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Wave Propagation in Stringy Black Hole

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ABSTRACT

We further study the nonperturbative formulation of two-dimensional black holes. We find a nonlinear differential equation satisfied by the tachyon in the black hole background. We show that singularities in the tachyon field configurations are always associated with divergent semiclassical expansions and are absent in the exact theory. We also discuss how the Euclidian black hole emerges from an analytically continued fermion theory that corresponds to the right side up harmonic oscillator potential.

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1. Introduction:

Two-dimensional string theory can be viewed as one dimensional-matter (time) coupled to two-dimensional gravity [1,2,3]. Since the latter has a nonperturbative formulation in terms of a one-dimensional matrix model [4] (nonrelativistic fermions) we have an opportunity to study nonperturbative aspects of two-dimensional string theory. In particular, two-dimensional string theory has a black hole solution [5,8,7] and one can begin to explore nonperturbative aspects of black holes, in particular the important question of the fate of the classical singularity.

In any attempt to understand the emergence of a non-trivial spacetime in the matrix model one has to contend with the fact that the non-relativistic fermions are formulated in a flat spacetime [8,9,10]. However, the "spacetime" of the perturbative collective excitation is a half-plane, the boundary of which is associated with the classical turning point of the fermions [8,9,11,12,10]. On the other hand it is well-known that the graviton-dilaton system $(G_{\mu\nu}, \Phi)$ for the two-dimensional black hole is equivalent to a metric $\widetilde{G}_{\mu\nu} = G_{\mu\nu} \exp(-2\Phi)$ which corresponds to a spacetime that is flat but has a boundary determined by the condition $\exp(-2\Phi) \geq 0$. It is this reasoning that makes it plausible that one may be able to describe a black hole spacetime in the semiclassical limit of the matrix model.

Recently [13] we have discussed a scalar (tachyon) field theory which in the semi-classical limit reduces to a scalar field coupled to the two-dimensional dilaton-black hole. Other works in this direction include [14,15,16,17,18]. References [14] and [18] deal with the continuum formulation and relate the coset black holes with 'Liouville' theory. References [15],[16] and [17] deal directly with the matrix model. The field theory discussed in [13] is obtained from the nonperturbative formulation of the $c = 1$ matrix model in terms of the phase-space distribution operator

$$\widehat{\mathcal{U}}(p, q, t) = \int_{-\infty}^{\infty} dx\, \psi^\dagger(q - \frac{x}{2}) e^{-ipx} \psi(q + \frac{x}{2}).$$

Since the correlations of this operator are exactly calculable we are able to define an exact nonperturbative quantization of the scalar field. The main result of [13] was that singularities at the position of the black hole singularity ($uv = \mu/2$ in Kruskal coordinates) that occur in the computation of the tachyon ground state do not survive quantization: *the singularities are only a malady of the semiclassical expansion.*

In the present paper we generalize the last result to show that, even for generic field configurations, singularities at the centre of the black hole appear only in semiclassical expansions and like in [13] the exact results are non-singular. We also explore in greater detail the semiclassical expansion applied to the tachyon field to show that perturbatively it satisfies a closed non-linear differential equation (the linear part being the equation of motion for a free field coupled to the dilaton-black hole). The picture that emerges is the following. In some regions of space-time where the semiclassical expansion is valid, the classical physics is described by the above equation of motion for the tachyon; in those regions where the semiclassical expansion shows singularity, clearly the differential equation as well as the attendant notion of spacetime backgrounds are to be discarded in favour of the exact quantum theory that is defined in terms of the fermion theory. We describe the working rules of the quantum theory and explain some unusual features of the theory associated with non-trivial commutation rules of the "tachyon field". We also demystify the "hyperbolic transform" by exhibiting some of its physically interesting properties and showing that it is the unique transform which satisfies these properties. Finally we discuss in brief the Euclidian black hole obtained from an analytically continued fermion theory.

2. Quantized Tachyon in Black Hole

Let us first briefly review some salient points of [13]. The basic construct is the fermion bilinear $\widehat{\phi}(p, q, t)$ and is defined by a "hyperbolic transform" of the quantum phase space density $\widehat{\mathcal{U}}(p, q, t)$:

$$\widehat{\phi}(p, q, t) = \int dp' dq' \, K(p, q|p', q') \, \widehat{\mathcal{U}}(p', q', t) \tag{1}$$

where

$$K(p,q|p',q') \equiv |(p-p')^2 - (q-q')^2|^{-1/2} \tag{2}$$

Equation (1) can be regarded as a relation between Heisenberg operators, where

$$\widehat{\phi}(p,q,t) = e^{iHt}\widehat{\phi}(p,q,0)e^{-iHt}, \;\; \widehat{\mathcal{U}}(p,q,t) = e^{iHt}\widehat{\mathcal{U}}(p,q,0)e^{-iHt} \tag{3}$$

where

$$H = \int \frac{dpdq}{2\pi} h(p,q)\widehat{\mathcal{U}}(p,q), \qquad h(p,q) = \frac{1}{2}(p^2 - q^2) \tag{4}$$

Clearly equation (1) is also valid for expectation values

$$\langle\psi|\widehat{\phi}(p,q,t)|\psi\rangle = \int dp'dq'\, K(p,q|p',q')\langle\psi|\widehat{\mathcal{U}}(p',q',t)|\psi\rangle \tag{5}$$

where $|\psi\rangle$ is any state in the Fermi theory.

For the Heisenberg operator $\widehat{\mathcal{U}}(p,q,t)$ we have the equation of motion

$$(\partial_t + p\partial_q + q\partial_p)\widehat{\mathcal{U}}(p,q,t) = 0 \tag{6}$$

Used in the definition (1), together with (2), this leads to

$$(\partial_t + p\partial_q + q\partial_p)\widehat{\phi}(p,q,t) = 0 \tag{7}$$

The last equation implies that if we define the variables

$$u = e^{-t}(p+q)/2, \;\; v = e^{t}(p-q)/2 \tag{8}$$

or equivalently

$$p = ue^{t} + ve^{-t}, \;\; q = ue^{t} - ve^{-t} \tag{9}$$

then

$$\partial_t \widehat{\phi}(ue^t + ve^{-t}, ue^t - ve^{-t}, t) = 0 \tag{10}$$

This means that we can define

$$\begin{aligned}\widehat{T}(u,v) \equiv& \widehat{\phi}(ue^t + ve^{-t}, ue^t - ve^{-t}, t) \\ =& \int du'dv' \widetilde{K}(u,v|u',v') \widehat{\mathcal{U}}(u'e^t + v'e^{-t}, u'e^t - v'e^{-t}, t),\end{aligned} \tag{11}$$

which is actually independent of t. Here

$$\widetilde{K}(u,v|u',v') \equiv \ |(u-u')(v-v')|^{-1/2} \tag{12}$$

In [13] we observed that if one considers states $|\psi\rangle$ in the fermion theory such that $\langle\psi|\widehat{\mathcal{U}}(p,q,t)|\psi\rangle$ differs from $\langle\psi_0|\widehat{\mathcal{U}}(p,q,t)|\psi_0\rangle$ at most in a small neighbourhood of the fermi surface $p^2 - q^2 = 2\mu$, then

$$\begin{aligned}\delta T(u,v) \equiv& \langle\psi|\delta\widehat{T}(u,v)|\psi\rangle = \langle\psi|\widehat{T}(u,v)|\psi\rangle - T_0(u,v) \\ \delta\widehat{T}(u,v) \equiv& \widehat{T}(u,v) - T_0(u,v)\end{aligned} \tag{13}$$

satisfies

$$[4(uv - \mu/2)\partial_u\partial_v + 2(u\partial_u + v\partial_v) + 1]\delta T(u,v) = 0 + o(\frac{\delta E}{\mu}) \tag{14}$$

