

Stringy quantum effects in 2-dimensional Black-Hole

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ABSTRACT

We discuss the classical 2-dim. black-hole in the framework of the non-perturbative formulation (in terms of non-relativistic fermions) of $c = 1$ string field theory. We identify an off-shell operator whose classical equation of motion is that of tachyon in the classical graviton-dilaton black-hole background. The black-hole ‘singularity’ is identified with the fermi surface in the phase space of a single fermion, and as such is a consequence of the semi-classical approximation. An exact treatment reveals that stringy quantum effects wash away the classical singularity.

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1. Introduction

The semiclassical quantization of black hole solutions of classical general relativity leads to a well-known paradox (which was first raised by Hawking) of pure states evolving into mixed states. Such a circumstance occurs when a neutral black-hole evaporates completely due to Hawking radiation. There are various viewpoints which either call for a modification of the present formulation of quantum mechanics or present a resolution within the existing framework of quantum mechanics. However, it is fair to say that presently this issue remains unresolved. For recent reviews see [1-3].

One of the proposals to resolve Hawking's paradox is that we view gravitation as naturally embedded in string theory. Then one may be tempted to argue that in the process of evaporation when the mass of the black-hole is approximately planck mass, stringy fluctuations would have to be included and these may smoothen out the classical curvature singularity of the black-hole. The problem with this proposal has been that the known formulations of string theory in physical dimensions are perturbative and as such are not equipped to address the question of singularities.

However, it is very fortunate that 2-dimensional string theory has a black-hole solution [4,5] and at the same time it has a non-perturbative formulation in terms of non-interacting non-relativistic fermions whose single particle hamiltonian is $h(p, q) = \frac{1}{2}(p^2 - q^2)$ [6,7]. This fortunate circumstance can be exploited to answer some of the issues of black-hole physics.

In this note we discuss the classical black-hole of the 2-dimensional string theory in the non-perturbative framework of the fermion field theory that is dictated by the W_∞ symmetry algebra which is a one parameter deformation of the algebra of area preserving diffeomorphisms. The deformation parameter is the string coupling $g_{str} = \hbar \sim \frac{1}{\mu}$. The fundamental operator of this formulation is $u(p, q, t) = \int dx \psi^+ \left(q - \frac{x}{2} \right) e^{-ipx} \psi \left(q + \frac{x}{2} \right)$, whose classical limit gives the phase space distribution of fermions. $u(p, q, t)$ transforms covariantly under W_∞ . The string coupling, among other things, labels the co-adjoint orbits of W_∞ . Motivated

by the fact that the classical fermion trajectories are hyperbolas in the (p, q) phase plane we introduce a hyperbolic measure on the phase plane and define a transform of $u(p, q, t)$: $\phi(p, q, t) = \int \frac{dp' dq'}{|p'^2 - q'^2|^{1/2}} u(p' + p, q' + q, t)$.

The classical equation for $\phi(p, q, t)$ in the phase plane turns out to be that of ‘massless’ tachyon propagating in classical black-hole background of the 2-dimensional string theory. On-shell, $\phi(p, q, t)$ is a function of only the two combinations $u = \frac{1}{2}(p + q)e^{-t}$ and $v = \frac{1}{2}(p - q)e^t$. The black-hole singularity $uv = \frac{\mu}{2}$ precisely maps onto the hyperbola that defines the fermi surface in the phase plane, $\frac{1}{2}(p^2 - q^2) = \mu$. However, in this non-perturbative formulation we can go beyond the semi-classical calculation. An exact computation of the tachyon background gives an answer that is finite at $uv = \mu/2$. From a detailed analysis we find that near the black-hole singularity higher loop stringy effects become important and in fact wipe out the singularity. This provides an explicit demonstration that stringy effects (at least in the lower dimensional model) remove the curvature singularity. The qualitative picture of black-holes in the fermi fluid picture has been previously discussed by us [8].

The plan of this paper is as follows. In the next section we review some relevant features of the non-perturbative formulation of $c = 1$ string field theory developed in [9]. In Sec. 3 we derive the Wheeler-deWitt equation for the loop operator in the classical limit. In Sec. 4 we introduce the new tachyon operator and the mapping from the single fermion phase space to the u, v plane. Using this mapping and the Wheeler-deWitt equation we then show that the tachyon satisfies equation (39) of [4]. We also discuss the W_∞ algebraic structure related to the tachyon operator. In Sec. 5 we present an exact calculation of the one-point function of this operator in the fermion field theory. This one-point function is finite at the classical black-hole singularity. We discuss in detail how stringy loop corrections wipe out the singularity.

