TIFR-TH-94/09

String Field Theory of Two Dimensional QCD: A Realization of W_∞ Algebra

Avinash Dhar, Gautam Mandal and Spenta R. Wadia
Tata Institute of Fundamental Research
Homi Bhabha Road, Bombay 400 005, INDIA

Abstract

We consider the formulation of two dimensional QCD in terms of gauge invariant bilocal operators (string field) which satisfy a W_∞ algebra. In analogy with our work on the $c = 1$ string field theory we derive an action and associated constraints for the bilocal field using the method of coadjoint orbits. The $1/N$ perturbation theory around a classical solution that corresponds to the filled Dirac sea leads to the 'tHooft equation for meson fluctuations. It is shown that the spectrum of mesons, which are the higher string modes, transform as a representation of the wedge subalgebra $W_{\infty+} \otimes W_{\infty-}$. We briefly discuss the baryon as a stringy solitonic configuration and its characterization in terms of W_∞ algebra.

Introduction

Gauge theories and string theory have a long standing symbiotic relationship. It was the string model that inspired the $1/N$ expansion of non-abelian gauge theories [1]. In this work 'tHooft discovered the connection between Feynman diagrams of matrix valued field theories and two dimensional Riemann surfaces. Subsequently stringlike equations were derived for Wilson loops in the large N limit [4].

More recently matrix models have been used to define and exactly solve low dimensional string theories [5, 6]. In particular there has been much work in studying the $c = 1$ string field theory. This theory has an exact representation in terms of non-relativistic fermions [7] and is characterized by the infinite dimensional Lie algebra W_∞ [8]. An exact boson representation of this theory can be constructed in terms of the bilocal field $\Phi(x, y) = \psi(x)\psi^+(y)$ where $\psi(x)$ is the non-relativistic fermion field. This was done by constructing the co-adjoint orbit of W_∞ (characterized by a quadratic constraint, and the fermion number) and using Kirillov's method to write down the action[9]. A rigorous basis for this construction was given by the method of W_∞ coherent states [10].

In this paper we apply the above method to two dimensional QCD [3]. Using gauge invariance we can write the theory entirely in terms of the gauge invariant bilocal operator $M_{\alpha\beta,ij}(x, y) = (1/N)\sum_{a=1}^{N} \psi^a_{i\alpha}(x)\psi^{+a}_{j\beta}(y)$, which satisfies the infinite dimensional algebra $W^f_\infty = W_\infty \otimes U(n_f)$. This is the string field of two dimensional QCD (see Sec. 1 for notation). Gauge invariance leads to a quadratic constraint on the string field. This constraint and the baryon number characterize the co-adjoint orbit of W^f_∞ and one can once again construct the action using Kirillov's method (Sec. 2). In this formulation, as expected, N (the number of colours) plays the role of $1/\hbar$. Bilocal operators in two-dimensional QCD have been previously considered in [11] and [12]. More recently there have been several papers [13, 14, 15] on the connection between two-dimensional Yang-Mills theory and string theories.

In this reformulation of two-dimensional QCD as an open string field theory we get several new insights into both string theory and gauge theories. In a sense many promises which were not realised in $c = 1$ string field theory are fulfilled here[1]. For one thing, we find that there is an infinite tower of physical, stable mesons (in the large N limit) which form the excitations of the string field theory and transform into one another as a representation of the group $W_{\infty +} \otimes W_{\infty -} \otimes U(n_f)$ (Sec. 3). Recall that in contrast in $c = 1$ string field theory, which also formed a representation of W_∞, there was only one physical particle, the tachyon, along with discrete moduli. It appears that in the present model the dynamical gauge field which gives rise to effective interactions between fermions provides a far richer structure to the theory.

The second remark concerns solitonic configurations [11] in this model. In the large N limit the string field $M_{\alpha\beta,ij}(x, y)$ becomes classical and is related very simply to the self-consistent Hartee-Fock potential in which the fermions move. Nontrivial classical solutions of the equation of motion for the string field correspond to different self-consistent Hartree-Fock potentials which arise out of populating quasi-particle wavefunctions above the Dirac sea. These correspond to baryons in the theory (Sec. 4). In the language of the group W_∞^f these different classical solutions correspond to different *representations* of the group. This brings forth the essential point about a *stringy soliton* which, in contrast with solitons in local field theories which are characterized by discrete or finite dimensional groups, must be characterized by infinite dimensional groups generated by loop operators. We also mention that the amplitude for creation of a soliton-antisoliton pair in this theory goes as [11] $e^{-N} \sim e^{-1/\sqrt{\hbar}}$ which is indicative of stringy nonperturbative effect [18].

1. Hamiltonian Formulation of 2-dimensional QCD

In this section we review the Hamiltonian formulation of two-dimensional QCD (QCD_2) in the light-cone gauge to set up our notation (the discussion here is similar to the one in [12]). We consider the gauge group $SU(N)$ and the corresponding gauge fields $A_\mu^{ab}(\mu = 0, 1)$ which are traceless hermitian matrices. The fermions are as usual denoted by $\psi_{i\alpha}^a(x)$, where $a = 1, \ldots, N$ is the colour index, $i = 1, \ldots, n_f$ is the flavour index and $\alpha = 1, 2$ is the

[1] We should point out that the QCD strings considered here are *a priori* not the same as continuum strings represented by an integral over world-sheet metrics on smooth surfaces. We make some remarks about possible connections in Sec. 4.

