluid by specifying the boundary as the map $x^i(\sigma)$ from a one-dimensional parameter space $\sigma \in [0, 2\pi]$ or $\sigma \in (-\infty, +\infty)$ (depending on whether the boundary is closed or open respectively) to the phase plane x^i , $(x^1 = x, x^2 = p)$. Indeed since we are considering filled (simply connected) regions, let us invent a two-dimensional parameter space σ, τ (which is a disc or a half-plane) which maps onto the two-dimensional region filled by the fermi fluid. The map is $(\sigma, \tau) \mapsto x^i(\sigma, \tau)$. If we parametrize the disc by $\tau \in [0, 1]$ then the boundary is given by $\tau = \tau_0 \equiv 1$ (in the case of the half-plane, parametrized by $\sigma \in (-\infty, \infty), \tau \in (-\infty, 0]$ the boundary is $\tau = \tau_0 \equiv 0$). The ‘string’ is simply the image of the one-dimensional boundary $\tau = \tau_0$ of the parameter space.

Action of w_∞ on the ‘string’:

Again, according to (58), we evolve the point (x, p) of the phase plane (“target space”) under the hamiltonian $h = f$: $x^i \rightarrow x^i + \epsilon^{ik} \partial_k f$, which means that entire two-dimensional image $x^i(\sigma, \tau)$ changes by

$$\tilde{a}(f).x^i(\sigma, \tau) \equiv \epsilon^{ik} \partial_k f(x(\sigma, \tau), p(\sigma, \tau)) \quad (70)$$

and the one-dimensional image of the τ -boundary (the string $x^i(\sigma, \tau_0)$) transforms by

$$\tilde{a}(f).x^i(\sigma, \tau_0) \equiv \epsilon^{ik} \partial_k f(x(\sigma, \tau_0), p(\sigma, \tau_0)) \quad (71)$$

We should remark that it is only the ‘string’ that is the real dynamical variable. If we make any transformation that moves the fermions *inside* the fermi fluid, leaving the boundary unchanged, then the physical state of the system is unchanged (because the phase space density is unchanged— the original reason for this of

course is the indistinguishability of identical fermions). In concrete terms, any transformation in the target space $(x, p) \rightarrow (x', p')$ that leaves the image $x^i(\sigma, \tau_0)$ invariant, does not change the “fermi fluid”. Indeed the statement is even stronger. Even if the map $x^i(\sigma, \tau_0)$ is changed in a manner such that a reparametrization of the boundary $\sigma \rightarrow \sigma'$ can account for the change, then we havent really changed the fluid profile.

It is interesting to ask what are the canonical transformations that change the string only upto reparametrization. The answer is, all those transformations h that satisfy

$$\tilde{a}(h).x^i(\sigma, \tau_0) \propto \partial_\sigma x^i(\sigma, \tau_0) \quad (72)$$

The proportionality ‘constant’ can be a function of σ (indeed if the function is $g(\sigma)$ the reparametrization that is implied here is $d\sigma'/d\sigma = g(\sigma)$). To see what these functions precisely are, let’s combine (71) and (72). We get

$$\partial_i h \partial_\sigma x^i = \partial_\sigma h(x^i(\sigma, \tau_0)) = 0 \quad (73)$$

which implies that the “string” is a surface of constant h . It is clear that for the fermi fluid in the ground state, any function of the energy is a candidate h . Since such h ’s do not change the fermi fluid, the space of fermi fluids is actually a coset G/H [15].

The scalar product: in this parametrization we have

$$\langle \chi_R | f \rangle = \int_R dx dp f(x, p) = \int_D d\sigma d\tau \epsilon_{ik} \partial_\sigma x^i \partial_\tau x^k f(x(\sigma, \tau), p(\sigma, \tau)) \quad (74)$$

where D denotes the entire parameter space σ, τ (disc in case of closed strings and half-plane in case of open strings).

To write the action consider x^k as a function of time t and the additional variable s . Then, (40) applied to our case is

$$\frac{\partial x^k}{\partial b} = \tilde{a}(f_b).x^k = \epsilon^{kl}\partial_l f_b, \quad b = s, t \quad (75)$$

The classical action (62) now reads

$$S_0 = \int dt ds \int_D d\sigma d\tau (\partial_\sigma f_s \partial_\tau f_t - (s \leftrightarrow t)) \quad (76)$$

Using (75), we can evaluate the partial derivatives involved in (76). The result is

$$S_0 = \int dt ds d\sigma d\tau F_{\sigma\tau} F_{st} \quad (77)$$

The field strength is defined as

$$F_{\mu\nu} = \epsilon_{ik} \partial_\mu x^i \partial_\nu x^k \quad (78)$$

which can be derived from a gauge potential

$$A_\mu = \epsilon_{ik} x^i \partial_\mu x^k \quad (79)$$

Here μ, ν run over all coordinates of the four-dimensional space $\xi^\mu = (s, t, \sigma, \tau)$.