2. $c = 1$ String Field Theory:

We briefly review the non-perturbative formulation of the $c = 1$ string field theory. As is well-known this theory is exactly described by non-relativistic fermions moving in a background hamiltonian. The double scaled field theory corresponds to the hamiltonian $h(p, q) = \frac{1}{2}(p^2 - q^2)$. Since the fermion number is held fixed the basic excitations are described by the bilocal operator $\phi(x, y, t) = \psi(x, t)\psi^+(y, t)$ or equivalently its transform

$$u(p, q, t) = \int_{-\infty}^{+\infty} dx \psi^+ \left(q - \frac{x}{2} \right) e^{-ipx} \psi \left(q + \frac{x}{2} \right) \quad (1)$$

The expectation value of this operator in a state is the fermion distribution function in phase space. Eqn. (1) also has the important property that given a ‘‘classical function’’ $f(p, q, t)$ in the phase space, we have an operator in the fermion field theory

$$\mathcal{O}_f = \int dp dq f(p, q, t) u(p, q, t) = \int dx \psi^+(x, t) \hat{f}(\hat{x}, \hat{p}, t) \psi(x, t)$$

where $\hat{f}(\hat{x}, \hat{p})$ is the Weyl-ordered operator corresponding to the classical function $f(p, q, t)$. For example, vector fields corresponding to the functions $f_{\alpha\beta}(p, q) = e^{i(p\beta - q\alpha)}$ satisfy the classical algebra ω_∞ of area-preserving diffeomorphisms. The corresponding quantum operators in the fermion field theory

$$\tilde{u}(\alpha, \beta, t) = \int \frac{dp dq}{(2\pi)^2} e^{i(p\beta - q\alpha)} u(p, q, t) \quad (2)$$

satisfy the W_∞ algebra (a one-parameter deformation of ω_∞) [9][★]

$$[\tilde{u}(\alpha, \beta, t), \tilde{u}(\alpha', \beta', t)] = 2i \sin \frac{\hbar}{2} (\alpha\beta' - \beta\alpha') \tilde{u}(\alpha + \alpha', \beta + \beta', t) \quad (3)$$

An exact boson representation of the fermion field theory that reflects the W_∞ symmetry can be achieved in terms of the 3-dim. field $u(p, q, t)$, provided we impose

★ We have in our previous works also used the notation $W(\alpha, \beta, t)$ for $\tilde{u}(\alpha, \beta, t)$.

the constraints that follow from its microscopic definition

$$\cos \frac{\hbar}{2} (\partial_q \partial_{p'} - \partial_{q'} \partial_p) u(p, q, t) u(p', q', t) \Big|_{\substack{p'=p \\ q'=q}} = u(p, q, t) \quad (4)$$

$$\int \frac{dp dq}{2\pi\hbar} u(p, q, t) = N \quad (5)$$

where N is the total number of fermions. Also the equation of motion that follows from the definition (1) is

$$(\partial_t + p\partial_q + q\partial_p)u(p, q, t) = 0 \quad (6)$$

The equation for $\tilde{u}(\alpha, \beta, t)$ is

$$(\partial_t + \alpha\partial_\beta + \beta\partial_\alpha)\tilde{u}(\alpha, \beta, t) = 0 \quad (7)$$

The constraints (4) and (5) in fact specify a co-adjoint orbit of W_∞ , and the classical action is constructed using the method of Kirillov

$$\begin{aligned} S[u, h] = & \int ds dt \int \frac{dp dq}{2\pi\hbar} u(p, q, t, s) \hbar^2 \{ \partial_s u(p, q, t, s), \partial_t u(p, q, t, s) \}_{MB} \\ & + \int dt \int \frac{dp dq}{2\pi\hbar} u(p, q, t) h(p, q). \end{aligned} \quad (8)$$

when $\{ , \}_{MB}$ is the Moyal bracket (for details see [9]).

Let us now briefly indicate the classical limit of the string theory ($\hbar \rightarrow 0$). In this limit the constraint (4) implies that $u(p, q, t)$ is a characteristic function of a region of phase space specified by a boundary [10-12]. For example the ground state corresponds to the static solution $u(p, q) = \theta(h(p, q) - \mu)$, $\mu \sim -\frac{1}{\hbar}$. The massless excitation (tachyon) corresponds to a curve that is a small deviation from the fermi surface $\frac{1}{2}(p^2 - q^2) = \mu$.