In the above, $|\psi_0\rangle$ refers to the fermion ground state, $T_0(u,v) = \langle\psi_0|\widehat{T}(u,v)|\psi_0\rangle$ and δE is the maximum spread of energy relative to the fermi surface in the support of $\langle\psi|\widehat{\mathcal{U}}(p,q,t)|\psi\rangle$.

To this order, therefore, the construct (13) provides us with classical solutions for a free scalar field coupled to the two-dimensional black hole. We shall see in the next section that the higher order terms in $(\delta E/\mu)$ correspond to non-linear terms in $\delta T(u,v)$.

Now, as we emphasized in [13], expression (13) does more than just to provide perturbative solutions to (14). $\delta T(u,v)$ is in fact exactly computable in the fermion theory (in principle for any state $|\psi\rangle$). We will see that typically the exact expression has a bad perturbative expansion in $(\delta E/\mu)$ and the divergences in solutions of (14) at the "black hole singularity" $uv = \mu/2$ are a result of such bad expansions. Indeed the exactly computed $\delta T(u,v)$ is free of singularities. In [13] we have already explcitly computed one such example.

We will thus take the attitude that the fermion theory, through equations such as (13), defines for us quantization of a scalar field coupled to the two-dimensional black hole. Such a definition is of course automatically nonperturbative since the fermion theory is so. In the next few sections we will explore in detail the semiclassical physics by considering in the fermion theory small fluctuations near the fermi surface and show how to use the nonperturbative formalism of the fermion theory to extract the nonperturbative behaviour.

3. Properties of the Hyperbolic Transform

Before launching into properties of the "tachyon field" (11) it is useful to understand its definition in some more detail. In this section we shall mention some remarkable properties of the kernel (2) that defines for us the "hyperbolic transform" (1). The basic properties of (2) (equivalently, of (12)) are:

(i) Lorenz covariance: $K(P,Q|P',Q') = K(p,q|p',q')$, where $P = p\cosh\theta + q\sinh\theta, Q = p\sinh\theta + q\cosh\theta$ and similarly for P', Q'.

(ii) Translational invariance: $K(p,q|p',q') = f(p-p', q-q')$

(iii) Differential equation:

$$\begin{aligned}&[4(uv-\mu/2)\partial_u\partial_v + 2(u\partial_u + v\partial_v) + 1]\widetilde{K}(u,v|u',v') = o[u'v' - \mu/2]\\ &\widetilde{K}(u,v|u',v') = K(\frac{u+v}{2}, \frac{u-v}{2}|\frac{u'+v'}{2}, \frac{u'-v'}{2})\end{aligned} \tag{15}$$

The precise form of the last equation is

$$[4(uv - \mu/2)\partial_u\partial_v + 2(u\partial_u + v\partial_v) + 1]\widetilde{K}(u, v|u'v') \\ = (u'v' - \mu/2)\partial_u\partial_v K(u, v|u', v') \tag{16}$$

As we shall see, it is equation (15) (or (16)) that is responsible for the black hole interpretation of the low energy physics of the tachyon. This is because if one considers a state $|\psi\rangle$ such that the support of $\langle\psi|\widehat{\mathcal{U}}(p', q', t)|\psi\rangle$ is near the fermi surface, then $\langle\psi|\widehat{T}(u, v)|\psi\rangle$ vanishes to leading order according to (11) and (15).

Clearly, property (iii) is very desirable from the point of view of black hole physics. Property (i) implies that the equations of motion are the same for $\phi(p, q, t)$ and $\mathcal{U}(p, q, t)$; this implies that the reduced variables u, v used in both cases have the same physical interpretation. Property (ii) is directly related to the fact that the hyperbolic transform becomes local in Fourier space; in other words, the (double) Fourier transform of $\phi(p, q, t)$'s ($\widetilde{\phi}(\alpha, \beta, t)$'s) are basically rescalings of $W(\alpha, \beta, t)$'s [19, 13] which ensure that the $\widetilde{\phi}$'s have an algebra that has a classical limit. The latter property implies that the classical action written in terms of $\phi(p, q, t)$ has an $\hbar \to 0$ limit which, once again is rather crucial.

We now prove that (2) is the **unique kernel** satisfying all the three properties mentioned above.

Proof:
Properties (i) and (ii) imply that K is some function of only the combination $(p - p')^2 - (q - q')^2$:

$$K(p, q|p', q') = g((p-p')^2-(q-q')^2)) \longleftrightarrow \widetilde{K}(u, v|u', v') = f(x),\ x = (u-u')(v-v') \tag{17}$$

Property (iii) states that if we choose $u' = q_0e^{\theta}/2, v' = -q_0e^{-\theta}/2, q_0 \equiv \sqrt{-2\mu}$ so that $u'v' = \mu/2$ (note that μ, the fermi energy, is negative in our convention), then

$$[4(uv - \mu/2)\partial_u\partial_v + 2(u\partial_u + v\partial_v) + 1]\widetilde{K}(u, v|q_0e^{\theta}/2, -q_0e^{-\theta}/2) = 0 \tag{18}$$

For this choice of u', v' we have $x = uv - \mu/2 + [q_0/2](ue^{-\theta} - ve^{\theta})$. Using (17) for

$\widetilde{K}$, and introducing the notation $y = uv - \mu/2$, we get from (18)

$$2xf'(x) + f(x) + y[6f'(x) + 4xf''(x)] = 0 \tag{19}$$

This equation must be identically satisfied for all y, which implies that both the y-independent term and the coefficient of y must vanish in (19). Curiously the first condition implies the second condition, so we need not separately consider the second condition. Thus we get

$$2xf'(x) + f(x) = 0 \tag{20}$$

The above equation is solved by

$$f(x) = \text{constant } |x|^{-1/2} \tag{21}$$

which proves that (modulo an overall constant)

$$K(p, q|p', q') = |(p-p')^2 - (q-q')^2|^{-1/2} \longleftrightarrow \widetilde{K}(u, v|u', v') = |(u-u')(v-v')|^{-1/2}$$

4. Non-linear Differential Equation for the Tachyon

In this section we discuss the semiclassical physics of the tachyon $\widehat{T}(u, v)$ in detail and derive a closed non-linear differential equation for $\delta T(u, v) = \langle\psi|\widehat{T}(u, v)|\psi\rangle - T_0(u, v)$ in a semiclassical expansion.