Some interesting features of this gauge theory are:

(1) The gauge transformations $A_\mu \rightarrow A_\mu + \partial_\mu \theta$ correspond to making canonical transformations $x^i \rightarrow x^i + \epsilon^{ik} \partial_k f$ where

$$\theta = (2 - x^i \partial_i) f \quad (80)$$

(2) In the notation of differential forms, the gauge potential is given by a

one-form

$$A = A_\mu d\xi^\mu = \epsilon_{ik} x^i dx^k = x dp - p dx \quad (81)$$

and the gauge field by the two-form

$$F = \epsilon_{ik} dx^i \wedge dx^k = dx(\xi) \wedge dp(\xi) = X_*(\Omega) \quad (82)$$

where $X_*(\Omega)$ denotes the pull-back of the symplectic form Ω in the phase plane onto the four-dimensional space ξ^μ (the map X refers to the embedding $\xi^\mu \rightarrow x^i(\xi)$). In other words, the symplectic form $\Omega = dx \wedge dp$ in the target space induces a symplectic form in the four-dimensional space. Our field strength is precisely equal to that.

In this notation the action looks like

$$S_0 = \int_M d\sigma d\tau ds dt (X_*(\Omega))_{\sigma\tau} (X_*(\Omega))_{st} \quad (83)$$

Note that the induced symplectic structure degenerates where the embedding map X is singular. This is precisely what happens at the turning points of the profiles. We shall come back to the issue of the turning point shortly.

The hamiltonian term (66) in the string representation can be written as

$$S_h = \int_M B \wedge X_*(\Omega) \quad (84)$$

where B is defined as the two-form $B = (\partial_x h \partial_s x + \partial_p h \partial_s p) ds \wedge dt$.

Let us now see under what conditions we can derive the collective field theory from the above considerations. For this purpose it is more useful to use the form (65) of the classical action and use the string parametrization in it. The second

line of (65) reads as

$$S_0 = \int dt \int_R f_t \quad (85)$$

where f_t satisfies

$$\partial_i f_t = \epsilon_{ik} \partial_t x^k \quad (86)$$

Let us convert the area integral over the region R in (85) into a line integral by thinking of f_t as a magnetic field and inventing a vector potential $a_i, i = x, p$. That is, let us find a_i such that

$$\partial_x a_p - \partial_p a_x = f_t \quad (87)$$

In that case (85) becomes

$$S_0 = \int dt \int_{\partial R} (a_p dp + a_x dx) = \int dt d\sigma (a_p \partial_\sigma p + a_x \partial_\sigma x) \quad (88)$$

Now this reduction is true for any choice of a_i which satisfies (87). Let us choose the gauge $a_p = 0$, then from (87) we have

$$a_x = -(\partial_p)^{-1} f_t \quad (89)$$

and so we get

$$S_0 = \int dt d\sigma \partial_\sigma x [-(\partial_p)^{-1} f_t] \quad (90)$$

In order to make connection with the collective field action we would now like to specify points on the “string” by their x -values rather than by the parameter σ . In other words, we attempt a change of variable in (90) from σ to $x(\sigma)$. Now since $x(\sigma)$ is actually $x(\sigma, t)$, how does one make the change of variable at all t ?

We will see that the way out is suggested by reparametrization invariance. Note that a given fluid profile (more precisely the density function χ_R) does not change if one simply makes a reparametrization of the boundary of the fluid. Therefore given a classical path of a string described as $x^i(\sigma, t)$, physically it is the same as another path $x^i(\sigma'(\sigma, t), t)$. Indeed one can check that the above action (90) is invariant under $x^i(\sigma, t) \rightarrow x^i(\sigma'(\sigma, t), t)$. Note that we are talking here about arbitrary *time-dependent* reparametrizations $\sigma \rightarrow \sigma'(\sigma, t)$.