Dirac index.

The Lagrangian of QCD_2 is

$$\mathcal{L} = -\frac{1}{4}\ trF_{\mu\nu}F^{\mu\nu} + \bar{\psi}i\not{D}\psi + m\bar{\psi}\psi \tag{1}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i(g/\sqrt{N})[A_\mu, A_\nu]$, $\not{D} = \gamma^\mu \left(\partial_\mu + i(g/\sqrt{N})\ A_\mu\right)$ and γ^μ are the gamma matrices in two dimensions. We will find it convenient to use light-cone coordinates[2] in which the Lagrangian gets rewritten as:

$$\begin{aligned}\mathcal{L} = {} & 2\mathrm{tr}F_{+-}^2 + 2\psi_-^\dagger(i\partial_+ - (g/\sqrt{N})A_+)\psi_- \\ & + 2\psi_+^\dagger(i\partial_- - (g/\sqrt{N})A_-)\psi_+ + m(\psi_+^\dagger\psi_- + \psi_-^\dagger\psi_+)) \\ F_{+-} = {} & \partial_+A_- - \partial_-A_+ + i(g/\sqrt{N})[A_+, A_-]\end{aligned} \tag{2}$$

In the following we shall use the light-cone gauge[3]

$$A_+ = \frac{1}{2}A^- \equiv \frac{1}{2}(A_0 + A_1) = 0 \tag{3}$$

Note that in this gauge we still have symmetry under local gauge transformations which depend only on the variable x^-. Let us regard x^+ as the "time" and x^- as "space". To find the hamiltonian we need to go through the usual procedure of finding out the canonical momenta and the constraints. The constraint in the gauge sector (Gauss law, eq. (5)) comes in the usual fashion by imposing the requirement that the hamiltonian evolution preserves the gauge (3). In the fermionic sector the use of the light-cone coordinates leads to an additional constraint coming from the fact that there are no x^+-derivatives on ψ_+ in (2); hence the equation of motion for ψ_+, (6) turns out actually to be a constraint on the theory. Summarizing, we have the hamiltonian

$$\begin{aligned}& H \equiv P_+ = \int dx^-[(1/8)\mathrm{tr}\ E^2 - (m/2)(\psi_-^\dagger(x)\psi_+(x) + \psi_+^\dagger(x)\psi_-(x))] \\ & [A_-(x^-, x^+), E(y^-, x^+)] = i\delta(x^- - y^-)\end{aligned} \tag{4}$$

along with the gauss law constraint

$$G_-^{ab} \equiv \partial_- E^{ab} + i\frac{g}{\sqrt{N}}\,[A_-, E]^{ab} - 2\frac{g}{\sqrt{N}}\left(\psi_-^{\dagger b}\psi_-^a - \frac{1}{N}\ \delta^{ab}\psi_-^{\dagger c}\psi_-^c\right) = 0 \tag{5}$$

[2]Our notation for light-cone variables is summarized in the Appendix.

[3]The reason we prefer to work in this gauge is that it is Lorentz-covariant.

and the fermionic constraints

$$\begin{aligned} &2(i\partial_- - (g/\sqrt{N})A_-)\psi_+ + m\psi_- = 0 \\ &2\psi_+^\dagger(i\overleftarrow{\partial_-} - (g/\sqrt{N})A_-) + m\psi_-^\dagger = 0 \end{aligned} \tag{6}$$

The fermionic constraint just means that so far as calculation of correlation functions involving the ψ_-'s is concerned we can forget about this constraint; in correlation functions involving the ψ_+'s we can use the constraint to express them in terms of ψ_-'s. We shall henceforth assume that this has been done.

In the quantum theory the Hamiltonian and the constraint act on the Schrödinger wave functional. In particular $G_-^{ab}|\Psi\rangle = 0$ expresses the gauge invariance of $|\Psi\rangle$ under local gauge transformations which are functions of x^-. Equivalently $|\Psi\rangle$ is only a functional of the orbit $A_-^{[\Omega]} = \Omega A_- \Omega^+ + i\Omega\partial_-\Omega^+, \psi_-^{[\Omega]} = \Omega\psi_-$ which can be parametrized by the choice of the gauge $A_- = 0$. In this case we can solve for the electric field E from (5) and substitute in (4) to get

$$\begin{aligned} H = (g^2/4N) \int dx^- dy^- \Big(&\psi_{i-}^a(x^-)\psi_{j-}^{\dagger a}(y^-)|x^- - y^-| \; \psi_{j-}^b(y^-) \; \psi_{i-}^{\dagger b}(x^-) \\ &- \tfrac{1}{N} \; \psi_{i-}^a(x^-)\psi_{i-}^{\dagger a}(x^-)|x^- - y^-| \; \psi_{j-}^b(y^-)\psi_{j-}^{\dagger b}(y^-) \\ &- (im^2/4) \int dx^- dy^- \mathrm{sgn}(x^- - y^-)\psi_{i-}^a(x^-)\psi_{i-}^{\dagger a}(y^-) \Big) \end{aligned} \tag{7}$$

where m is a cutoff dependent constant that includes the bare quark mass.