3. Equation of motion for the density field:

In this section we shall derive a second order equation for the fourier transform of the fermion density field

$$\tilde{u}_0(\alpha, t) \equiv \tilde{u}(\alpha, 0, t) = \int \frac{dp dq}{(2\pi)^2} e^{-iq\alpha} u(p, q, t) \quad (9)$$

Consider the Taylor expansion

$$\tilde{u}(\alpha, \beta, t) = \sum_{n=0}^{\infty} \frac{\beta^n}{n!} \tilde{u}_n(\alpha, 0, t), \quad \tilde{u}_n = \partial_{\beta}^n \tilde{u}(\alpha, \beta, t)|_{\beta=0}.$$

Then equation (7) implies that

$$\partial_t \tilde{u}_n(\alpha, t) + \alpha \tilde{u}_{n+1}(\alpha, t) + n \partial_{\alpha} \tilde{u}_{n-1}(\alpha, t) = 0, \quad n = 0, 1, \dots \quad (10)$$

For $n = 0$ and $n = 1$ we have

$$\partial_t \tilde{u}_0 + \alpha \tilde{u}_1 = 0, \quad \partial_t \tilde{u}_1 + \alpha \tilde{u}_2 + \partial_{\alpha} \tilde{u}_0 = 0$$

Eliminating \tilde{u}_1 from these, and taking a derivative w.r.t. μ , the fermi energy, we get the equation

$$(\partial_t^2 - \alpha \partial_{\alpha}) \partial_{\mu} \tilde{u}_0(\alpha, t) = \alpha^2 \partial_{\mu} \tilde{u}_2(\alpha, t) \quad (11)$$

Now using the definition $h(p, q) = \frac{1}{2}(p^2 - q^2)$, it is easy to see that

$$\partial_{\mu} \tilde{u}_2(p, t) = (\partial_{\alpha}^2 - 2\mu) \partial_{\mu} \tilde{u}_0(p, t) + \tilde{\epsilon}(\alpha, t)$$

where

$$\tilde{\epsilon}(\alpha, t) = \int \frac{dp dq}{(2\pi)^2} e^{-i\alpha q} (h(p, q) - \mu) \partial_{\mu} u(p, q, t) \quad (12)$$

$\tilde{\epsilon}$ is essentially a measure of the deviation of the energy from the fermi level μ .

Using (12) in (11) we get the basic operator equation for $\partial_{\mu} \tilde{u}_0(p, t)$,

$$\left(\partial_t^2 - (\alpha \partial_{\alpha})^2 + 2\alpha^2 \mu \right) \partial_{\mu} \tilde{u}_0(\alpha, t) = -2\alpha^2 \tilde{\epsilon}(\alpha, t) \quad (13)$$

Note that the static part of the operator on the l.h.s. is the Wheeler-deWitt operator.

Before proceeding further let us make a simple application of the operator eqn. (13) and derive the Wheeler-deWitt equation. If we define the time-independent ground state expectation value of $\tilde{u}_0(\alpha, t)$ to be $\psi_0(\alpha, \mu)$, then we see that in the classical limit (13) reduces to

$$[(\alpha\partial\alpha)^2 - 2\alpha^2\mu]\partial_\mu\psi_0(\alpha, \mu) = 0 \quad (14)$$

which is the Wheeler-deWitt equation for the ‘loop operator’ (for imaginary loop lengths) [13]. In arriving at (14) we have used that the ground state expectation value $\langle u(p, q, t) \rangle_0 = \theta(h(p, q) - \mu)$ in the classical limit, so that $\tilde{\epsilon}$ on the r.h.s. of (13) vanishes in this limit.

Let us now derive the classical equation for the operator $\tilde{u}_0(\alpha, t)$ in a state that is a small deviation from the classical ground state. As is well-known, such a state is described by a characteristic function of phase space whose boundary is a small area-preserving deviation from the fermi surface $\frac{1}{2}(p^2 - q^2) = \mu$ [10-12]. Call the region bounded by the hyperbola $\frac{1}{2}(p^2 - q^2) = \mu$, $q < 0$, R_0 , and the deformed region R . By small deformation we mean $|\frac{\delta E}{\mu}| \equiv |\frac{h(p, q) - \mu}{\mu}| \ll 1$, for $(p, q) \in \delta R \equiv R - R_0$. (Note that $\int_{\delta R} \delta E / \mu$ is in fact the expansion parameter of the string perturbation theory).