Let us consider states $|\psi\rangle$ which satisfy

$$\langle\psi|\widehat{\mathcal{U}}(p, q, t)|\psi\rangle = \vartheta([p_+(q, t) - p][p - p_-(q, t)]) + o(\hbar). \tag{22}$$

where $\vartheta(x) = 1$ if $x > 0$ and $= 0$ otherwise. This corresponds to the "quadratic profile" ansatz [12,20]. We recall that in the $\hbar \to 0$ limit, the classical $\mathcal{U}(p, q, t)$

(expectation value in a state) satisfies $\mathcal{U}^2(p,q,t) = \mathcal{U}(p,q,t)$ which implies that $\mathcal{U}(p,q,t)$ is the characteristic function for some region; the quadratic profile ansatz assumes that there is one connected region with boundary given by $(p-p_+(q,t))(p-p_-(q,t)) = 0$. As is well-known, in the ground state $|\psi_0\rangle$, the boundary is the fermi surface itself: $p^2 - q^2 - 2\mu = 0$, which correspond to

$$p^0_\pm(q,t) = p^0_\pm(q) = \pm\sqrt{q^2+2\mu}\,\vartheta(q^2+2\mu). \tag{23}$$

We shall use the notation

$$p_\pm(q,t) \equiv p^0_\pm(q) + \eta_\pm(q,t) \tag{24}$$

Let us compute $\delta T(u,v)$ in the state (22)(ignoring the $o(\hbar)$ terms for the moment), using (13). We get

$$\begin{aligned}\delta T(u,v) = &\int_{-\infty}^{\infty} dq \int_0^{\eta_+(q,t)} \frac{dp'}{|[2ue^t - (p' + p^0_+(q) + q)][2ve^{-t} - (p' + p^0_-(q) - q)]|^{1/2}} \\ &- \int_{-\infty}^{\infty} dq \int_0^{\eta_-(q,t)} \frac{dp'}{|[2ue^t - (p' + p^0_-(q) + q)][2ve^{-t} - (p' + p^0_-(q) - q)]|^{1/2}}\end{aligned} \tag{25}$$

Let us now make some further assupmtions about the state $|\psi\rangle$, namely that the fluctuations $\eta_\pm(q,t)$ are non-zero only for $q < -q_0 \equiv -\sqrt{-2\mu}$ and that in this region $|\eta_\pm(q,t)| << |p^0_\pm(q)|$. In other words, we are considering "small" fluctuations near the left branch of the fermi surface in the classically allowed region. Under these assumptions we can expand the integrand in a Taylor series in p' around $p' = 0$. The p'-integrals now are easy to do, giving powers of $\eta_\pm(q)$. The q-integrals that remain are now effectively between $-\infty$ and $-q_0$ [$q_0 = \sqrt{-2\mu}$], so one can use a different integration variable τ, the "time of flight", given by

$$q = -q_0\cosh\tau,\ p^0_\pm(q) = \pm q_0\sinh\tau \tag{26}$$

If we also define the rescaled functions $\bar{\eta}_\pm(\tau,t) = |p^0_\pm(q)|\eta_\pm(q,t)$, we ultimately get

$$\delta T(u,v) = \frac{1}{2}\int\limits_0^\infty d\tau \Big[k_+(u,v|\tau,t)\bar{\eta}_+(\tau,t) - k_-(u,v|\tau,t)\bar{\eta}_-(\tau,t)]$$
$$-\frac{1}{8}\int\limits_0^\infty \frac{d\tau}{p^0_+(q)}(e^{-t}\partial_u + e^t\partial_v)k_+(u,v|\tau,t)\bar{\eta}_+(\tau,t)^2 - (e^{-t}\partial_u + e^t\partial_v)k_-(u,v|\tau,t)\bar{\eta}_-(\tau,t)^2\Big] + o(\bar{\eta}^3_\pm) \tag{27}$$

where

$$k_\pm(u,v|\tau,t) \equiv |(ue^t + \frac{q_0}{2}e^{\mp\tau})(ve^{-t} - \frac{q_0}{2}e^{\pm\tau})|^{-1/2}, \quad q_0 \equiv \sqrt{-2\mu} \tag{28}$$

Relations between $\bar{\eta}_\pm$ and $\delta T(u,v)$:

Equation (27) is important in that it builds a correspondence between the semiclassical quantities of the fermion theory and those of the $\delta T(u,v)$-theory. To understand it better, let us first choose a different coordinatization for the u,v-space. Let us define[⋆]

$$\mathbf{x} = \frac{1}{2}\ln|\frac{uv}{\mu/2}|, \quad \mathbf{t} = \frac{1}{2}\ln|v/u| \tag{29}$$

This is not a one-to-one map. Let us consider for the moment the quadrant of the u,v-space where $u>0, v>0$. In that case we can write down the inverse maps as follows:

$$u = \frac{q_0}{2}e^{\mathbf{x}-\mathbf{t}}, \quad v = \frac{q_0}{2}e^{\mathbf{x}+\mathbf{t}} \tag{30}$$

Now recall that by definition of $\delta T(u,v)$ (*cf.* the remark about the t-independence of the right hand side of (11)), if $\bar{\eta}_\pm(\tau,t)$ satisfy their equations of motion [these

⋆ We use boldface letters so as to distiguish $\mathbf{t}$ from the time t of the fermion theory.

can be derived by tracing their definition back to $\mathcal{U}(p,q,t)$ and they read as

$$\frac{(\partial_t - \partial_\tau)\bar{\eta}_\pm = \frac{1}{2}\partial_\tau\big(\bar{\eta}^2_\pm}{[p^0_+]^2\big)} \tag{31)]}$$

then the right hand side of (27) is actually t-independent. This means that we can choose t to be anything we like. It is most useful to choose

$$t = \mathbf{t} \tag{32}$$

on the right hand side of (27). This equation now reads, to leading order, as

$$\delta T(\mathbf{x},\mathbf{t}) = \frac{1}{2}\int\limits_0^\infty d\tau[\widetilde{k}_+(\mathbf{x}|\tau)\bar{\eta}_+(\tau,\mathbf{t}) - \widetilde{k}_-(\mathbf{x}|\tau)\bar{\eta}_-(\tau,\mathbf{t})] + o(\bar{\eta}^2_\pm) \tag{33}$$

where

$$\widetilde{k}_\pm(\mathbf{x},\tau) \equiv \frac{2}{q_0}|(e^{\mathbf{x}} + e^{\mp\tau})(e^{\mathbf{x}} - e^{\pm\tau})|^{-1/2} \tag{34}$$