Let us now introduce the gauge invariant string field

$$M_{\alpha\beta,ij}(x^-, y^-; x^+) = \frac{1}{N} \sum_{a=1}^{N} \psi_{i\alpha}^a(x^-, x^+)\psi_{j\beta}^{\dagger a}(y^-, x^+) \tag{8}$$

Here α, β are spinor indices taking values $+, -$. Note that $M_{\alpha\beta,ij}(x^-, y^-; x^+)$ is gauge invariant in $A_- = 0$ gauge. Its manifestly gauge invariant form is

$$M_{\alpha\beta,ij}(x^-, y^-; x^+) = \psi_{i\alpha}^a(x^-, x^+) \left(e^{i\int_{x_-}^{y^-} A_-(z^-, x^+)dz^-} \right)_{ab} \psi_{i\alpha}^{\dagger b}(y^-, x^+) \tag{9}$$

The hamiltonian (7) can be expressed entirely in terms of the string field

$M_{\alpha\beta,ij}(x^-, y^-; x^+)$,

$$H = N \int dx^- dy^- \Big[(g^2/4) M_{--,ij}(x^-, y^-) |x^- - y^-| M_{--,ij}(x^-, y^-) \\ - (g^2/4N) M_{--,ii}(x^-, x^-) |x^- - y^-| M_{--,jj}(y^-, y^-) \\ - i(m^2/4)\, \mathrm{sgn}(x^- - y^-) M_{--,ii}(x^-, y^-) \Big] \tag{10}$$

The fermionic constraint (6) reads in this language (after we have put $A_- = 0$):

$$\begin{aligned} 2i\partial_{y^-} M_{++,ij}(x^-, y^-) + m M_{+-,ij}(x^-, y^-) = 0 \\ 2i\partial_{x^-} M_{+-,ij}(x^-, y^-) + m M_{--,ij}(x^-, y^-) = 0 \end{aligned} \tag{11}$$

Note that N factors out of the hamiltonian and the subleading term which arises because we are dealing with $SU(N)$ *rather than* $U(N)$ drops out in the large N limit.

Fixing the Global Gauge Invariance

The choice of the gauge $A_- = 0$ still allows for global $SU(N)$ colour rotations, and in principle the wavefunction can carry a representation of this global group or its subgroup depending on the dynamical situation. In 2-dimensional QCD if we restrict to finite energy states then the linear Coulomb potential will ensure that coloured asymptotic states do not appear. We summarize this as a condition that the wave function is invariant under global $SU(N)$ symmetry

$$\begin{aligned} &[E^{ab}(x^- = +\infty) - E^{ab}(x^- = -\infty)]|\Psi\rangle \\ &= 2\int dx^- \left(\psi^{\dagger b}_{i-}(x^-)\psi^a_{i-}(x^-) - \tfrac{1}{N}\delta^{ab}\psi^{\dagger c}_{i-}(x^-)\psi^c_{i-}(x^-) \right) |\Psi\rangle = 0 \end{aligned} \tag{12}$$

The baryon number operator on the light-cone is defined by

$$\begin{aligned} B^{(-)}_- &= \tfrac{1}{N}\sum_{a=1}^N \int dx^- \bar{\psi}^{+a}_{i\alpha}(x)(\gamma^0\gamma^+)^{\alpha\beta}\psi^a_{i\beta}(x) \\ &= \tfrac{1}{N}\sum_{a=1}^N \int dx^- \bar{\psi}^{+a}_{i-}(x)\psi^a_{i-}(x) \end{aligned} \tag{13}$$

The appearance of the $\gamma^0\gamma^+$ is due to the fact that our baryon number is an integral on the light cone rather than on a space-like surface. We use the superscript $(-)$ to explicitly remind ourselves of this fact. The subscript says that fermions of only the $-$ chirality appear in the integral. One can show that this operator commutes with the hamiltonian in (10) and so is conserved.

Operator algebra of the string field

We can easily calculate the closed algebra satisfied by the string field $M_{\alpha\beta,ij}(x,y)$ using the fermion anti-commutation relation $\{\psi_-(x^-,x^+),\psi_-^\dagger(y^-,x^+)\} = \frac{1}{2}\delta(x^- - y^-)$[4]. Since in our case only the M_{--} are dynamical we present the algebra satisfied by them[5]:

$$\begin{aligned}\Big[M_{i_1j_1}(x_1^-,y_1^-;x^+), &M_{i_2j_2}(x_2^-,y_2^-;x^+)\Big] \\ &= \tfrac{1}{2N}\ \delta_{j_1i_2}\delta(y_1^- - x_2^-)M_{i_1j_2}(x_1^-,y_2^-;x^+) \\ &- \tfrac{1}{2N}\ \delta_{j_2i_1}\delta(y_2^- - x_1^-)\ M_{i_2j_1}(x_2^-,y_1^-;x^+)\end{aligned} \tag{14}$$

We recognize (14) as the infinite dim algebra $W_\infty^f \equiv W_\infty \otimes U(n)$. We recall that in the $c=1$ string field theory we had constructed the bilocal operator $\Phi(x,y) = \psi(x)\psi^+(y)$ in terms of non-relativistic fermions which satisfied the W_∞ algebra [9]:

$$[\Phi(x,y),\Phi(x',y')] = \delta(x'-y)\ \Phi(x,y') - \delta(y'-x)\Phi(y,x') \tag{15}$$

There the analogue of the baryon number was the fermion number $N = \int dx\ \psi^\dagger\psi$. We mention here that $W_\infty \otimes U(n)$ algebras have previously appeared in a different context in [19].