Let us consider the expectation value of $\tilde{u}_0(\alpha, t)$ in the state $|R\rangle$. Then from (13) we have

$$(\partial_t^2 - (\alpha\partial\alpha)^2 + 2\alpha^2\mu) \partial_\mu \langle \tilde{u}_0(\alpha, t) \rangle_R = -2\alpha^2 \langle \tilde{\epsilon}(\alpha, t) \rangle_R \quad (15)$$

Now

$$\langle \tilde{\epsilon}(\alpha, t) \rangle_R = \int \int \frac{dp dq}{(2\pi)^2} e^{-i\alpha q} (h(p, q) - \mu) \partial_\mu \chi_{R(t)}(p, q) \quad (16)$$

where $\chi_{R(t)}(p, q) = \langle u(p, q, t) \rangle_R$ is the characteristic function of the region R . De-

noting $\chi_{R(t)} - \chi_{R_0}$ by $\delta\chi(p, q, t)$, we have

$$\begin{aligned} \langle \tilde{\epsilon}(\alpha, t) \rangle_R &= \int \frac{dp dq}{(2\pi)^2} e^{-i\alpha q} (h(p, q) - \mu) \partial_\mu \chi_{R_0}(p, q) \\ &+ \int \frac{dp dq}{(2\pi)^2} e^{-i\alpha q} (h(p, q) - \mu) \partial_\mu \delta\chi(p, q, t) \end{aligned} \quad (17)$$

Since $\chi_{R_0}(p, q) = \theta(h(p, q) - \mu)$, the first term in the above vanishes. Also since $(h(p, q) - \mu) / \mu \ll 1$, for $\delta R(t)$ a small region around the fermi surface, the second term can be neglected compared to μ . The neglected terms represent higher order terms in string perturbation theory. Hence we have

$$(\partial_t^2 - (\alpha \partial_\alpha)^2 + 2\alpha^2 \mu) \partial_\mu \langle \tilde{u}_0(\alpha, t) \rangle_R = (\text{higher orders in string perturbation}). \quad (18)$$

Defining $\psi(\alpha, t) = \langle \tilde{u}_0(\alpha, t) \rangle_R - \psi_0(\alpha, \mu)$, where $\psi_0(\alpha, \mu)$ is a solution of the classical Wheeler-deWitt equation (14), we get the basic classical equation for $\psi(\alpha, t)$

$$(\partial_t^2 - (\alpha \partial_\alpha)^2 + 2\alpha^2 \mu) \partial_\mu \psi(\alpha, t) = 0 + \text{“corrections”} \quad (19)$$

The r.h.s. of the above equation is of course zero upto higher order string loop “corrections”, which can in principle be calculated from the fermion theory.

4. The hyperbolic transform

So far we have discussed the phase space density field $u(p, q, t)$ and the generators $\tilde{u}(\alpha, \beta, t)$ of W_∞ algebra. We now introduce another field, $\phi(p, q, t)$, through a “hyperbolic” transform of $u(p, q, t)$:

$$\phi(p, q, t) \equiv \int \frac{dp' dq'}{|p'^2 - q'^2|^{1/2}} u(p' + p, q' + q, t) \quad (20)$$

The choice of the measure in (20) is motivated by the fact that the classical trajectories of the fermions are hyperbolas in phase space. Also note that the transform

is defined off-shell. Using the equation of motion (6) of $u(p, q, t)$ it is easy to see that $\phi(p, q, t)$ satisfies an identical equation:

$$(\partial_t + p\partial_q + q\partial_p)\phi(p, q, t) = 0 \quad (21)$$

In other words, on-shell, $\phi(p, q, t)$ is a function of only the two combinations[★]

$$u \equiv \frac{1}{2}(p + q)e^{-t}, \quad v \equiv \frac{1}{2}(p - q)e^t. \quad (22)$$

We will denote this function suggestively by $T(u, v)$, i.e.

$$\phi(p, q, t) \stackrel{\text{on-shell}}{\equiv} T(u, v) \quad (23)$$

The remarkable point is that the function $T(u, v)$ satisfies, in the classical approximation, the equation of motion of tachyon in the background of black-hole metric and dilaton fields in the conformal gauge, the variables u and v being identified with the conformal coordinates of 2-dimensional string theory [4].

