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# $W_\infty$ coherent states and path-integral derivation of bosonization of non-relativistic fermions in one dimension

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## ABSTRACT

We complete the proof of bosonization of noninteracting nonrelativistic fermions in one space dimension by deriving the bosonized action using $W_\infty$ coherent states in the fermion path-integral. This action was earlier derived by us using the method of coadjoint orbits. We also discuss the classical limit of the bosonized theory and indicate the precise nature of the truncation of the full theory that leads to the collective field theory.

## 1. **Introduction and Summary**

Bosonization of non-relativistic fermions is an important problem with a long history. It was observed by Bloch [1] many years ago in his famous calculation of the stopping power of charged particles that the low energy excitations of a fermi gas can be described more suitably (within certain approximations) in terms of density fluctuations of the fermi gas ("sound waves") rather than in terms of individual excited particles and holes. He also observed that if the sound waves are quantized the quanta obey bose statistics under these approximations. In these treatments the fermions were considered to be basically free. Bohm *et al* [2] considered the effect of Coulomb interactions between the fermions and found a new kind of collective oscillation ("plasma oscillation") which had a characteristic frequency independent of the wave-number for low wave-numbers. The corresponding quanta (plasmons) again were found to be bosonic under some approximations. Tomonaga wrote a comprehensive article [3] in which he showed the formal equivalence of the low energy sector of a system of free non-relativistic fermions with that of a free relativistic boson in the case of one dimension under a set of well-defined approximations. The approximations basically consisted of (1) considering only those states of the fermi theory which are built from holes or excited particles (either left-moving or right-moving) which have wave-numbers between $3k_F/4$ and $5k_F/4$, where $k_F$ is the magnitude of the wave-number for the fermi surface, and (2) ignoring sound quanta which have wave-numbers greater than $k_F/2$. The sound quanta under these approximations were the same as quantized density waves in the fermi gas. Tomonaga was also able to incorporate the effect of interactions under this scheme, exploiting the beautiful observation that interactions between fermions, which typically require four-fermion terms in the hamiltonian, involve quadratic (density-density) terms in the bose hamiltonian, thus keeping the theory linear. The effect of interactions was thus incorporated in the Bose language in terms of a simple rediagonalization of the Bose field (a Bogoliubov transformation); the plasma oscillations for instance are trivially deduced in this way.

The question of a complete bosonization has subsequently been addressed by many authors. Except in the case of the Luttinger model [4] where the fermion has a linear dispersion relation, a complete bosonization of non-relativistic fermions has always been elusive, even in one dimension. The bosonization of relativistic fermions is similar in spirit to Tomonaga's approximate bosonization because the dispersion relation in the low energy band around the fermi surface is approximately linear. Works on bosonization of relativistic fermions include Lieb and Mattis [5], Luther and Peschel [6], Haldane [7] and, from the field theory point of view, Mandelstam [8]. A different approach to approximate bosonization of non-relativistic fermions was taken by Jevicki and Sakita [9] who exploited the equivalence of the fermion problem (in one dimension) to matrix models and used the method of collective variables.

A clue to the full solution of the bosonization problem can be obtained by looking at the semi-classical picture of a fermi gas, which describes the states of the fermi theory in terms of a fermi fluid of various shapes (with the same area as the ground-state configuration of the fluid, if we insist on fermion number conservation). This fermi fluid exists in the two-dimensional phase space of the single fermion. We see therefore that in this semi-classical approximation changes of the state of the fermi theory correspond to area-preserving shape changes of the fermi fluid. This is similar in spirit to Bloch's observation mentioned in the last paragraph except that we are now talking about fluctuations of the phase space density rather than that of the ordinary density. This classical picture has been elaborated in [10]. In [11] we extended this bosonization in terms of the phase space density to the quantum theory. In this paper we present a first principles proof of the bosonization starting from the fermion path integral using the techniques of coherent states. We will also present a brief discussion of the precise nature of the truncation of the bosonized theory that leads to collective field theory [9, 12].

## 2. Derivation of the Path-Integral using Coherent States

In the following we will consider the specific example of the fermion field theory which emerges in the double scaling limit of the $c = 1$ matrix model. The discussion is however easily generalized to other one-dimensional fermi systems.