2. The co-adjoint orbit of $W_\infty^f = W_\infty \otimes U(n_f)$ and the classical action

We now show that the requirement of global colour invariance (12) implies a quadratic constraint on $M_{ij}(x,y)$. Using fermion anti-commutation relation

[4]The factor of $\frac{1}{2}$ is due to the factor of 2 present in the kinetic term for ψ_- in the action (2).

[5]Henceforth we shall drop the subscripts $_{--}$ from M since those will be the only ones we will be interested in (unless otherwise stated).

(at equal x^+) we can prove the identity,

$$\int dz^- M_{ik}(x^-, z^-) M^{kj}(z^-, y^-)$$
$$= \tfrac{1}{2} M_{ij}(x^-, y^-) + (1/N^2) \psi^a_{i-}(x^-) \psi^{\dagger b}_{j-}(y^-) \int dz^- \textstyle\sum_k \psi^{a+}_{k-}(z^-) \psi^b_{k-}(z^-) \tag{16}$$

Now using the requirement of global colour invariance (12) it is easy to see that the following quadratic constraint is satisfied in the physical space of states,

$$M^2 = \frac{1}{N^2} M \int dz^- \sum_k \psi^{\dagger c}_{k-}(z^-) \psi^c_{k-}(z^-) + \frac{1}{2} M \tag{17}$$

where we have used a compact matrix notation, $M_{ij}(x^-, y^-)$ being the (ix^-, jy^-) element of a matrix M. This may be rewritten as

$$M^2 = M(\frac{}{2}1 + \frac{1}{N} \mathrm{Tr}(1 - M)) \tag{18}$$

where the 'Tr' refers to a combined flavour and 'space' (in the sense of x^-) summation. The quantity $\mathrm{Tr}(1 - M))$ is simply the baryon number operator (13). Since it is conserved we may simply replace it by its eigenvalue. We shall denote this by B, that is

$$\frac{1}{N} \sum_{a=1}^{N} \int dx \psi^{\dagger a}_{i-}(x) \psi^a_{i-}(x) |\Psi\rangle = \mathrm{Tr}(1 - M)|\Psi\rangle = B|\Psi\rangle \tag{19}$$

The constraint now becomes

$$M^2|\Psi\rangle = \left(\frac{1}{2} + \frac{B}{N}\right) M|\Psi\rangle \tag{20}$$

We recall that the analogues of (20) and (19) have appeared previously in the discussion of $c = 1$ field theory [9]. The classical analogues of these constraints specify the co-adjoint orbit of $W^f_\infty = W_\infty \otimes U(n)$, in the limit of large N.

Let us now construct the classical action in the large N limit. In view of the constraints on M the procedure is identical to that we employed in $c = 1$ field theory. The classical phase space is constructed in terms of the expectation value $\langle M\rangle$ of the operator M in the coherent states of the algebra W^f_∞. Analogous to the operator constraint (**??**) on physical states the classical phase space satisfies a constraint [10] in terms of the expectation value $\langle M\rangle$; it reads $\langle M\rangle^2 = \langle M\rangle$. Henceforth for simplicity of notation we will denote $\langle M\rangle$ by M itself.

The construction of the action now proceeds in exactly the same way as was discussed for the $c = 1$ model [9, 10]. Here we have the co-adjoint orbit of the group W^f_∞ described by the constraints

$$\begin{aligned} M^2 &= M \\ B &= \mathrm{Tr}(1 - M) \end{aligned} \tag{21}$$

The path-integral can be constructed by regarding the four-fermi interaction perturbatively. Like in [10] the information about the *filling of the Fermi sea* is contained in the choice of the particular co-adjoint orbit. The action is given by

$$S = N\Big[2i\int_\Sigma ds dx^+ \,\mathrm{Tr}(M\,[\partial_+ M, \partial_s M]) - \int_{-\infty}^{\infty} dx^+ \mathrm{Tr}(\frac{im^2}{4} SM + \frac{g^2}{4} M\widetilde{M})\Big] \tag{22}$$

where

$$\widetilde{M}_{ij}(x^-, y^-) \equiv |x^- - y^-| M_{ij}(x^-, y^-),\ S(x^-, y^-) = \mathrm{sgn}(x^- - y^-) \tag{23}$$

The region Σ of (s, x^+) integration is the lower half plane $x^+ \in (-\infty, +\infty)$, $s \in (-\infty, 0]$ and we have the boundary condition that $M(x^+, s = 0) = M(x^+)$ and $M(x^+, s) \to$ a constant (x^+-independent) matrix as $s \to -\infty$.