It is not very difficult to prove the above. The first step is to recast the r.h.s. of (20), on-shell (i.e. using the equation of motion of $u(p, q, t)$), as explicitly a function of u and v only. To do so one needs to make the change of variables

$$p' = E \cosh \theta, \quad q' = E \sinh \theta, \quad \text{for } p'^2 - q'^2 \geq 0$$

and (24)

$$p' = E \sinh \theta, \quad q' = E \cosh \theta, \quad \text{for } p'^2 - q'^2 \leq 0$$

in the integral. Then, after some manipulations one can show that

$$T(u, v) = \int_{-\infty}^{+\infty} dE \int_{-\infty}^{+\infty} d\theta \left[u(E, ue^\theta - ve^{-\theta}, \theta) + u(ue^\theta + ve^{-\theta}, E, \theta) \right] \quad (25)$$

This provides an explicit demonstration of the fact that $\phi(p, q, t)$ satisfies the equa-

★ We hope there is no confusion between the notation u used below and that for the phase space density $u(p, q, t)$. The reason for using this notation is to facilitate contact with earlier work.

tion (21). An operator similar to the first term in (25) has previously been interpreted as tachyon in black-hole background in [14]. We, however, believe that our symmetrical form is the more natural one to use. It is also important to realize that the form (25) for our proposed tachyon operator has been derived from the off-shell definition (20) after using the equation of motion for the u field.

The second step is to recast (25) in terms of the field $\tilde{u}(\alpha, \beta, t)$ by using

$$u(p, q, t) = \int d\alpha d\beta e^{i(q\alpha - p\beta)} \tilde{u}(\alpha, \beta, t) \quad (26)$$

This gives

$$T(u, v) = 2\pi \int_{-\infty}^{+\infty} d\theta \int_{-\infty}^{+\infty} d\alpha \left[e^{i\alpha(ue^\theta - ve^{-\theta})} \tilde{u}(\alpha, 0, \theta) + e^{-i\alpha(ue^\theta + ve^{-\theta})} \tilde{u}(0, \alpha, \theta) \right] \quad (27)$$

In the final step, let us define $\eta(u, v) \equiv T(u, v) - T_0(u, v)$, where T_0 is the vacuum value of T obtained from (27) by substituting the vacuum value of $\tilde{u}(\alpha, \beta, \theta)$ in it. Then, shifting $\tilde{u}(\alpha, \beta, \theta)$ by its vacuum value and using the equations (14) and (19) one can show that the fluctuations $\eta(u, v)$ satisfy the equation

$$\left[4 \left(uv - \frac{\mu}{2} \right) \partial_u \partial_v + 2(u\partial_u + v\partial_v) + 1 \right] \partial_\mu \eta(u, v) = 0 + \text{“corrections”} \quad (28)$$

This is identical to equation (38) of ref. [4] in the conformal gauge. The metric and dilaton background that enter in (28) are those of the classical 2-dim. black-hole in the conformal gauge of the 2-dim. gravity. It should be remarked that the full Kruskal extension emerges because we are working with the entire (p, q) phase plane of the fermion problem.

This identification of the tachyon field in the background of black-hole metric and dilaton system with the (matrix model) fermion field theory operator $\phi(p, q, t)$ defined in (20) is quite remarkable. What is more remarkable, however, is that (28) has solution which is singular at $uv = \frac{\mu}{2}$, while equation (25), from which

(28) was derived using “reasonable” assumptions, apparently has nothing singular about it at $uv = \frac{\mu}{2}$. What does $uv = \frac{\mu}{2}$ correspond to on the phase plane? The mapping we have constructed from the phase plane (at each instant of time) to the (u, v) space given by (22) enables us to answer this question easily. The $uv = \frac{\mu}{2}$ singularity corresponds to the classical single particle trajectory with energy $h(p, q) = \frac{1}{2}(p^2 - q^2) = \mu$. This identifies the classical black-hole mass with $\mu/2$ where μ is the fermi level. We mention that in [15], using continuum methods, the classical black-hole mass was identified with the scale of the Liouville theory (which is essentially μ). We thus arrive at the result that the fermi surface of the classical fermi fluid theory gets mapped on to the black-hole singularity in the (u, v) plane. Since there is nothing singular about the fermi surface, it would seem that the singularity at $uv = \frac{\mu}{2}$ in (28) is a result of neglecting the “correction” terms in the classical approximation. To resolve this issue one needs to do an exact calculation. Such a calculation can be done for the tachyon background $T_0(u, v)$. Before turning to it, we close this section with a discussion of the algebra satisfied by the operator $\phi(p, q, t)$.