Note that for large $\mathbf{x}$,

$$\frac{q_0}{2}\widetilde{k}_-(\mathbf{x},\tau)\exp[\mathbf{x}] = |1 + \exp(\tau - \mathbf{x})|^{-1/2} + o(e^{-\mathbf{x}})$$

The first term is similar to a low-temperature Fermi-Dirac distribution (the fact that the power is $-1/2$ instead of -1 does not materially affect the arguments). In fact, one can show that the for very large $\mathbf{x}$ it behaves like $\vartheta(\tau - \mathbf{x})$ and corrections to it are like increasing powers of $\partial_{\mathbf{x}}$ on the ϑ-function. Similar expansions are also available for $k_+(\mathbf{x},\tau)$. The precise statements for these $\partial_{\mathbf{x}}$-expansions are the following:

$$\mathcal{T}(\mathbf{x},\mathbf{t}) = \frac{1}{2}\{\mathcal{D}_+\bar{\eta}_+ - \mathcal{D}_-\bar{\eta}_-\} + o(e^{-\mathbf{x}}\bar{\eta}_\pm) + o(\bar{\eta}^2_\pm) \tag{35}$$

where

$$\mathcal{T}(\mathbf{x},\mathbf{t}) \equiv |uv|^{1/2}\delta T(u,v) \tag{36}$$

$$\mathcal{D}_\pm \equiv I_\pm(\partial_{\mathbf{x}}) + I_\pm(1/2 - \partial_{\mathbf{x}}) \tag{37}$$

$$I_\pm(\alpha) = \alpha^{-1}\ {}_2F_1(\frac{1}{2}, \alpha; \alpha+1; \mp 1)$$

where ${}_2F_1$ is the standard Hypergeometric function [21]. Using its properties one can write down an expansion for $\mathcal{D}_\pm$ in $\partial_{\mathbf{x}}$. The expansion begins with $\partial_{\mathbf{x}}^{-1}$. Defining

$$\eta_\pm = \pi_\eta \pm \partial_\tau \eta, \tag{38}$$

where η is the "tachyon" field that is associated with the standard $c = 1$ matrix model and π_η is its conjugate momentum, we can write down a derivative expansion for $\mathcal{T}(\mathbf{x}, \mathbf{t})$:

$$\mathcal{T}(\mathbf{x}, \mathbf{t}) = \eta(\mathbf{x}, \mathbf{t}) + o(\partial_{\mathbf{x}}\eta, \pi_\eta). \tag{39}$$

The identification of the "black hole tachyon" field with the standard $c = 1$ tachyon field in the asymptotic ($\mathbf{x} \to \infty$) is rather remarkable. As a result of this n-point functions of $\mathcal{T}(\mathbf{x}, \mathbf{t})$ are the same as those of the $c = 1$ tachyon at extreme low energy.

Finally, relation (35) can be inverted to give $\bar{\eta}_\pm$ in terms of $\mathcal{T}$:

$$\bar{\eta}_\pm = (\mathcal{D}_\pm \partial_{\mathbf{x}})^{-1} \partial_\pm \mathcal{T}(\mathbf{x}, \mathbf{t}) + o(e^{-x}\mathcal{T}) + o(\mathcal{T}^2), \quad \partial_\pm = \partial_{\mathbf{t}} \pm \partial_{\mathbf{x}} \tag{40}$$

We will use this relation below to obtain a nonlinear differential equation for $\mathcal{T}(\mathbf{x}, \mathbf{t})$.

We wish to emphasize that the above analysis can be repeated in other coordinates which are valid all through the Kruskal diagram (for instance in the light cone coordinates themselves) However, the formulae look more complicated.

Differential Equation:

Let us now go back to the other consequences of Eq. (27). Eq. (18) of Sec. 3 ensures that

$$[4(uv-\mu/2)\partial_u\partial_v+2(u\partial_u+v\partial_v)+1]k_\pm(u,v|\tau,t)=0$$

Using this, and applying the above differential opertor to (27) we get

$$\begin{aligned}&[4(uv-\mu/2)\partial_u\partial_v+2(u\partial_u+v\partial_v)+1]\delta T(u,v)\\&\qquad=\frac{1}{2}\partial_u\partial_v\int_0^\infty d\tau[k_+\bar\eta_+^2+k_-\bar\eta_-^2]+o(\bar\eta_\pm^3)\end{aligned}\tag{41}$$

Note that the linear term dropped out because of the special properties of the kernel. Using the $\mathbf{x},\mathbf{t}$-coordinatization, (41) can be written as

$$D_{\mathbf{x},\mathbf{t}}\mathcal{T}(\mathbf{x},\mathbf{t})=-\frac{e^{-2\mathbf{x}}}{|\mu|}[e^{\mathbf{x}/2}\partial_\mathbf{x}\{e^{-\mathbf{x}/2}(\mathcal{D}_+\bar\eta_+^2+\mathcal{D}_-\bar\eta_-^2)\}]+o(e^{-3\mathbf{x}}\bar\eta_\pm^2)+o(\bar\eta_\pm^3)\tag{42}$$

where

$$\begin{aligned}D_{\mathbf{x},\mathbf{t}}=&e^{\mathbf{x}}[4(uv-\mu/2)\partial_u\partial_v+2(u\partial_u+v\partial_v)+1]e^{-\mathbf{x}}\\=&(1+e^{-2x})(\partial_\mathbf{x}^2-\partial_\mathbf{t}^2)+e^{-2\mathbf{x}}(2\partial_\mathbf{x}+1)\end{aligned}\tag{43}$$

Finally, by using (40) to convert the $\bar\eta_\pm$ back into $\mathcal{T}$, we get the following closed differential equation in $\mathcal{T}$ upto quadratic order:

$$\begin{aligned}D_{\mathbf{x},\mathbf{t}}\mathcal{T}(\mathbf{x},\mathbf{t})=-\frac{e^{-2\mathbf{x}}}{|\mu|}[e^{\mathbf{x}/2}\partial_\mathbf{x}\{e^{-\mathbf{x}/2}(\mathcal{D}_+[(\mathcal{D}_+\partial_\mathbf{x})^{-1}\partial_+\mathcal{T}]^2+\mathcal{D}_-[(\mathcal{D}_-\partial_\mathbf{x})^{-1}\partial_-\mathcal{T}]^2)\}]\\+o(\mathcal{T}^3)+o(e^{-3x}\mathcal{T}^2)\end{aligned}\tag{44}$$

5. **Exact Quantum Theory Does Not See Black Hole Singuarity:**

In this section we will analyze the nature of singularities that occur in $\delta T(u,v)$ at $uv=\mu/2$ (which is the position of the curvature singularity of the black hole metric $ds^2 = dudv/(uv-\mu/2)$). We will see that these singularities occur as a result of making badly divergent semiclassical expansions and they are not present in $\delta T(u,v)$ when calculated exactly.