It is well-known [13-16] that the $c = 1$ matrix model is described by the field theory of noninteracting nonrelativistic fermions in one space dimension, defined by the action

$$S = \int_{-\infty}^{+\infty} dt \int_{-\infty}^{+\infty} dx \; \psi^+(x,t)(i\partial_t - h_x)\psi(x,t) \tag{1}$$

where the single-particle hamiltonian $h$ is given by

$$\begin{aligned} h_x &= \frac{1}{2}\left(-\partial_x^2 + V(x)\right) \\ V(x) &= -x^2 + \frac{g_3}{\sqrt{N}}x^3 + \cdots \\ N &= \int_{-\infty}^{+\infty} dx \; \psi^+(x,t)\psi(x,t) \end{aligned} \tag{2}$$

In the above we have chosen the zeros of energy and $x$-axis such that the (quadratic) maximum of the potential occurs at $x = 0$ and that $V_{max} = V(0) = 0$. The continuum (double scaling) limit is obtained by letting $N \to \infty$ and the bare fermi energy $\epsilon_F \to 0$ while keeping the renormalized fermi energy (measured from the top of the potential) $\mu \sim N\epsilon_F$ fixed. The string coupling $g_{str}$ is then given by $g_{str} \sim \frac{1}{|\mu|}$. ($\mu$ is negative in our conventions.)

In a previous work [11] we presented a bosonization of (1) using the method of coadjoint orbits of $W_\infty$. A heuristic derivation of this action, starting from the fermion path integral, was also discussed previously in [17]. Here we will complete the proof of bosonization of (1) by deriving the boson action of ref. [11] using the method of coherent states of $W_\infty$ in the fermion path-integral.

The use of $W_\infty$ coherent states in the fermion path-integral is made possible by the observation [17] that the bosonized problem is analogous to that of a spin in a magnetic field. Let us recall this analogy here. The most general (elementary) boson operator is the fermion bilocal

$$\phi(x,y,t) \equiv \psi^{+}(x,t)\psi(y,t) \tag{3}$$

By virtue of the fermion anticommutation relation, $\phi$ satisfies the closed operator algebra

$$\begin{aligned}[\phi(x,y,t),\phi(x',y',t')] = \phi(x,y',t)\delta(x'-y)& \\ -\phi(x',y,t)\delta(x-y')&\end{aligned} \tag{4}$$

Also, using the fermion equation of motion one can derive an equation of motion for $\phi$. Introducing the compact "matrix" notation,

$$\langle x|\Phi(t)|y\rangle \equiv \phi(x,y,t), \tag{5}$$

this equation of motion can be written as

$$i\partial_t\Phi + [h,\Phi] = 0, \tag{6}$$

where the matrix elements of $h$ are given by $\langle x|h|y\rangle = h_x\delta(x-y)$. Equations (4) and (6) describe a $W_\infty$ 'spin' system, with $h$ acting like an external magnetic field.

The $W_\infty$ algebra [18,11] (4) can be written in a more familiar form in terms of the new operator

$$W(\alpha,\beta,t) \equiv \int dx\; e^{i\alpha x}\phi(x+\beta/2, x-\beta/2, t) \tag{7}$$

which satisfies the algebra

$$[W(\alpha,\beta,t),W(\alpha',\beta',t)] = 2\pi i\sin\frac{1}{2}(\alpha\beta'-\alpha'\beta)W(\alpha+\alpha',\beta+\beta',t) \tag{8}$$

In the single-particle Hilbert space the $W_\infty$ algebra is generated by all differential operators in one-dimension, i.e. by operator of the type $\hat{x}^n\hat{p}^m$, where $[\hat{x},\hat{p}] = i$. A convenient basis is given by

$$g(\alpha,\beta) = e^{i(\alpha\hat{x}-\beta\hat{p})} \tag{9}$$

which satisfies the algebra

$$[g(\alpha,\beta), g(\alpha',\beta')] = 2\pi i \sin\frac{1}{2}(\alpha\beta' - \alpha'\beta) g(\alpha+\alpha',\beta+\beta') \tag{10}$$

The $W(\alpha,\beta,t)$ realize this algebra in the fermion fock space.