Analogous to the generators $\tilde{u}(\alpha, \beta, t)$ of W_∞ algebra corresponding to $u(p, q, t)$, let us introduce the operators $\tilde{\phi}(\alpha, \beta, t)$ corresponding to $\phi(p, q, t)$:

$$\tilde{\phi}(\alpha, \beta, t) \equiv \int \frac{dp dq}{(2\pi)^2} e^{i(p\beta - q\alpha)} \phi(p, q, t) \quad (29)$$

Using (20), we get

$$\tilde{\phi}(\alpha, \beta, t) = \left(\int \frac{dp' dq'}{|p'^2 - q'^2|^{1/2}} e^{i(q'\alpha - p'\beta)} \right) \tilde{u}(\alpha, \beta, t)$$

The integral can be easily done using the hyperbolic parametrization given in (24). The result is

$$\tilde{\phi}(\alpha, \beta, t) = \frac{2}{|\alpha^2 - \beta^2|^{1/2}} \tilde{u}(\alpha, \beta, t) \quad (30)$$

This simple proportionality of $\tilde{\phi}$ and \tilde{u} leads to the remarkable result that $\tilde{\phi}$ also

satisfies a W_∞ algebra:

$$\left[\tilde{\phi}(\alpha, \beta, t), \tilde{\phi}(\alpha', \beta', t) \right] = 4i \sin \frac{\hbar}{2} (\alpha\beta' - \alpha'\beta) \frac{|(\alpha + \alpha')^2 - (\beta + \beta')^2|^{1/2}}{|\alpha^2 - \beta^2|^{1/2} |\alpha'^2 - \beta'^2|^{1/2}} \tilde{\phi}(\alpha + \alpha', \beta + \beta', t) \quad (31)$$

The structure constants are now, however, very different and in fact singular. But this singularity is merely a reflection of the singular factor relating \tilde{u} to $\tilde{\phi}$ in (30). That there is a ω_∞ algebra underlying the black-hole geometry has been recently discussed in [16] in the context of $SL(2, R)/U(1)$ coset model of the black-hole. Here we see that this is true for the exact quantum problem (except that the ω_∞ algebra gets modified to its quantum counterpart W_∞).

5. Exact evaluation of the tachyon background

The tachyon background $T_0(u, v)$ is just the one-point function of the operator given in (28) in the fermi vacuum. The one-point function of the operator $\tilde{u}(\alpha, \beta, \theta)$ in the fermi vacuum is a function of $(\alpha^2 - \beta^2)$ only and can be shown to satisfy an exact quantum generalization of the Wheeler-deWitt equation (14). Because of the dependence through $(\alpha^2 - \beta^2)$ only, it is enough to write down the equation for $\langle \tilde{u}(\alpha, \theta) \rangle_0 \equiv \psi(\alpha, \mu)$:

$$[(\alpha \partial_\alpha)^2 - 2\mu\alpha^2 + \alpha^4/4] \partial_\mu \psi(\alpha, \mu) = 0 \quad (32)$$

The other function $\psi'(\beta, \mu) \equiv \langle \tilde{u}(0, \beta, \theta) \rangle_0$ can then be obtained from $\psi(\alpha, \mu)$ by the formal substitution $\alpha \rightarrow i\beta$. Equation (32) was first obtained in [13]. It can be shown to be a direct consequence of the fermion equation of motion [17].

Using equation (32) and a similar one for $\partial_\mu \psi'(\beta, \mu)$ in (28) one can easily derive the following exact equation for the tachyon background:

$$\left[\frac{1}{4} \partial_u^2 \partial_v^2 + 4 \left(uv - \frac{\mu}{2} \right) \partial_u \partial_v + 2(u \partial_u + v \partial_v) + 1 \right] \partial_\mu T_0(u, v) = 0 \quad (33)$$

Since $T_0(u, v)$ is a function of the variable $\xi = uv$ only (as can be easily deduced from (28) using that $\langle \tilde{u}(\alpha, \beta, \theta) \rangle_0$ is independent of θ), (33) can be simplified to

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$$\left[\frac{1}{4}(\partial_\xi \xi \partial_\xi)^2 + 4(\xi - \mu/2)\partial_\xi \xi \partial_\xi + 4\xi \partial_\xi + 1 \right] T_0(\xi) = 0 \quad (34)$$

where we have defined $\partial_\mu T_0(u, v) \equiv T_0(\xi)$.