Now though in the initial parametrization σ , $x(\sigma, t)$ depended on t , by changing over to σ' we may try to keep $x(\sigma'(\sigma, t), t)$ invariant in time by compensating between the two sources of time-variation. In other words, let us see if we can satisfy

$$\frac{dx}{dt} = \partial_t x + \partial_t \sigma' \partial_\sigma x = 0 \quad (91)$$

Clearly except when $\partial_\sigma x = 0$, we can choose the reparametrization $\sigma'(\sigma, t)$ to satisfy (91). Which means that except at these points we have

$$x(\sigma'(\sigma, t), t) = x(\sigma) \quad (92)$$

What is the significance of the points where $\partial_\sigma x$ vanishes? Well, these are precisely the turning points of the fluid profile on the x -axis. The “gauge choice” (92) cannot be validly made at these turning points. Physically this means that at all points except where the fluid boundary has turning points in the x -direction, one can always make a combination of vertical and horizontal motions (of the fermions living at the boundary) appear as a purely vertical motion by giving the fermion a suitable velocity component along the boundary (such motions do not change the state of the system, hence one is always allowed to add them without changing anything). At the turning point, since the tangent is vertical, adding any amount of tangential motion will change only the vertical component, and will never “gauge away” a horizontal component of the velocity there.

Thus, (92) is a valid gauge choice only when one restricts to fluid profiles which never move their x -turning points, that is, the ‘turning points’ are allowed to move purely vertically (mathemtically, one is saying that for (91) to be valid at a point where $\partial_\sigma x = 0$ one must have $\partial_t x = 0$). Let us assume such a restriction for the moment so that (92) is valid. Let us also assume that in the entire range of σ there is only one turning point on the x -axis, that is, there is only one value $\sigma = \sigma_0$ such that $\partial_\sigma x = 0$ at σ_0 ^{*}. This assumption is equivalent to the assumption of quadratic profiles[15], that is, that the fluid boundary is given by an equation $F(x, p) = 0$ where F is at most quadratic in p . Now, in the intervals $(-\infty, \sigma_0)$ and (σ_0, ∞) the map $\sigma \rightarrow x(\sigma)$ is separately invertible. We shall call the inverse maps $\sigma_-(x)$ and $\sigma_+(x)$ in the two intervals, respectively. We shall also use the notation $p(\sigma_+(x), t) = p_+(x, t)$ and $p(\sigma_-(x), t) = p_-(x, t)$.

Note that using (86) and (92) we have $\partial_p f_t = \dot{x} = 0 \Rightarrow (\partial_p)^{-1} f_t = p f_t$. Since $\partial_x f_t = -\dot{p}$, we have $f_t = -(\partial_x)^{-1} \dot{p}$. In these relations, $p(\sigma, t)$ is to be interpreted as $p_\pm(x, t)$ depending on whether $\sigma \geq \sigma_0$ or $\sigma \leq \sigma_0$.

Putting in all of the above, we see that (90) reduces to the ”kinectic” term of the collective field theory action:

$$\int dt \int dx [p_+ \frac{1}{\partial_x} \frac{dp_+}{dt} - p_- \frac{1}{\partial_x} \frac{dp_-}{dt}]$$

The hamiltonian term also agrees using similar reasoning and in this way we get the complete collective field theory action:

$$S = \int dt \int dx [p_+ \frac{1}{\partial_x} (\frac{dp_+}{dt}) - p_- \frac{1}{\partial_x} (\frac{dp_-}{dt}) + \frac{p_+^3 - p_-^3}{6} - \frac{x^2}{2} (p_+ - p_-)]$$

The above method of derivation clearly indicates the limitation of the collective field description of the theory. Besides the unwarranted restriction to quadratic

* We are considering for the moment the case of open string, i.e. non-compact σ ; for the closed-string case we have to assume that there are two turning points

profiles one needs to assume that the x -turning point remains static. In general the fluid profile will move in such a way that the turning point itself will be dynamical—the collective field description clearly misses this dynamics of the turning point.

6. Concluding remarks:

We hope that our classical action (62)(or (83)) brings new insights into formulating general principles of classical two-dimensional string theory. We wish to emphasize that the “classical” action for the quantum W_∞ -symmetry and the corresponding action for its classical limit w_∞ are different. In fact, the “classical” action corresponding to w_∞ is only the leading term (in powers of \hbar) of the “classical” action corresponding to W_∞ . Hence in this circumstance one will not obtain the correct quantum theory by quantizing this classical action. Such a circumstance also occurs in string theory. See, for example [30].

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Note added: While completing this work we received [31] which also discusses the fluid picture and the limitations of the collective field approach.

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