The equation of motion can be derived by considering infinitesimal motion on the co-adjoint orbit which preserves the constraints (21). Such a motion is given by $\delta M = i[\epsilon, M]$, which in expanded form reads

$$\begin{aligned} \delta M_{ij}(x^-, y^-) &= i\int dx_1^- \sum_{i_1} \epsilon_{i,i_1}(x^-, x_1^-) M_{i_1,j}(x_1^-, y^-) \\ &- i\int dx_1^- \sum_{i_1} M_{i,i_1}(x^-, x_1^-)\epsilon_{i_1,j}(x_1^-, y^-) \end{aligned} \tag{24}$$

By making such a variation in the action we can derive the following equation of motion

$$i\partial_+ M = (im^2/8)[M, S] + (g^2/4)\left[\widetilde{M}, M\right] \tag{25}$$

3. Classical Solution and Fluctuations: Representation of $W_{\infty+} \otimes W_{\infty-} \otimes U(n_f)$

For simplicity we first consider the case of one flavour. It is trivial to construct static (x^+-independent) solutions which are translationally invariant, in other words solutions of the form

$$M(x^-, y^-; x^+) = (1/2\pi) \int dk_- dk'_- M(k_-, k'_-; x^+) \exp(-ik_- x^- + ik'_- y^-)$$
$$M(k_-, k'_-; x^+) = f(k_-)\delta(k_- - k'_-) \tag{26}$$

The equation of motion (25) is automatically satisfied. To satisfy the quadratic constraint (21) we need

$$f(k_-)^2 = f(k_-) \tag{27}$$

The other (Baryon number) constraint basically defines the filling of the Dirac sea.

Vacuum Solution

Note that the solution $f(k_-) = \theta(k_- - k_F)$ satisfies (27). Indeed this precisely corresponds to the filled Dirac sea of free relativistic fermions in which the Fermi level k_F is determined by the Baryon number: $\int_{-\infty}^{k_F} dk_- = B$. Clearly B is defined only through a regularization. Let us choose a regularization such that the choice $k_F = 0$ corresponds to the regularized Baryon number $= 0$.

We therefore have a vacuum solution in the $B = 0$ sector given by

$$M_0(k_-, k'_-) = \theta(k_-)\delta(k_- - k'_-) \tag{28}$$

We emphasize that though this solution corresponds to the filled Dirac sea for a system of non-ineracting fermions this is indeed a solution in our case of the fully interacting theory (the interaction term in the equation of motion trivially vanishes for the condition (26)).

Fluctuations: Meson Spectrum as Representation of $W_{\infty+} \otimes W_{\infty-}$

Let us parametrize fluctuations around (28) as

$$M = e^{iW/\sqrt{N}} M_0 e^{-iW/\sqrt{N}} = M_0 + (i/\sqrt{N})[W, M_0] - (1/2N)[W, [W, M_0]] + o(N^{-3/2}) \tag{29}$$

where $e^{iW/\sqrt{N}}$ are arbitrary W_∞^f-group elements. Note that this parametrization automatically ensures that the constraints are satisfied. We put in the factor of $\sqrt{N}$ explicitly to characterize fluctuations.

In the momentum-space notation (29) reads[6]

$$\begin{aligned} M(k,k';x^+) &= \theta(k)\delta(k-k') + (i/\sqrt{N})(\theta(k')-\theta(k))W(k,k';x^+) \\ &\quad - \textstyle\int_{k''}(1/2N)(\theta(k)+\theta(k')-2\theta(k''))W(k,k'';x^+)W(k'',k';x^+) + o(N^{-3/2}) \end{aligned} \tag{30}$$

We see that upto linear order in (30) only the $W(k,k')$'s with mixed signs of k,k' appear. This leads to a set of very interesting observations.

Let us introduce the following notation (the discussion in the rest of the subsection is for a given x^+ which we do not write explicitly)

$$\begin{aligned} W^{++}(k,k') &= W(k,k') \\ W^{+-}(k,k') &= W(k,-k') \\ W^{-+}(k,k') &= W(-k,k') \\ W^{--}(k,k') &= W(-k,-k') \\ \text{where } \ k,k' &> 0 \end{aligned} \tag{31}$$

and similarly for the $M(k,k')$'s[7].

We note that the quantum algebra of the $M(k,k')$'s

$$[M(k,k'),M(l,l')] = (1/2N)(\,-\delta(k-l')M(l,k') + \delta(k'-l)M(k,l')) \tag{32}$$

leads to the following structure of commutation relation between the $M^{\pm\pm}$'s

$$[M^{++}(k,k'),M^{++}(l,l')] = (1/2N)(-\delta(k-l')M^{++}(l,k')+\delta(k'-l)M^{++}(k,l')) \tag{33}$$

$$[M^{--}(k,k'),M^{--}(l,l')] = (1/2N)(-\delta(k-l')M^{--}(l,k')+\delta(k'-l)M^{--}(k,l')) \tag{34}$$

$$[M^{++},M^{--}] = 0 \tag{35}$$

$$\begin{aligned} [M^{++}(k,k'),M^{+-}(l,l')] &= (1/2N)\delta(k'-l)M^{+-}(k,l') \\ [M^{++}(k,k'),M^{-+}(l,l')] &= -(1/2N)\delta(k-l')M^{-+}(l,k') \end{aligned} \tag{36}$$

$$\begin{aligned} [M^{--}(k,k'),M^{+-}(l,l')] &= -(1/2N)\delta(k-l')M^{+-}(l,k') \\ [M^{++}(k,k'),M^{-+}(l,l')] &= (1/2N)\delta(k'-l)M^{-+}(k,l') \end{aligned} \tag{37}$$

[6]In the following we shall skip the subscript $_-$ from the momentum labels for simplicity of typing.