The vacuum of the fermion theory is easily constructed by filling the fermi sea to a certain fermi level, which is determined by the number of fermions. Let us denote this vacuum state by $|F_0\rangle$. Coherent states are constructed by the action of the $W_\infty$ group elements on $|F_0\rangle$ [19]

$$|F_\theta\rangle = u(\theta)|F_0\rangle, \quad u \in \mathcal{G}W_\infty \tag{11}$$

$\mathcal{G}W_\infty$ is the Lie group corresponding to the $W_\infty$ algebra. In terms of the generators $W(\alpha,\beta)$, $u(\theta)$ can be parametrized as

$$u(\theta) = \exp[i \int d\alpha\, d\beta\; W(\alpha,\beta)\theta(\alpha,\beta)] \tag{12}$$

In general, for certain functions $\theta(\alpha,\beta)$, $u(\theta)$ would leave $|F_0\rangle$ invariant. This subset of $u(\theta)$'s clearly forms a subgroup $H$ of $\mathcal{G}W_\infty$. So, the distinct coherent states $|F_0\rangle$ in (11) are given by the elements of the coset $\mathcal{G}W_\infty/H$. This coset depends on the filling of the fermi sea. (To illustrate this point, consider the simpler case of a finite level system instead of $W_\infty$ (e.g. one may consider $u(N)$). Then, it is clear for example that in the extreme case of filling of all levels, the coset consists of a single point, the fermi vacuum. For partial filling there is clearly a nontrivial coset and it may be verified by the reader in simple cases that the coset depends on the filling.)

Having specified the $\mathcal{G}W_\infty$ coherent states defined on the fermi vacuum $|F_0\rangle$, let us explain their significance in bosonizing the fermion path integral. Firstly, let us note that we are interested in evaluating correlation functions involving only the bilocal boson operator $\phi(x,y,t)$ or some (fourier) transform of it. Because of this it is sufficient to consider intermediate states in the path-integral from the linear span of $\{\prod_i \phi(x_i,y_i)|F_0\rangle\} \equiv \mathcal{F}$. These states form a complete set and give a resolution of the identity. On the other hand, we may consider the linear span of the set of coherent states $\{|F_\theta\rangle\} \equiv \mathcal{E}$. Clearly, any element in the linear span of $\mathcal{F}$ is in the linear span of $\mathcal{E}$ and vice versa. Hence, we may consider a resolution of the identity in terms of the coherent states, even though they form an overcomplete set,

$$\int d\mu(\theta)|F_\theta\rangle\langle F_\theta| = 1 \tag{13}$$

The derivation of the path-integral now rests on the evaluation of the short time kernal

$$K_{t+\epsilon,\epsilon} = \langle F_{\theta(t+\epsilon)}|e^{i\epsilon H}|F_{\theta(t)}\rangle \tag{14}$$

where $H = \int dx\ \psi^+ h_x \psi$ is the hamiltonian of the fermion field theory and is an element of the $W_\infty$ algebra. One may equivalently write it in terms of the bilocal operator $\Phi$ as $H = tr(h\Phi)$, where we have introduced the notation $tr(AB) = \int dx\ dy\langle x|A|y\rangle\langle y|B|x\rangle$. Expanding (14) in $\epsilon$, we get

$$K_{t+\epsilon,\epsilon} = \langle F_{\theta(t+\epsilon)}|F_{\theta(t)}\rangle + i\epsilon\langle F_{\theta(t)}|H|F_{\theta(t)}\rangle + O(\epsilon^2) \tag{15}$$

To evaluate the first term in (15) we use (11) and expand in $\epsilon$. We get

$$\langle F_{\theta(t+\epsilon)}|F_{\theta(t)}\rangle = 1 + i\epsilon\langle F_0|u^+(\theta(t))i\partial_t u(\theta(t))|F_0\rangle + O(\epsilon^2) \tag{16}$$

The operator inside the expectation value in (16) is an element of the $W_\infty$ algebra and so it can be expanded in the basis provided by $W(\alpha,\beta)$,

$$u^+(\theta(t))i\partial_t u(\theta(t)) = \int d\alpha\ d\beta\ C_{\alpha\beta}(\theta(t),\partial_t\theta(t))W(\alpha,\beta) \tag{17}$$

Let us now define the single-particle analogue of (12),

$$g(\theta) = \exp[i\int d\alpha\ d\beta\ g(\alpha,\beta)\theta(\alpha,\beta)] \tag{18}$$