Let us consider a state $|\psi\rangle$ in the fermion theory which, like in the previous section, represents fluctuations in the neighbourhood of the left branch of the fermi surface $p^2-q^2=2\mu$ (generalizations are obvious). In this region the fermion phase space can be coordinatized by $(p,q) = (R\sinh\theta, -R\cosh\theta), R>0$. As explained earlier, the support of $\langle\psi|\widehat{\mathcal{U}}(p,q,t)|\psi\rangle$ in the limit $\hbar\to 0$ defines a region $\mathcal{R}(t)$ occupied by the fermi fluid at time t. For the ground state $|\psi_0\rangle$ this region is given by $\mathcal{R}_0 = \{\infty > R \geq R_0 \equiv \sqrt{-2\mu}, \infty > \theta > -\infty\}$ (plus its mirror image on the right half of the phase plane). For simplicity of calculation, let us choose for the moment a state $|\psi\rangle$ so that the region $\mathcal{R}(t)$ has a particularly simple geometry. To be specific, we choose that $\mathcal{R}(t=0)$ is obtained from $\mathcal{R}_0$ by adding a region $\delta\mathcal{R} = \{R_0 \geq R \geq R_1, \theta_1 \geq \theta \geq \theta_2\}$ and subtracting a region $\widetilde{\delta\mathcal{R}} == \{\widetilde{R}_1 \geq R \geq R_0, \widetilde{\theta}_1 \geq \theta \geq \widetilde{\theta}_2\}$. We will call $\delta\mathcal{R}$ the "blip" and $\widetilde{\delta\mathcal{R}}$ the "antiblip"; basically the state $|\psi\rangle$ is created from the ground state $|\psi_0\rangle$ by removing fermions from the region $\widetilde{\delta\mathcal{R}} \subset \mathcal{R}_0$ and placing them in the region $\delta\mathcal{R}$ just outside the filled fermi sea. Fermion number conservation is achieved by choosing the areas of $\delta\mathcal{R}$ and $\widetilde{\delta\mathcal{R}}$ to be the same, which is equivalent to the condition that

$$(\theta_1-\theta_2)(R_1^2-R_0^2) = (\widetilde{\theta}_1-\widetilde{\theta}_2)(R_0^2-\widetilde{R}_1^2)$$

The region $\mathcal{R}(t)$ at non-zero times t is simply obtained by shifting the θ-boundaries of both the blip and the antiblip by t. Using this, one can easily write down the expression for $\delta T(u,v)$ for this state:

$$\delta T(u,v) = \delta T_b(u,v) + \widetilde{\delta} T_b(u,v) + o(\hbar) \tag{45}$$

where

$$\delta T_b(u,v) = \frac{1}{2}\int_{R_1}^{R_0} RdR \int_{\theta_1}^{\theta_2} d\theta |(u+Re^{-\theta})(v-Re^{\theta})|^{-1/2} \tag{46}$$

represents contribution of the 'blip'. The contribution of the 'antiblip', $\widetilde{\delta T}_b(u,v)$, is a similar expression involving the tilde variables. It is important to note that there are $\hbar$-corrections to (45), as $\mathcal{U}(p,q,t)$ is actually a characteristic function plus $o(\hbar)$ terms.

A remark is in order about two seemingly different expansions that we are making in this paper. One is in $\hbar$, and the other is in $|\delta E/\mu|$. Ultimately, as it turns out, both are expansions in $g_{\rm str}$. For the moment, in (45) we have made an explicit $\hbar$-expansion, and no $|\delta E/\mu|$ expansion yet. What we will show in the present example is that it is this latter expansion that is badly divergent and results in increasingly singular behaviour at $uv=\mu/2$, and if one treats expressions like (46) without a $|\delta E/\mu|$ expansion, then $\delta T(u,v)$ does not have any singularities at $uv=\mu/2$. More generally, we will argue that singularities are invariably absent whenever one performs the $|\delta E/\mu|$ resummation; the $\hbar$-corrections coming from corrections like those present in (45) do not affect the conclusions *vis-a-vis* singularities.

Let us analyze the singularities of (46) (treatment of $\widetilde{\delta T}_b(u,v)$ is similar). Clearly singularities can arise only when the expression inside the square root vanishes. It is also clear that a linear zero inside the square root is not a singularity (recall that $\int dx\, x^{-1/2}$ is not singular), we must have a quadratic zero. In other words, both factors must vanish. This can happen only if $u<0, v>0$. Let us choose the parametrization $u=-re^{-\chi}/2, v=re^{\chi}/2$. We get

$$\delta T_b(u,v) = \frac{1}{2}\int_{R_1}^{R_0} RdR \int_{\theta_1}^{\theta_2} d\theta |(R-r)^2 - 4Rr\sinh^2(\frac{\theta-\chi}{2})|^{-1/2} \tag{47}$$

If we look at any one of the integrals separately, over θ or R, we see a logarithmic singularity at $R=r, \theta=\chi$ provided this point is included in the range of integration. Let us do the θ integral first. If r is outside the range $[R_1, R_0]$ we do not have any singularities. If $r \in (R_1, R_0)$, and $\chi \in (\phi_1, \phi_2)$, it is easy to see that for $R \approx r$ the θ integral behaves as $\ln|R-r|$. Let us assume that the range $[R_0, R_1]$ is small (compared to R_0, say, which means that the blip consists of small energy fluctuations compared to the fermi energy) so that $R \approx r$ through the range of integration (this is only a simplifying assumption and the conclusions do not depend on it). We get (for $\phi_2 > 1/2\ln(-v/u) > \phi_1$)

$$\begin{aligned}\delta T_b(u,v) \sim & \int_{R_1}^{R_0} R dR \ln|R-r| \\ =&(R_0-r)\ln|R_0-r| - (R_1-r)\ln|R_1-r| - (R_1-R_0) \\ \approx&(-\mu/2)^{-1/2}[(uv-\mu/2)\ln|uv-\mu/2| - (uv-\mu/2-\Delta/2)\ln|uv-\mu/2-\Delta/2|\end{aligned} \tag{48}$$

In the last line, we have put in the values $R_0 = \sqrt{-2\mu}, r = \sqrt{-4uv}$ and defined $R_1 \equiv \sqrt{2(-\mu-\Delta)}, \Delta > 0$ (Δ thus measures the maximum energy fluctuation of the blip from the fermi surface). Since $R_0 \approx r \approx R_1$ we have used $R_{0,1} - r \approx (R_{0,1}^2 - r^2)/(2R_0)$ and also $R_0 - R_1 \approx (R_0^2 - R_1^2)/(2R_0)$.