[7]The $\pm$ signs here denote the sign of momenta and are not to be confused with chirality.

$$
\begin{array}{l}
[M^{+-}(k,k'),M^{-+}(l,l')] = (1/2N)(\,-\,\delta(k-l')M^{--}(l,k') + \delta(k'-l)M^{++}(k,l')) \\
{}[M^{+-},M^{+-}] = 0 = [M^{-+},M^{-+}]
\end{array}
\tag{38}
$$

The commutation relations (33),(34),(35) simply state that there are two commuting sub-algebras of W^f_∞ with identical structure constants generated respectively by the M^{++}'s and M^{--}'s. Let's call these algebras $W_{\infty+}$ and $W_{\infty-}$. Let us imagine for the moment that in (38) the generators M^{++} and M^{--} were replaced by their vacuum expectation values. In that case the commutation relations (36),(37), (38) would simply mean that the M^{+-} and M^{-+} are canonical conjugates which form representation of the direct product of the 'wedge' algebras $W_{\infty+}\otimes W_{\infty+}$. We shall argue below that when we use the expansion (30) in terms of the fluctuations $W(k,k')$ in effect such a replacement of operators by their vacuum expectation values can be done in the large N limit.

Let us rewrite (30) in the $\pm$ notation. We get, keeping upto the first non-trivial order in fluctuations

$$
\begin{array}{l}
M^{++}(k,k') = \delta(k-k') - (1/N)\int_0^\infty dk'' W^{+-}(k,k'')W^{-+}(k'',k') + o(N^{-3/2}) \\
M^{+-}(k,k') = -(i/\sqrt{N})W^{+-}(k,k') + o(N^{-1}) \\
M^{-+}(k,k') = (i/\sqrt{N})W^{-+}(k,k') + o(N^{-1}) \\
M^{--}(k,k') = (1/N)\int_0^\infty dk'' W^{-+}(k,k'')W^{+-}(k'',k)
\end{array}
\tag{39}
$$

We find that the commutation relations for the M's imply that

$$
\begin{array}{l}
[W^{+-}(k,k'),W^{-+}(l,l')] = (1/2)\delta(k-l')\delta(k'-l) + o(N^{-1/2}) \\
{}[W^{+-},W^{+-}] = [W^{-+},W^{-+}] = 0 + o(N^{-1/2})
\end{array}
\tag{40}
$$

and that

$$
\begin{array}{l}
[\overline{M}^{++}(k,k'),W^{+-}(l,l')] = (1/2)\delta(k'-l)W^{+-}(k,l') + o(N^{-1/2}) \\
{}[\overline{M}^{++}(k,k'),W^{-+}(l,l')] = -(1/2)\delta(k-l')W^{-+}(l,k') + o(N^{-1/2})
\end{array}
\tag{41}
$$

Here $\overline{M}^{++} \equiv NM^{++}$. Similar stataments are true for M^{--}. Note that it is the $\overline{M}$'s that in the large N-limit have an N-independent structure constant. Thus, in the limit $N=\infty$ we get a Heisenberg algebra of the W^{+-},W^{-+} which forms a module of $W_{\infty+}\otimes W_{\infty-}$.

In the following we shall analyze the equations of motion for the fluctuations W^{+-},W^{-+} to identify an infinite tower of mesons. Thus the Heisenberg algebra can essentially be identifed with the algebra of canonical commutation relations of those fields.

Spectrum

The action (22) based on W_∞^f coadjoint orbit can be rewritten in terms of the fluctuations W as

$$\begin{aligned}
S = -2 \textstyle\int dx^+ \int_0^\infty dk \int_0^\infty dk' \Big[& W^{+-}(k',k;x^+) i\partial_+ W^{-+}(k,k';x^+) \\
& - (m^2/4)(1/k + 1/k') W^{+-}(k,k';x^+) W^{-+}(k',k;x^+) \\
& + (g^2/8\pi) W^{+-}(k',k;x^+) \textstyle\int_k^{-k'} (dp/p^2)(W^{-+}(k-p,k'+p;x^+) - W^{-+}(k,k';x^+)) \\
& + (g^2/8\pi) W^{-+}(k',k;x^+) \textstyle\int_k^{-k'} (dp/p^2)(W^{+-}(k'+p,k-p;x^+) - W^{+-}(k',k;x^+)) \Big] \\
& + 0(N^{-1/2})
\end{aligned} \tag{42}$$