Because $W(\alpha,\beta)$ and $g(\alpha,\beta)$ satisfy an identical algebra it follows that the single-particle operator $g^+(\theta(t))i\partial_t g(\theta(t))$ has an expansion in $g(\alpha,\beta)$ with coefficients identical to $C_{\alpha\beta}$ in (17):

$$g^+(\theta(t))i\partial_t g(\theta(t)) = \int d\alpha\ d\beta\ C_{\alpha\beta}(\theta(t),\partial_t\theta(t))g(\alpha,\beta) \tag{19}$$

Now using $\langle x|g(\alpha,\beta)|y\rangle = \delta(x-y+\beta)e^{i\alpha(\frac{x+y}{2})}$ and (7) it can be easily shown that

$$\langle F_0|W(\alpha,\beta)|F_0\rangle = tr(g(\alpha,\beta)\phi_0) \tag{20}$$

where we have defined

$$\langle F_0|\Phi|F_0\rangle \equiv \phi_0 \tag{21}$$

From (17), (19) and (20) we then get

$$\langle F_0|u^+(\theta(t))i\partial_\epsilon u(\theta(t))|F_0\rangle = tr(\phi_0 g^+(\theta(t))i\partial_t g(\theta(t))) \tag{22}$$

Let us now evaluate the second term in (15). Since the single-particle hamiltonian $h$ is an element of the $W_\infty$ algebra we can expand it in the basis $g(\alpha,\beta)$ as $h \equiv \int d\alpha\ d\beta\ h_{\alpha\beta}g(\alpha,\beta)$. This implies that the fermion field theory hamiltonian

$H$ has the expansion $H = \int d\alpha \; d\beta \; h_{\alpha\beta} W(\alpha, \beta)$, where we have used (7) to set $tr(g(\alpha, \beta)\Phi)$ equal to $W(\alpha, \beta)$. We may now write the second term in (15) as

$$
\begin{aligned}
\langle F_{\theta(t)} | H | F_{\theta(t)} \rangle &= \int d\alpha \; d\beta \; h_{\alpha\beta} \langle F_0 | u^+(\theta(t)) W(\alpha, \beta) u(\theta(t)) | F_0 \rangle \\
&= \int d\alpha \; d\beta \; h_{\alpha\beta} \int d\alpha' \; d\beta' \; C_{\alpha\beta, \alpha'\beta'}(\theta(t)) \langle F_0 | W(\alpha', \beta') | F_0 \rangle \\
&= \int d\alpha \; d\beta \; h_{\alpha\beta} \int d\alpha' \; d\beta' \; C_{\alpha\beta, \alpha'\beta'}(\theta(t)) tr(g(\alpha', \beta')\phi_0)
\end{aligned}
$$

where in the second step above we have used that $u^+ W u$ is an element of $W_\infty$ algebra to reexpand it in $W(\alpha, \beta)$ and in the last step we have used (20). Denoting the single-particle representative of $u(\theta(t))$ by $g(\theta(t))$ as before, and using in the above an argument similar to the one given for the identity of coefficients in (17) and (19), we get

$$
\begin{aligned}
\langle F_{\theta(t)} | H | F_{\theta(t)} \rangle &= \int d\alpha \; d\beta \; h_{\alpha\beta} tr(g^+(\theta(t)) g(\alpha, \beta) g(\theta(t)) \phi_0) \\
&\equiv tr(g^+(\theta(t)) h g(\theta(t)) \phi_0)
\end{aligned}
\tag{23}
$$

Putting together (14)– (16), (22) and (23) we get for the short time kernel

$$
K_{t+\epsilon, \epsilon} = \exp[i\epsilon tr\{\phi_0(g^+ i\partial_t g + g^+ h g)\}]
$$

Hence the finite time kernal is

$$
K = \int \prod_\epsilon d\mu(g(\theta(t))) \exp[i \int dt \; tr\{\phi_0(g^+ i\partial_t g + g^+ h g)\}] \tag{24}
$$

where $d\mu(g(\theta(t)))$ is an appropriate measure over the coset $\mathcal{G}W_\infty/H$. The path-integral in (24) had earlier been heuristically argued for in [17].