It is easy to see that $\delta T_b(u,v)$ has no singularities*. However it is also easy to see that it develops singurities as soon as one attempts a semiclassical expansion in Δ/μ; to be precise one gets

$$\delta T_b(u,v) \sim |\Delta/\mu| \ln|uv-\mu/2| + |\Delta/\mu|^2 (uv-\mu/2)^{-1} + \cdots \tag{49}$$

where once again $1/2\ln(-v/u) \in (\phi_1, \phi_2)$ (for $1/2\ln(-v/u)$ outside this range there are no singularities).

Thus, we see that the tachyon solution (46) develops a singularity at $uv = \mu/2$ at the level of a Δ/μ expansion, though the full solution does not.

What does the above example teach us for the general scenario? In general,

$\star$ This statement is true all over u, v-space including the horizon.

the tachyon solution would be given by expressions like

$$\delta T(u,v) = \frac{1}{2}\int RdR \int d\phi f(R,\theta) |4(u + Re^{-\theta})(v - Re^{\theta})|^{-1/2} \tag{50}$$

where $f(R,\theta) \equiv \delta\mathcal{U}(R\sinh\theta, -R\cosh\theta)$ and the only thing that we have assumed is that the support of $\delta\mathcal{U} \equiv \langle\psi|\widehat{\mathcal{U}}|\psi\rangle - \langle\psi_0|\widehat{\mathcal{U}}|\psi_0\rangle$ is confined to the region $p^2 < q^2$ (the generalization to the other region is straightforward). Note that the expression (46) can be recovered by putting

$$\delta\mathcal{U} = \vartheta[(R - R_0)(R_1 - R)]\vartheta[(\theta - \theta_1)(\theta_2 - \theta)] \tag{51}$$

where $\vartheta(x) = 1$ if $x > 0$ and 0 otherwise. The integral (50) consists of contributions from $R > 0$ (left half of the phase plane) and from $R < 0$. Let us look at the $R > 0$ part first. Once again the singularities can only come from $u < 0, v > 0$ region of the u, v space. Using the same parametrization as in (47) we get

$$\delta T(u,v) = \frac{1}{2}\int RdR \int d\theta f(R,\theta) |(R - r)^2 - 4Rr\sinh^2(\frac{\theta - \chi}{2})|^{-1/2} \tag{52}$$

The basic lesson of the previous example is that even though each one-dimensional integral, taken separately over R or θ, has a logarithmic singularity if the point $R = r, \theta = \chi$ is included in the support of $f(R,\theta)$, the second integral smooths out that singularity. (In the previous example we did the θ integral first to find logarithmic singularity and the R-integration smoothed that out; it could as easily have been done the other way around.) In fact the issue is that of a **two-dimensional** integration of the sort $\int dxdy\, f(x,y)|x^2 - y^2|^{-1/2}$. For a smooth function $f(x,y)$ the possible singularity at $x = y = 0$ (a linear zero in the denominator) is washed away by a stronger (quadratic) zero in the integration measure. A singularity can be sustained only if $f(x,y)$ has a pole or a stronger singularity at $x = y = 0$. However, in our case the quantum phase space density $\mathcal{U}(p,q,t)$ cannot have such singularities. The reason is that the fermion field theory states that we are concerned with are W_∞-rotations of the ground state, and therefore the corresponding

$\mathcal{U}(p,q,t)$ is also a W_∞-rotation of the ground state density $\mathcal{U}_0(p,q)$. Since the latter is a smooth distribution and a unitary rotation cannot induce singularities, we see that $\mathcal{U}(p,q,t)$ must be non-singular. The tachyon field configurations constructed by integrals such as (52) are therefore non-singular too. Another way to think of this is to use a two-step argument: the $\hbar \to 0$ limit of $\mathcal{U}(p,q,t)$ is obtainable by a classical area-preserving diffeomporphism (element of w_∞) and is certainly nonsingular, being given by the characteristic function of a region equal to the w_∞-transformed Fermi sea. This implies that the tachyon field constructed from it is already non-singular; incorporation of $\hbar$-effects further smoothen these disctributions.

We conclude therefore that the exact quantum theory does not permit any singlarities in the tachyon field configuration.

6. Some Novel Features of the Quantum Field Theory of $\widehat{T}(u,v)$

So far we have discussed the semiclassical physics of $\delta T(u,v)$ including its low energy differential equation and in the last section we have seen how to compute the expectation values of $\delta\widehat{T}(u,v)$ beyond the semiclassical expansion using the fermion theory. In this section we discuss in more detail some novel features of the two-dimensional quantum field theory of $\delta\widehat{T}(u,v)$ defined by the underlying fermion theory.

Let us first discuss how $\delta\widehat{T}(u,v) \equiv \widehat{T}(u,v) - T_0(u,v)$ which is *a priori* defined in terms of an on-shell three-dimensional field $\widehat{\mathcal{U}}(p,q,t)$ defines a Heisenberg operator in a *two-dimensional* field theory. Basically we use the fact that the right hand side of (11) is actually independent of t to put t equal to the "time" of the u,v-space. To fix ideas, let us consider the coordinatization (29)-(30) of the u,v-space where $\mathbf{x},\mathbf{t}$ correspond to space and time. This of course limits the discussion to the quadrant $\{u > 0, v > 0\}$ but the discussion holds just as well for more global choices of "time" coordinates also. Now, in these coordinates, writing $\delta\widehat{T}(u(\mathbf{x},\mathbf{t}),v(\mathbf{x},\mathbf{t}))$ as

$\delta\widehat{T}(\mathbf{x},\mathbf{t})$ by an abuse of notation, we have

$$\begin{aligned}
&\delta\widehat{T}(\mathbf{x},\mathbf{t}) \\
&= \int dudv\ |(e^x - ue^{\mathbf{t}})(e^x - ve^{-t})|^{-1/2}\delta\widehat{\mathcal{U}}(ue^t + ve^{-t}, ue^t - ve^{-t}, t) \\
&= \int dudv\ |(e^x - u)(e^x - v)|^{-1/2}\delta\widehat{\mathcal{U}}(ue^{t-\mathbf{t}} + ve^{\mathbf{t}-t}, ue^{t-\mathbf{t}} - ve^{\mathbf{t}-t}, t)
\end{aligned} \tag{53}$$

Here

$$\delta\widehat{\mathcal{U}}(p,q,t) \equiv \widehat{\mathcal{U}}(p,q,t) - \langle\psi_0|\widehat{\mathcal{U}}(p,q,t)|\psi_0\rangle$$

Since these expressions are actually independent of t, we can choose the "gauge"

$$t = \mathbf{t} \tag{54}$$

which gives

$$\delta\widehat{T}(\mathbf{x},\mathbf{t}) = \int dudv\ |(e^x - u)(e^x - v)|^{-1/2}\delta\widehat{\mathcal{U}}(u+v, u-v, \mathbf{t}) \tag{55}$$