Evidently eqn. (34) does not show any singular behaviour for $\xi \rightarrow \mu/2$. One can reduce this equation to a hypergeometric equation in momentum space (conjugate to ξ) and thus solve it. We will not do so here, but instead we will directly substitute the expression for $\psi(\alpha, \mu)$ obtained from a computation in the fermion field theory in (27) and evaluate $T_0(\xi)$. It is important to realize that this obviates the need to discuss boundary conditions in solving (34), since the fermion field theory gives a definite answer for $\psi(\alpha, \mu)$. An integral representation for $\partial_\mu \psi(\alpha, \mu)$ has been given in [18]^{*}. It is

$$\partial_\mu \psi(\alpha, \mu) = \frac{1}{2\pi} \operatorname{Re} \int_0^\infty d\lambda \frac{e^{-i\mu\lambda + \frac{i}{4}\alpha^2 \coth \lambda/2}}{\sinh \lambda/2} \quad (35)$$

The integral in (35) is defined for complex α^2 with a small positive imaginary part. The result is then analytically continued to real α^2 . Similarly,

$$\partial_\mu \psi'(\beta, \mu) = \frac{1}{2\pi} \operatorname{Re} \int_0^\infty d\lambda \frac{e^{i\mu\lambda + \frac{i}{4}\beta^2 \coth \frac{\lambda}{2}}}{\sinh \lambda/2} \quad (36)$$

Using (35) and (36) in (27), we get after some manipulations

$$T_0(\xi) = (2\pi)^{3/2} \operatorname{Im} \int_0^\infty d\lambda \frac{e^{i\mu\lambda - 2i\xi \tanh \frac{\lambda}{2}}}{\sqrt{\sinh \lambda}} \left[-e^{-i\frac{\pi}{4}} H_0^{(1)}\left(2|\xi| \tanh \frac{\lambda}{2}\right) + e^{i\frac{\pi}{4}} H_0^{(2)}\left(2|\xi| \tanh \frac{\lambda}{2}\right) \right] \quad (37)$$

where $H_0^{(1)}(z)$ and $H_0^{(2)}(z)$ are standard combinations of Bessel function [19].

^{*} The prefactor given in front of the integral in this reference is incorrect. The correct one has been given below. Also, note the different sign convention for the fermi level μ .

Equation (37) reveals the source of the $uv = \xi \rightarrow \mu/2$ singularity in (28). The classical limit in which (28) is valid corresponds to restricting the range of λ -integration in (37) to the region $\frac{1}{|\mu|} \ll \lambda \ll 1$, $|\xi|\lambda \gg 1$. In this range of λ we can approximately replace $2 \tanh \frac{\lambda}{2}$ and $\sinh \lambda$ by λ and use the asymptotic expansions of $H_0^{(1)}$ and $H_0^{(2)}$ [20],

$$\begin{aligned} H_0^{(1)}(z) &\sim \sqrt{\frac{2}{\pi z}} e^{i(z-\pi/4)} \\ H_0^{(2)}(z) &\sim \sqrt{\frac{2}{\pi z}} e^{-i(z-\pi/4)} \end{aligned}$$

to get

$$T_0(\xi) \sim \text{Re} \int_{1/\mu}^1 \frac{d\lambda}{\lambda} e^{i(\mu-\xi)\lambda} (e^{-i|\xi|\lambda} + e^{i|\xi|\lambda}) \quad (38)$$

which gives the singular solution

$$T_0(\xi) \sim \ell n \left(\xi - \frac{\mu}{2} \right) \quad (39)$$

obtained in [4,5]. The singularity in (38) at $\xi = \mu/2$ is, however, clearly a result of the approximations made in deriving it since the full expression in (37) is finite at $\xi = \mu/2$. At this value of ξ there is a cancellation of the terms in the exponent linear in λ in (37), so one must retain higher order terms in λ in the expansion of $\tanh \lambda/2$. The correct result is