Let us now make contact with the boson action and path-integral measure given in ref. [11], which we write below (see eqns. (44) $-$ (46) of this ref.):

$$\tilde{K} = \int \prod_t d\mu(\phi_t) \exp iS[\phi] \tag{25}$$

$$S[\phi] = i\int ds\ dt\ tr(\phi[\partial_t\phi, \partial_s\phi] + \int dt\ tr(\phi h) \tag{26}$$

$$d\mu(\phi) = \delta(tr\ \phi - N)\prod_{x,y}\delta(\phi_{xy}^2 - \phi_{xy})\prod_{x,y}d\phi_{xy}. \tag{27}$$

In the above, $\phi$ is a hermitian matrix with elements $\phi_{xy} = \langle x|\phi|y\rangle$. Also, $\phi(t,s)$ is an extension of $\phi(t)$ such that for $-\infty < t < \infty$ we have $-\infty < s \leq 0$ and the boundary conditions $\phi(t,s)\big|_{s=0} = \phi(t)$ and $\phi(t,s)\big|_{s=-\infty}$ = time-independent constant matrix. We note that if we set $\phi = g^+\phi_0 g$, $g \in \mathcal{G}W_\infty$ and $\phi_0$ fixed by (21), then the action (26) and the measure (27) reduce to that appearing in (24). The reason for the measures being identical is that if we fix $\phi_0$ then the integration in (25) is only over the coset obtained by modding out $\mathcal{G}W_\infty$ by that subgroup which commutes with $\phi_0$ (i.e. satisfies for all elements $v$, $v^+\phi_0 v = \phi_0$). But this is precisely the coset $\mathcal{G}W_\infty/H$ over which the integration in (24) is done. Let us prove this. Consider the definition (21) of $\phi_0$. Let $v$ be an element of $H$ and let $V$ be its representative in the fermion fock space. Then, using $V|F_0\rangle = |F_0\rangle$, for $V \in H$, we get

$$\begin{aligned}\phi_0 &\equiv \langle F_0|\Phi|F_0\rangle \\ &= \langle F_0|V^+\Phi V|F_0\rangle \\ &= v\phi_0 v^+\end{aligned}$$

The last step follows from arguments similar to those used in deriving (22) and (23). Thus the two cosets are the same.

For complete identity of the path-integrals in (24) and (25) we must, then, explain what restricts the integration over hermitian matrices in (25) to only over the $W_\infty$ "angles". To see this let us make the change of variables in (25) to the "angles" and eigenvalues of the hermitian matrix $\phi$:

$$\phi = gV^0 g^+, \quad g \in \mathcal{G}W_\infty, \quad V^0 \text{ diagonal} \tag{28}$$

Putting in (26) it is easy to show that

$$S[\phi] = S[g, V^0] = \int dt\ tr\{V^0(g^+ i\partial_t g + g^+ hg)\} \tag{29}$$

The measure changes to

$$d\mu(\phi) = \delta(\int d\nu V^0_\nu - N) \prod_\nu [dV^0_\nu \delta(V^{0^2}_\nu - V^0_\nu)] J(V^0) d\mu(g) \tag{30}$$

Here $d\mu(g)$ is the measure over the coset obtained by modding out the subgroup from $\mathcal{G}W_\infty$ that commutes with $V^0$ and the label $\nu$ displays the basis in which $\phi$ is diagonal (typically this is the energy basis). $J(V^0)$ is the Jacobian of change of variable and depends only on the eigenvalue matrix $V^0$. The important point to note is that the $V^0$ integration can be restricted to a single instant of time. This is because the $\delta$-function imposing $\phi^2 = \phi$ implies that the eigenvalues of $\phi$ are only 0 and 1. Because of the other $\delta$-function, there are always precisely $N$ number of ones (fermion number conservation). This means that the diagonal matrix $V^0$ at all times has 1 in $N$ number of places and the rest zeros. Time-dependence can come only in shuffling of the positions of these zeros and ones. Since that is achieved by a $\mathcal{G}W_\infty$ Weyl transformation, it is already included in the "angle" integration. Thus the integration over $V^0$ may be restricted to a single instant of time. Finally, let us discuss the last ingredient needed to completely fix the functional integral in (25), namely, a boundary condition on $\phi(t)$. This may be given in the form of its value at, say, infinite past. For example, a complete specification would be to set

$$\phi(t)\Big|_{t\to-\infty} = \phi_0 \tag{31}$$

where $\phi_0$, defined by (21), corresponds to fermi vacuum. Clearly, other specifications correspond to different fillings of energy levels i.e. to excited states. Hence the complete equivalence of (24) and (25)−(27) requires specifying a boundary condition on $\phi(t)$ in (25) (in particular, in this case this is (31)).