Note that the fields on both sides of (55) are evaluated at the same time $\mathbf{t}$. More explcitly we see that

$$\delta\widehat{T}(\mathbf{x},\mathbf{t}) = e^{iH\mathbf{t}}\delta\widehat{T}(\mathbf{x},0)e^{-iH\mathbf{t}} \tag{56}$$

where

$$\delta\widehat{T}(\mathbf{x},0) = \int dudv\ |(e^x - u)(e^x - v)|^{-1/2}\delta\widehat{\mathcal{U}}(u+v, u-v, 0) \tag{57}$$

The hamiltonian is the same as in (4). Eq. (56) tells us that $\delta\widehat{T}(\mathbf{x},\mathbf{t})$ is a Heisenberg operator in a two-dimensional field theory. Eq. (55) allows us to write down time-ordered products of $\delta\widehat{T}(\mathbf{x},\mathbf{t})$'s in terms of time-ordered products of the $\delta\widehat{\mathcal{U}}(p,q,t)$'s:

thus

$$\langle\psi|\mathcal{T}(\delta\widehat{T}(\mathbf{x}_1,\mathbf{t}_1)\cdots\delta\widehat{T}(x_n,\mathbf{t}_n))|\psi\rangle$$
$$=\int du_1 dv_1\cdots du_n dv_n |(e^{x_1}-u_1)(e^{x_1}-v_1)|^{-1/2}\cdots|(e^{x_n}-u_n)(e^{x_n}-v_n)|^{-1/2}\times$$
$$\mathcal{T}(\delta\widehat{\mathcal{U}}(u_1+v_1,u_1-v_1,\mathbf{t}_1)\cdots\delta\widehat{\mathcal{U}}(u_n+v_n,u_n-v_n,t_n)) \tag{58}$$

In this way the expectation values of time-ordered products of $\widehat{T}(u,v)$ can be computed from the fermion theory. As we have stressed earlier, in principle these contain answers to all dynamical questions in the theory. However, these correlation functions are not related to usual particle-scattering amplitudes in the standard fashion, parimarily because:

• $[\delta\widehat{T}(\mathbf{x}_1,\mathbf{t}),\delta\widehat{T}(\mathbf{x}_2,\mathbf{t})]\neq 0$, *i.e.*, the field $\delta\widehat{T}(\mathbf{x},\mathbf{t})$ does not commute with itself at equal times.

In fact the non-trivial commutation relation is a direct consequence of the W_∞ algebra. As we should expect, the field $\widehat{T}(u,v)$ bears a close resemblance to the spin operator in a magnetic field. In both cases the symplectic structures (ETCR in the quantum theory) are non-trivial. We know that in case of the spin, dynamical questions are better formulated in terms of coherent states $|\mathbf{n}\rangle$ satisfying $\langle\mathbf{n}|\widehat{\mathbf{S}}|\mathbf{n}\rangle=\mathbf{n}$. Questions such as how $|\mathbf{n}\rangle$ evolves in time are equivalent to calculating the dynamical trajectory $\langle\widehat{\mathbf{S}(\mathbf{t})}\rangle$. In the present case $\delta T(u,v)\equiv\langle\delta\widehat{T}(u,v)\rangle$ plays a role exactly similar to this object.

• Two-point function $\neq$ "Propagator".

We have seen in Sec. 4 that $\delta T(u,v)$, or equivalently $\mathcal{T}(\mathbf{x},\mathbf{t})$, satisfies the 'black hole' differential equation (44). In an ordinary scalar field theory such a thing would imply

$$D_{\mathbf{x},\mathbf{t}}G_2(\mathbf{x},\mathbf{t}|\mathbf{x}',\mathbf{t}')=(1+e^{-2\mathbf{x}})\delta(\mathbf{x}-\mathbf{x}')\delta(\mathbf{t}-\mathbf{t}') \tag{59}$$

where

$$G_2(\mathbf{x},\mathbf{t}|\mathbf{x}',\mathbf{t}')\equiv\langle\psi_0|T(\widehat{\mathcal{T}}(\mathbf{x},\mathbf{t})\widehat{\mathcal{T}}(\mathbf{x}',\mathbf{t}')|\psi_0\rangle \tag{60}$$

The prefactor $1+\exp(-2\mathbf{x})$ is equal to $1/\sqrt{\det g}$ in $(\mathbf{x},\mathbf{t})$ coordinates, required to make the δ-function covariant. The operator $\widehat{\mathcal{T}}$ is defined as

$$\widehat{\mathcal{T}}(\mathbf{x},\mathbf{t}) \equiv |uv|^{-1/2}\delta\widehat{T}(u,v). \tag{61}$$

The symbol T in (60) denotes time-ordering in the time $\mathbf{t}$.

In our case, (59) is not true because of the non-standard commutation relations of the $\widehat{\mathcal{T}}$ field. It is easy to derive that

$$D_{\mathbf{x},\mathbf{t}}G_2(\mathbf{x},\mathbf{t}|\mathbf{x}',\mathbf{t}') = (1+e^{-2\mathbf{x}})\big(\delta(\mathbf{t}-\mathbf{t}')[\widehat{\mathcal{T}}(\mathbf{x},t),\widehat{\mathcal{T}}(\mathbf{x}',t)]+\partial_{\mathbf{t}}\delta(\mathbf{t}-\mathbf{t}')[\widehat{\mathcal{T}}(\mathbf{x},\mathbf{t}),\widehat{\mathcal{T}}(\mathbf{x}',\mathbf{t})]\big) \tag{62}$$

The commutation relation of the $\widehat{\mathcal{T}}$ fields can be derived from the definition (61) and the $\mathcal{U}(p,q,t)$ commutation relations. Neglecting corrections of order e^{-x} we have

$$\begin{aligned} [\widehat{\mathcal{T}}(\mathbf{x},\mathbf{t}),\partial_t\widehat{\mathcal{T}}(\mathbf{x}',\mathbf{t})] &\propto [\mathcal{D}'_+\mathcal{D}_+ + \mathcal{D}'_-\mathcal{D}_-]\partial_{\mathbf{x}}^2\delta(\mathbf{x}-\mathbf{x}') \\ [\widehat{\mathcal{T}}(\mathbf{x},\mathbf{t}),\widehat{\mathcal{T}}(\mathbf{x}',\mathbf{t})] &\propto [\mathcal{D}'_+\mathcal{D}_+ - \mathcal{D}'_-\mathcal{D}_-]\partial_{\mathbf{x}}\delta(\mathbf{x}-\mathbf{x}') \end{aligned} \tag{63}$$

where $\mathcal{D}_\pm$ are as defined in (37) (the primes refer to $\mathbf{x}'$). In the limit of extremely large $\mathbf{x}$, the operators $\mathcal{D}_\pm$ go as $(\partial_{\mathbf{x}})^{-1}$ and we recover canonical communication relations. This is essentially because in this limit the field $\widehat{\mathcal{T}}(\mathbf{x},\mathbf{t})$ becomes the same as the "tachyon" η of the standard $c=1$ matrix model (see Eq. (39)).