This gives rise to the equation of motion

$$\begin{aligned}
& i\partial_+ W^{-+}(k,k';x^+) = (m^2/4)(1/k + 1/k') W^{-+}(k,k';x^+) \\
& - (g^2/4\pi) \textstyle\int_k^{-k'} dp (1/p^2) [W^{-+}(k-p,k'+p;x^+) - W^{-+}(k,k';x^+)] + o(N^{-1/2})
\end{aligned} \tag{43}$$

where k and k' are both ≥ 0. Let us now define

$$\begin{aligned}
r_- &= k + k' \\
x &= k'/r_-
\end{aligned} \tag{44}$$

and also change the variable from p to y in the integral on the r.h.s. of (43), where y is defined by

$$y = \frac{p + k'}{r_-} \tag{45}$$

Clearly both x and y range over the interval $[0,1]$. Finally, writing

$$W^{-+}(k,k';x^+) = \int \frac{dr^+}{2\pi} \phi(x; r_-, r_+) e^{ir_+ x^+} \tag{46}$$

we get

$$4r_- r_+ \phi(x) = m^2 (\frac{1}{x} + \frac{1}{1-x}) - \frac{g^2}{\pi} \int_0^1 \frac{dy}{(y-x)^2} (\phi(y) - \phi(x)) \tag{47}$$

which is the same as the 'tHooft equation. This implies the existence of an infinite tower of relativistic particle spectrum [2].

We therefore conclude that *in the large N limit the infinite tower of mesons in two-dimensional QCD (*the higher string modes*) form a representation of the algebra $W_{\infty +} \otimes W_{\infty -}$.*

We remark that although the above algebra is represented in the space of fluctuations it does not commute with the hamiltonian. One interesting consequence of this, as seen by considering the action of the $\overline{M}^{++}$ or $\overline{M}^{--}$'s on the $\phi(x; r_-, r_+)$'s, is that *different mass levels transform into one another under the action of this algebra.* This is indeed as one would expect in a string field theory.

Generalization to many flavours

The above discussion has a simple generalization to $n_f > 1$. The vacuum solution is

$$M_{0,ij}(k,k') = \delta_{ij}\theta(k)\delta(k-k') \tag{48}$$

The 'wedge' subalgebra's are generated by $M^{++}_{ij}(k,k')$ and $M^{--}_{ij}(k,k')$. The 'dynamical variables' are $W^{+-}_{ij}(k,k')$ and $W^{-+}_{ij}(k,k')$. There is a 'tHooft equation for each hermitian generator of the $U(n_f)$ group. Thus we have a tower of meson corresponding to each generator. In other words, the small fluctuations around the vacuum (48) *form a repressentation of $W_{\infty +} \otimes W_{\infty -} \otimes U(n_f)$.*

4. Concluding Remarks

We make here a few observations about solitonic configurations in the theory. In the large N-limit the string field $M_{ij}(x^-, y^-)$ becomes classical which essentially means that the quartic fermion interaction in (7) can be considered equivalent to a quadratic potential term:

$$\begin{aligned} H = (g^2/4) \int dx^- dy^- \Big(& \psi^a_{i-}(x^-) \psi^{a+}_{j-}(y^-) V_{ij}(x^-, y^-) \\ & - \tfrac{1}{N}\, \psi^a_{i-}(x^-) \psi^{+a}_{i-}(x^-) V_{jj}(y^-, y^-) \Big) \\ V_{ij}(x^-, y^-) \equiv \widetilde{M}^c_{ij}(x^-, y^-) & = |x^- - y^-| M^c_{ij}(x^-, y^-) \end{aligned} \tag{49}$$

Here the superscript ^{c} on M denotes 'classical' solution. In the previous section we have discussed in detail a particular classical solution in the translationally invariant sector. It is clear that every classical solution of the equation of motion (29) for M would correspond to some 'Hartree-Fock' potential V. The fermions moving in these potentials are like quasi-particles of Landau theory of fermi liquids. We can try to construct solutions of the many-body problem where we place N quarks in a single localized quasi-particle state above a filled sea of quasi-particle levels. Such classical solutions would correspond to baryons. The quantitative details of this picture will shortly appear elsewhere [16]. The point that we want to make here is that we can again go through arguments similar to those used in the previous section to construct a representation of W_∞^f around each of these classical solutions $M^c_{ij}(x^-, y^-)$. Thus we find that *solitons in 'string theories' are characterized by infinite-dimensional groups like* W_∞^f *which are generated by non-local loop operators and are therefore capable of describing an infinite-parameter space of deformation of these solitons.*

We end with a few remarks comparing QCD_2 strings with continuum strings. As we remarked in the introduction, there are many differences *a priori*. In the matrix model formulation of continuum strings, the double scaling limit ensures that a smooth continuum limit of the triangulated surfaces can be taken. No such facility exists in the QCD_2 case in any obvious sense, the problem being compounded by the fact that in two dimensions the triple-gluon vertex, which is crucial in forming a surface, vanishes in an axial gauge. One possible reconciliation could be that in two dimensions the metric fluctuations on the world sheet for an open string theory essentially boil down to fluctuations of the length of the string. Presumably the meson spectrum reflects just this vibrating degree of freedom [17].

Acknowledgements: We would like to thank R. Shankar for participation in the early stages of this work.