This completes the *proof* of the bosonization. This was earlier done by us using the method of co-adjoint orbits of $W_\infty$ [11]. In fact the $\delta$-functions in the measure in (27) and (30) specify the co-adjoint orbit of $W_\infty$ corresponding to the representation in terms of $N$, non-relativistic fermions.

3.**Path integral in terms of phase space fluid density**:

In this section we will reexpress the action (26) in terms of phase space density of fermions since it is in terms of this variable that the semiclassical picture of bosonization in terms of a fermi fluid emerges. In terms of the fermionic variables of the action (1), the phase space density opearator is defined as

$$\widehat{\mathcal{U}}(p,q,t) = \int dx\, \psi^\dagger(q - x/2, t) e^{-ipx} \psi(q + x/2, t) \tag{32}$$

We shall denote its expectation value in a $\mathcal{G}W_\infty$ coherent state as $u(p,q,t)$. We also introduce a fourier transform of $u(p,q,t)$:

$$\tilde{u}(\alpha,\beta,t) = \int \frac{dp}{2\pi}\frac{dq}{2\pi} e^{i(p\beta - q\alpha)} u(p,q,t) \tag{33}$$

$\tilde{u}(\alpha,\beta,t)$ is essentially the expectation value in a $\mathcal{G}W_\infty$ coherent state of the generator $W(\alpha,\beta,t)$ of $W_\infty$ algebra.

Consider now the expansion of $\phi(\hat{x},\hat{p})$, which enters the path integral (26), in terms of a basis for $W_\infty$ algebra in the single-particle Hilbert space:

$$\phi(\hat{x},\hat{p},t) = \int d\alpha d\beta g(\alpha,\beta)\tilde{u}(\alpha,\beta,t) \tag{34}$$

This expansion is clearly valid since $\phi$ may be thought of as the expectation value in a $\mathcal{G}W_\infty$ coherent state of the operator $\Phi$. In fact (33) and (34) define the Weyl ordering of $\phi(\hat{x},\hat{p})$ corresponding to the classical function $u(p,q,t)$.

In order to express the action (26) in terms of $u(p,q,t)$, we state a lemma due to Moyal:

Lemma (Moyal):

Given two classical functions $f_1(p,q)$ and $f_2(p,q)$ and their corresponding Weyl ordered operators $\hat{f}_1(\hat{x},\hat{p})$ and $\hat{f}_2(\hat{x},\hat{p})$, the classical function corresponding to the commutator $[\hat{f}_1,\hat{f}_2]$ is the fourier transform of the Moyal bracket,

$$
\begin{aligned}
&[\hat{f}_1,\hat{f}_2] = \int d\alpha d\beta \hat{g}(\alpha,\beta)\widetilde{\{f_1,f_2\}}_{MB}(\alpha,\beta) \\
&\{f_1,f_2\}_{MB} = \left[2\sin\frac{1}{2}(\partial_p\partial_{q'}-\partial_{p'}\partial_q)(f_1(p,q)f_2(p',q'))\right]_{p'=p,q'=q}
\end{aligned}
\tag{35}
$$

Note that in the second equation above the first term in the expansion of $\sin(\partial_p\partial_{q'}-\partial_{p'}\partial_q)$ is just the Poisson bracket. In (35) the trace identity $tr[A,B]=0$ is implicit. Restricting to such operators is equivalent to requiring $\int\int dp\, dq\{a(p,q),b(p,q)\}_{MB} = 0$, for the corresponding classical functions. This can be achieved by requiring the boundary condition that $a(p,q)$ and $b(p,q)$ are constant as $p,q\to\infty$.