$$T_0(\xi = \mu/2) \sim \text{Re} \int_{1/\mu}^1 \frac{d\lambda}{\lambda} e^{i\frac{\mu}{12}\lambda^3} \quad (40)$$

which is obviously finite. In fact, for $(\xi - \mu/2)$ very small, $T_0 \sim a + b(\xi - \mu/2) +$ higher orders, where a and b are constants. Since the genus expansion of $T_0(\xi)$ is obtained by taking $\lambda \rightarrow \kappa\lambda$ in (37) and expanding in powers of κ , we see that stringy higher loop effects are responsible for the finiteness of $T_0(\xi)$ at $\xi = \mu/2$. We wish to emphasize that this has been possible because the exact answer (37)

already sums string perturbation theory to all orders. From point of view of the exact equation (33) the higher derivative term is important close to the classical singularity. One might wonder about possible geometrical interpretation of the exact equation (33). Is there a description at all in the neighbourhood of the classical singularity in terms of backgrounds corresponding to graviton, dilaton and the other higher modes of the string? Or is there a topological description without space time geometry?

In conclusion, we wish to point out that the identification of the tachyon operator, (20), in fermion field theory corresponding to the classical black-hole geometry in 2-dim. string theory, coupled with the fact that the fermion theory can be treated exactly, has opened up the possibility of understanding precisely how a consistent theory of quantum gravity washes away the classical black-hole singularity. It is to be hoped that a similar mechanism is operative in more realistic higher dimensional theories.

Acknowledgement: We would like to acknowledge A.M. Sengupta for many discussions on black-hole physics. One of us (SRW) would also like to thank S.R. Das for useful discussions.

Note Added:

(i) The second-order differential equation for the tachyon background in the classical approximation ((28) with $\eta(u, v)$ replaced by $T_0(u, v)$) has two independent solutions one of which is regular at $uv = \mu/2$ and the other singular. That the appropriate boundary condition is the one which picks up the singular solution can be seen by directly evaluating (20) for the fermi vacuum. In the classical limit, $\langle u(p, q, t) \rangle_0 = \theta(h(p, q) - \mu)$. So we have, in the classical limit,

$$\partial_\mu T_0(u, v) = - \int dp' dq' |(p - p')^2 - (q - q')^2|^{-1/2} \delta\left(\frac{1}{2}(p'^2 - q'^2) - \mu\right) \quad (41)$$

Since μ is negative in our convention, we can use the change of variable

$$p' = E \sinh \theta, \quad q' = E \cosh \theta$$

to do the above integration. The E -integration can be readily done using the δ -function. The result is

$$\begin{aligned} \partial_\mu T_0(u, v) = - \int_{-\infty}^{\infty} d\theta [& |4uv + 2\mu + 2\sqrt{-2\mu}(ue^{-\theta} - ve^\theta)|^{-1/2} \\ & + |4uv + 2\mu - 2\sqrt{-2\mu}(ue^{-\theta} - ve^\theta)|^{-1/2}] \end{aligned} \quad (42)$$

For $uv \leq 0$, we get

$$\begin{aligned} \partial_\mu T_0(u, v) = - \int d\theta [& |4uv + 2\mu + 4\sqrt{-2\mu}|uv|^{1/2} \cosh \theta|^{-1/2} \\ & + |4uv + 2\mu - 4\sqrt{-2\mu}|uv|^{1/2} \cosh \theta|^{-1/2}] \end{aligned} \quad (43)$$

It is easy to see that the θ -integration diverges logarithmically for $uv \rightarrow \mu/2$,

$$\partial_\mu T_0(u, v) \sim \ln(uv - \mu/2).$$

For $uv \geq 0$,

$$\begin{aligned} \partial_\mu T_0(u, v) = - \int d\theta [& |4uv + 2\mu + 4\sqrt{-2\mu}|uv|^{1/2} \sinh \theta|^{-1/2} \\ & + |4uv + 2\mu - 4\sqrt{-2\mu}|uv|^{1/2} \sinh \theta|^{-1/2}] \end{aligned} \quad (44)$$

This is obviously non-singular. We see from this calculation, directly from the classical fermi fluid theory, that the appropriate boundary conditions on the differential equation for the tachyon in the black hole metric is the one that picks out the singular solution.

(ii) We thank E. Martinec for pointing out to us that a transform similar to the first term in (25) has appeared in the context of the connection between Liouville theory and the $SL(2, R)/U(1)$ black hole theory in a note added in the published version of [15].

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