Appendix

We summarize our notation for the light-cone variables:

$$
\begin{array}{l}
\text{coordinates}: x^{\pm} \equiv x^0 \pm x^1 \\
\text{metric}: ds^2 = (dx^0)^2 - (dx^1)^2 = dx^+ dx^-;\ g_{+-} = g_{-+} = \frac{1}{2} \\
\text{gauge fields}: A_{\pm} = (\partial x^0/\partial x^{\pm})A_0 + (\partial x^1/\partial x^{\pm})A_1 = \frac{1}{2}(A_0 \pm A_1) \\
\text{momenta}: k_{\pm} = \frac{1}{2}(k_0 \pm k_1) \\
\text{mass-shell}: 4k_+k_- = m^2 \\
\text{gamma-matrices}: \gamma^{\pm} = \gamma^0 \pm \gamma^1
\end{array}
\tag{50}
$$

We use the following explicit representation for gamma-matrices:

$$
\gamma^0 = \sigma^1,\ \gamma^1 = -i\sigma^2,\ \gamma^5 \equiv \gamma^0\gamma^1 = \sigma^3 \tag{51}
$$

$$
\gamma^+ = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix};\ \gamma^- = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} \tag{52}
$$

We also use the following notation for the Dirac fermion ψ:

$$
\psi = \begin{pmatrix} \psi_- \\ \psi_+ \end{pmatrix} \tag{53}
$$

References

[1] G. 'tHooft, Nucl. Phys. B72 (1974) 461.

[2] G. 'tHooft, Nucl. Phys. B75 (1974) 461.

[3] *The Large N Expansion in Quantum Field Theory and Statistical Physics: From Spin Systems to 2-Dimensional Gravity*, Eds. E. Brezin and S.R. Wadia (World Scientific 1993).

[4] Y.M. Makeenko and A.A. Migdal, Phys. Lett. B88 (1979) 135; S.R. Wadia, Phys. Rev. D24 (1981) 970; B. Sakita, Phys. Rev. D21 (1980) 1067; for other references please see [3].

[5] F. David, Nucl. Phys. B257 (1985) 45; V.A. Kazakov, Phys. Lett. B150 (1985) 282; J. Ambjron, B. Durhuus and J. Frohlich, Nucl. Phys. B257 (1985) 433.

[6] E. Brezin and V.A. Kazakov, Phys. Lett. B236 (1990) 144; D.J. Gross and A.A. Migdal, Nucl. Phys. B340 (1990) 333; M.R. Douglas and S.H. Shenker, Nucl. Phys. B335 (1990) 635.

[7] E. Brezin, C. Itzykson, G. Parisi and J.B. Zuber, Comm. Math. Phys. 59 (1978) 35; A. Sengupta and S.R. Wadia, Int. J. Mod. Phys. A6 (1991) 1961; G. Mandal, A. Sengupta and S.R. Wadia, Mod. Phys. Lett. A6 (1991) 1465; D.J. Gross and I. Klebanov, Nucl Phys. B352 (1990) 671.

[8] S. Das, A. Dhar, G. Mandal and S.R. Wadia, Int. J. Mod. Phys. A7 (1992) 5165; J. Avan and A. Jevicki, Phys. Lett. 226B (1991) 35 and 272B (1991) 17; G. Moore and N. Seiberg, Int. J. Mod. Phys. A7 (1992) 2601.

[9] A. Dhar, G. Mandal and S.R. Wadia, Mod. Phys. Lett. A7 (1992) 3129.

[10] A. Dhar, G. Mandal and S.R. Wadia, Mod. Phys. Lett. A8 (1993) 3557.

[11] E. Witten, Nucl. Phys. B160 (1979) 57.

[12] K. Kikkawa, Phys. Lett. B92 (1980) 315.

[13] D.J. Gross, Nucl. Phys. B400 (1993) 161; D.J. Gross and W. Taylor, Nucl. Phys. B400 (1993) 181 and Nucl. Phys. B403 (1993) 395.

[14] M.R. Douglas and V.I. Kazakov, Phys. Lett. 319B (1993) 219; M.R. Douglas, preprint RU-93-13, NSF-ITP-93-39; J. Minahan and A. Polychronakos, Phys. Lett. B312 (1993) 155 (see also S.R. Wadia, Phys. Lett. B93 (1980) 403).

[15] J. Minahan, Phys. Rev. D47 (1993) 3430.

[16] A. Dhar, P. Lakdawala, G. Mandal and S.R. Wadia (in preparation).

[17] See W.A. Bardeen, I. Bars, A.J. Hanson and R.D. Peccei, Phys. Rev D13 (1975) 2364, where the 'tHooft spectrum is derived starting from an open Nambu-Goto type string whose essential dynamics is at the end-points.

[18] S.H. Shenker, in Proceedings of the Cargese Workshop in *Random Surfaces, Quantum Gravity and Strings 1990*, Eds. O. Alvarez, E. Marinari and P. Windey.

[19] I. Bakas and E. Kiritsis, Mod. Phys. Lett. A5 (1990) 2039; Prog. Theor. Phys. (Proc. Suppl.) 102 (1990) 15.