Using (33) , (34) and (35) we can easily see that the action (26) becomes,

$$
\begin{aligned}
S[u] = &\int dsdt\int \frac{dpdq}{2\pi}u(p,q,t,s)\left\{\partial_s u(p,q,t,s),\partial_t u(p,q,t,s)\right\}_{MB} \\
&+\int dt\int\frac{dpdq}{2\pi}h(p,q)u(p,q,t)
\end{aligned}
\tag{36}
$$

and the measure (27) becomes

$$
d\mu(u) = \delta\left(\int\frac{dpdq}{2\pi}u(p,q,t)-N\right)\prod_{p,q}\left[\delta(C(p,q,t))du(p,q,t)\right].
$$

This implies the constraints,

$$
C(p,q,t)\equiv\left[\cos\frac{1}{2}(\partial_p\partial_{q'}-\partial_{p'}\partial_q)(u(p,q,t)u(p',q',t))\right]_{p'=p,q'=q}-u(p,q,t)=0 \tag{37}
$$

$$
\int\frac{dpdq}{2\pi}u(p,q,t)=N \tag{38}
$$

To derive the equation of motion from (36) we make a variation $\delta u(p,q,t) =$

$\{\epsilon, u\}_{MB}$ that preserves the constraints (37), (38). This gives

$$\frac{\partial}{\partial t} u(p,q,t) + \{h, u\}_{MB}(p,q,t) = 0.$$

This is the 'quantum' version of Liouville's equation. It is worth mentioning that if $h = \frac{1}{2}(p^2 - q^2)$, the Moyal bracket equals the Poisson bracket and we get

$$\partial_t u + (p\partial_q + q\partial_p)u = 0 \tag{39}$$

4.**Weak coupling (semiclassical) limit**:

We are now in a position to discuss the weak coupling limit. We shall restrict our discussion to the single-particle hamiltonian $h = \frac{1}{2}(p^2 - q^2)$ which is obtained in the double scaling limit of $c = 1$ matrix model. The discussion can be easily generalized to other cases. As has been explained in detail in [11], the semiclassical limit of this theory is obtained by expanding in a power series both the 'sine' in the Moyal Bracket (35) and the 'cosine' in the constraint (37) and retaining only the first term in both cases. In this limit the constraint simplifies to $u^2(p,q,t) = u(p,q,t)$, implying that the configurations that enter the path integral are characteristic functions corresponding to regions in phase space. The only dynamical part of a characteristic function is the boundary of the region and the dynamics consists of changes of the boundary preserving the area enclosed. One can indeed give a precise description of this using the classical limit of $W_\infty$ algebra which is the algebra of classical canonical transformations in two dimensions. We refer the reader for details to ref. [11].

Let us now see under what precise approximations does collective field theory emerge from the above semiclassical limit. Consider defining the moments of

$u(p,q,t)$,

$$\begin{aligned}
\rho(q,t) &\equiv \frac{\tilde{\rho}(q,t)}{2\pi} = \int_{-\infty}^{+\infty} \frac{dp}{2\pi} u(p,q,t) \\
\pi(q,t)\rho(q,t) &= \int_{-\infty}^{+\infty} \frac{dp}{2\pi} p u(p,q,t) \\
\pi_2(q,t)\rho(q,t) &= \int_{-\infty}^{+\infty} \frac{dp}{2\pi} p^2 u(p,q,t) \qquad \text{, etc.}
\end{aligned} \tag{40}$$

The equations of motion for the moments $\rho(q,t), \pi(q,t), \pi_2(q,t)$ etc. can be derived from the equation of motion (39),

$$\begin{aligned}
&\partial_t \tilde{\rho} + \partial_q(\tilde{\rho}\pi) = 0 \\
&\partial_t \pi = \partial_q \left( \frac{\pi^2}{2} + \frac{q^2}{2} - \pi_2 \right) + \frac{\partial_q \rho}{\rho}(\pi^2 - \pi_2) \\
&\text{etc.}
\end{aligned} \tag{41}$$

Furthermore, the constraint (37) implies certain relations among the moments. In the semiclassical limit the ground state is described by

$$u_0(p,q) = \theta\left(\mu - \frac{p^2 - q^2}{2}\right) = \theta\left[(\sqrt{q^2 + 2\mu} - p)(p + \sqrt{q^2 + 2\mu})\right]$$

where the curve $\frac{p^2 - q^2}{2} = \mu$ defines the fermi surface. Collective field theory is defined by parametrizing $u(p,q,t)$ near $u_0(p,q)$ by [20]