The non-identification of the two-point function with the propagator is basically related to the fact that $\delta\widehat{T}(u,v)$ or $\widehat{\mathcal{T}}(\mathbf{x},\mathbf{t})$ cannot create particle states because they do not commute at equal times.

- How does one address the issue of propagation then?

As we have stressed, $\widehat{T}(\mathbf{x},\mathbf{t})$ can be regarded as a Heisenberg operator in a two-dimensional field theory. If H is a functional of $\widehat{T}(\mathbf{x},0)$ and $\partial_{\mathbf{t}}\widehat{T}(\mathbf{x},0)$, then given $\langle\psi|\widehat{T}(\mathbf{x},0)|\psi\rangle$ and $\langle\psi|\partial_{\mathbf{t}}\widehat{T}(\mathbf{x},0)|\psi\rangle$, one can determine in principle $\langle\psi|\widehat{T}(\mathbf{x},\mathbf{t})|\psi\rangle$. In general it is a difficult question whether H is a functional of only $\widehat{T}(\mathbf{x},0), \partial_{\mathbf{t}}\widehat{T}(\mathbf{x},0)$. However even if it is not clear how much initial data is required to get a unique dynamical trajetory (unique answer for, let's say, $\langle\psi|\widehat{T}(\mathbf{x},\mathbf{t})|\psi\rangle$), the fermionic construction **does** list **all** dynamical trajectories. Also, we do know from the analysis of small fluctuations (for instance using the language of $\bar{\eta}_\pm$) that data worth two real functions (e.g. $\bar{\eta}_\pm(\tau,t=0)$ are enough to determine the future evolution for the fermionic state, except perhaps some discrete data corresponding to "discrete states". Incidentally, it is also clear from the one-to-one correspondence between $\bar{\eta}_\pm(\tau)$ and $\widehat{T}(\mathbf{x},0),\partial_\mathbf{t}\widehat{T}(\mathbf{x},0)$ that we can choose **any** kind of initial data on any given spacelike (or lightlike) surface $\mathbf{t}=\mathbf{t}_0$ (this list includes the white hole horizon) by simply choosing the appropriate fermionic state, or equivalently the appropriate values for $\bar{\eta}_\pm(\tau,\mathbf{t}_0)$.

7. Analytically continued Fermion Theory and The Euclidian Black Hole

In this section we would like to study the implication for the tachyon theory of the analytic continuation of the Fermion field theory discussed earlier in [22], namely⋆

$$t\to it,\ p\to -ip \tag{64}$$

In this analytic continuation, the "hyperbolic transform" becomes an "elliptic transform",

$$\phi(p,q,t)=\int\frac{dp'dq'}{[(p-p')^2+(q-q')^2]^{1/2}}\,\mathcal{U}(p,q,t), \tag{65}$$

Then, by the analytically continued equation of motion,

$$(\partial_t+p\partial_q-q\partial_p)\,\mathcal{U}(p,q,t)=0 \tag{66}$$

we can show that $\phi(p,q,t)$ is of the form

$$\phi(p,q,t)=T(u,\bar{u}) \tag{67}$$

⋆ This is equivalent to the analytic continuation [25] of the harmonic oscillator frequency $w\to iw$ in the fermion potential.

where

$$u = \frac{q - ip}{2} \exp(-it), \; \bar{u} = \frac{q + ip}{2} \exp(it) \tag{68}$$

As in Sec. 2, one can show that in the low energy approximation $T(u, \bar{u})$ satisfies the equation of motion of a massless field in the Euclidian black hole:

$$[4(u\bar{u} - \mu/2)\partial_u \partial_{\bar{u}} + 2(u\partial_u + \bar{u}\partial_{\bar{u}}) + 1]T(u, \bar{u}) = 0 + o(T^2) \tag{69}$$

This corresponds to a dilaton-metric background that describes the Euclidian black hole (the "cigar").

It is remarkable that the analytic continuation (64) of the matrix model defines the usual analytical continuation of the black hole physics! This makes it tempting to believe that the thermal Green's functions of the latter may have a direct significance in terms of the matrix model in the forbidden region.

8. Concluding Remarks

In this concluding section we would like to comment on some issues that need deeper understanding.

Firstly there is the question of the S-matrix. The definition of the S-matrix requires the specification of the 'in' and 'out' states. These states can be inferred by analyzing the two-point correlation function of a complete commuting (equal time) set of field operators. A natural set is the density operator $\psi^\dagger(x, t)\psi(x, t) \equiv \partial_x \varphi$, because perturbatively we know that φ creates 'in' and 'out' massless particle states. Once this is done we can evolve the 'in' states to the 'out' states in the standard fashion and define the S-matrix. This has been previously calculated [23,12,10,24] The black hole interpretation of this theory and in particular the non-linear differential equation (44) seem to strongly suggest that the kernel that propagates 'in' states to the 'out' states has a representation in the $T(u, v)$ theory. It would be very interesting to see this explicitly.

One of the limitations of the $c = 1$ matrix model as a model of black holes is that this black hole is eternal and one cannot envisage any process (*e.g* formation and evaporation) that involves changing the mass of the black hole. The simple reason for this is that the mass of the black hole is equal to the fermi level which cannot be changed if the number of fermions is held fixed. It is an interesting question whether there is a way of circumventing this difficulty within the context of string theory.

Note Added: While this paper was being written up, we received the article by S.R. Das, "Matrix models and nonperturbative string propagation in two-dimensional black hole backgrounds", Enrico Fermi Institute preprint EFI-93-16. In this paper the author has raised the issue of the identification of the "correct tachyon operator" which gives rise to "physical scattering processes" from a black hole. We would like to point out here that none of the "tachyon" operators defined in Eq. (5) of that paper can create particle states because they do not commute with each other at spacelike distances since they are built out of the density operator at different 'matrix model times'. This reason is analogous to the reason why the operator $\widehat{T}(u, v)$ defined in [13] and the present paper cannot, as has been explained in great detail in Sec. 6. For this reason, one cannot interpret the object defined in Eq. (12) of that paper as the wavefunction sought by the author. We would also like to point out here that given the definitions in Eq. (5) of that paper, it is not clear to us how time-ordered (in some definition of time in the u, v-space) correlators of these operators are related to those of the collective field theory. For this reason the l.h.s. of Eq. (12) of that paper cannot be obtained simply by inserting the result (19) in the r.h.s. of (12). This needs some understanding of the connection between (some definition of) time in the u, v-space and the matrix model time. Our definition of the tachyon operator has afforded us an immediate connection between these two "times". In fact, as explained in this paper, we can use the t-independence in the right hand side of our Eq. (11) so that the matrix model time and the "time" of the u, v-space can be identified with each other and thus relate the time-ordered correlators of $\delta\widehat{T}(u, v)$ and those of the matrix model.

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