$$u(p,q,t) = \theta\left[(p_+(q,t) - p)(p - p_-(q,t))\right] \tag{42}$$

where $p_+(q,t)$ and $p_-(q,t)$ are such that $|p_\pm(q,t) - \sqrt{q^2 + 2\mu}|$ is small. In other words the collective field theory approximation to the semiclassical limit is described by those low energy excitations of the fermi fluid near the fermi surface which are described by a curve quadratic in $p$. This assumption, together with the semiclassical constraint $u^2 = u$, leads to specific relations between the moments $\rho, \pi, \pi_2$ etc. In particular, we have

$$\pi_2 = \pi^2 + \frac{1}{12}\tilde{\rho}^2 \tag{43}$$

Substituting this in (41) we obtain the equations of collective field theory:

$$\begin{aligned} &\partial_t \tilde{\rho} + \partial_q(\pi \tilde{\rho}) = 0 \\ &\partial_t \pi + \pi \partial_q \pi = -\partial_q \left( -\frac{q^2}{2} + \frac{\tilde{\rho}^2}{8} \right) \end{aligned} \tag{44}$$

It is easy to see that a generic boundary (*i.e.* not necessarily quadratic in $p$) violates the above equations. In fact, a generic boundary is not even described in terms of the first two moments. We have shown in [21] in an explicit example how this can result in physical quantities having different values from those calculated from collective field theory *even* in the semiclassical limit.

We thus see that if we restrict ourselves to those shapes of the fermi fluid that have a quadratic profile then the semiclassical approximation reduces to collective field theory.

To summarize, in this paper we have presented a first-principles proof of bosonization of noninteracting, nonrelativistic fermions in one space dimension and obtained the bosonic action. In the semiclassical limit the bosonized theory reduces to the dynamics of area-preserving fermi fluid profiles in the phase space. Restricting to quadratic profiles gives the collective field theory.

## REFERENCES


1. F. Bloch, ZS. Phys. 81 (1933) 363, Helv. Phys. Acta 7 (1934) 385.
2. See D. Bohm and D. Pines, Phys. Rev. 92 (1953) 609, and references therein.
3. S. Tomonaga, Prog. Theor. Phys. 5 (1950) 544.
4. J.M. Luttinger, J. Math. Phys. 4 (1963) 1154.
5. D. Mattis and E. Lieb, J. Math. Phys. 6 (1965) 375.
6. A. Luther and I. Peschel, Phys. Rev. B9 (1974) 2911.
7. F.D.M. Haldane, J. Phys. C 14 (1981) 2585.
8. S. Mandelstam, Phys. Rev. D11 (1985) 3026.
9. A. Jevicki and B. Sakita, Nucl. Phys. B165 (1980) 511.
10. A. Dhar, G. Mandal and S.R. Wadia, Int. J. Mod. Phys. A8 (1993) 325.
11. A. Dhar, G. Mandal and S.R. Wadia, Mod. Phys. Lett. A7 (1992) 3129.
12. S.R. Das and A. Jevicki, Mod. Phys. Lett. A5 (1990) 1639.
13. E. Brezin, C. Itzykson, G. Parisi and J.B. Zuber, Comm. Math. Phys. 59 (1978) 35; E. Brezin, V.A. Kazakov and Al.B. Zamolochikov, Nucl. Phys. B338 (1990) 673; D. Gross and N. Miljkovich, Nucl. Phys. B238 (1990) 217; P. Ginsparg and J. Zinn-Justin, Phys. Lett. B240 (1990) 333; G., Parisi, Europhys. Lett. 11 (1990) 595.
14. A.M. Sengupta and S.R. Wadia, Int. J. Mod. Phys. A6 (1991) 1961; G. Mandal, A.M. Sengupta and S.R. Wadia, Mod. Phys. Lett. A6 (1991) 1465.
15. D.J. Gross and I. Klebanov, Nucl. Phys. B352 (1990) 671.
16. G. Moore, Nucl. Phys. B368 (1992) 557.
17. S.R. Das, A. Dhar, G. Mandal and S.R. Wadia, Mod. Phys. Letts. A7 (1992) 71.

18. S.R. Das, A. Dhar, G. Mandal and S.R. Wadia, Int. J. Mod. Phys. A7 (1992) 5165.

19. A. Perelomov, Generalized Coherent States and Their Applications, Springer-Verlag.

20. J. Polchinski, Nucl. Phys. B346 (1990) 253.

21. A. Dhar, G. Mandal and S.R. Wadia, Int. J. Mod. Phys. A8 (1993) 3811.