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STRING BETA FUNCTION EQUATIONS FROM THE $c = 1$ MATRIX MODEL

Avinash Dhar[1],[2]
Theory Division, CERN, CH-1211, Geneva 23, SWITZERLAND

Gautam Mandal[3] and Spenta R. Wadia[4]
Tata Institute of Fundamental Research,
Homi Bhabha Road, Bombay 400 005, INDIA

Abstract

We derive the σ-model tachyon β-function equation of 2-dimensional string theory, in the background of flat space and linear dilaton, working entirely within the $c = 1$ matrix model. The tachyon β-function equation is satisfied by a nonlocal and nonlinear combination of the (massless) scalar field of the matrix model. We discuss the possibility of describing the 'discrete states' as well as other possible gravitational and higher tensor backgrounds of 2-dimensional string theory within the $c = 1$ matrix model. We also comment on the realization of the W-infinity symmetry of the matrix model in the string theory. The present work reinforces the viewpoint that a nonlocal (and nonlinear) transform is required to extract the space-time physics of 2-dimensional string theory from the $c = 1$ matrix model.

[1]On leave from Theoretical Physics Group, Tata Institute of Fundamental Research, Homi Bhabha Road, Bombay 400 005, India.
[2]e-mail: adhar@surya11.cern.ch
[3]e-mail: mandal@theory.tifr.res.in
[4]e-mail: wadia@theory.tifr.res.in

0 Introduction

Two-dimensional string theory bears, in many ways, the same relationship to its higher dimensional counterparts as do low-dimensional, exactly solvable field theory models to their less tractable counterparts in 4 dimensions. It is the 'simplest' string theory one can imagine; it has a massless propagating mode and vestiges of the massive string modes in the form of 'discrete states' [1, 2]; it possesses a very large symmetry group W_∞ [3-7]; it has a black hole solution to the classical beta-function equations [8]; and most remarkably it also has a nonperturbative formulation in terms of an integrable theory of noninteracting nonrelativistic fermions of the $c = 1$ matrix model [9-11,24]. The last feature makes it an ideal testing ground for discussing issues of strong coupling string theory and issues related to black hole evaporation and gravitational collapse.

To address these questions of string theory in the matrix model, however, we first need a detailed mapping between the two. This is a nontrivial problem since the natural 'space-time' parameters of the matrix model do not have the interpretation of the space-time variables of the string theory. This arises as a consequence of the fact [2, 16] that one has to use the so-called 'leg-pole' transformation to the asymptotic wave-functions to arrive at the S-matrix of the string tachyon from that of the matrix model scalar. The 'leg-pole' prescription in position space corresponds to a nonlocal transform of the asymptotic wave-functions. Using this nonlocal transform it has been shown in [15] that the S-matrix of the matrix model actually reproduces that of the string theory, which comes from a target space action in the background of flat space and linear dilaton [5].

In this work we will provide further evidence to reinforce the viewpoint that a nonlocal and, as we shall see, nonlinear mapping is required to extract the space-time physics of the string theory from the matrix model. More specifically, we will show that a specific nonlocal and nonlinear combination of the scalar field of the matrix model satisfies the σ-model tachyon β-function equation of 2-dimensional string theory [17, 18] in the background of flat space and linear dilaton. Moreover, the quantization inherited from the matrix model implies a canonical quantization for this particular combination, thus completing its identification with the tachyon of string theory (in flat background). We will discuss the possibility of describing the discrete states as well as other backgrounds, including those of higher tensor modes of the string, within the existing formulation of the matrix model, by employing similar transforms. Finally, we will show that the W-infinity symmetry of the matrix model has a nonlocal realization in the string theory, a result that was earlier derived using the methods of BRST cohomology in liouville string theory [7]. Throughout this paper we will be working in the framework of perturbation theory. For instance we will not attempt to generalize our mapping to situations in which the fermi fluid goes over to the other side of the potential barrier.

[5] The general idea of non-local transforms has previously appeared in the context of mapping of matrix model to string theory in other backgrounds, *e.g.* in [12-14].

1 Review of Some Aspects of the Matrix Model

Here we will briefly summarize some aspects of the $c = 1$ matrix model that will be relevant to the discussion in the following sections. For a more extended review which covers this, see [19].

In the double-scaling limit the $c = 1$ matrix model is mapped to a model of noninteracting nonrelativistic fermions in an inverted harmonic oscillator potential [9] in one space dimension. The single-particle hamiltonian for this model is

$$h(p, q) = \frac{1}{2}\left(p^2 - q^2\right) \tag{1.1}$$

where (p, q) label the single-particle phase space of the fermions. There is a convenient (for many calculations in the matrix model) field theoretic description for the double-scaled model in terms of free nonrelativistic fermions [11]. The fermion field, which we denote by $\psi(q, t)$, satisfies the equation of motion

$$i\partial_t \psi(q, t) = -\frac{1}{2}\left(\partial_q^2 + q^2\right)\psi(q, t) \tag{1.2}$$

and its conjugate $\psi^\dagger(q, t)$ satisfies the complex conjugate of eqn. (1.2). The ground state of this model is the fermi vacuum obtained by filling up to the energy level μ (< 0). The semiclassical limit is obtained as $|\mu| \to \infty$ and in this limit the fermi surface is described by the hyperbola

$$\frac{1}{2}\left(p^2 - q^2\right) = \mu = -|\mu| \; . \tag{1.3}$$

The basic building block for the present work will be the phase space density of fermions, which we denote by $u(p, q, t)$. In terms of the fermi field $\psi(q, t)$ it is defined as

$$u(p, q, t) \equiv \int d\lambda e^{-ip\lambda} \psi^\dagger\left(q - \frac{\lambda}{2}, t\right)\psi\left(q + \frac{\lambda}{2}, t\right) \tag{1.4}$$

and it satisfies the equation of motion

$$\left(\partial_t + p\partial_q + q\partial_p\right) u(p, q, t) = 0 \; , \tag{1.5}$$

which follows from eqn. (1.2), or by directly using the hamiltonian

$$H = \int \frac{dpdq}{2\pi} h(p, q) u(p, q, t) \tag{1.6}$$

and the equal-time commutation relation for the phase space density $u(p, q, t)$, which follows from its definition, eqn. (1.4), in terms of underlying fermions:

$$\begin{aligned}[u(p, q, t), u(p', q', t)] &= -4\int \frac{dp''dq''}{2\pi} u(p'', q'', t) \\ &\quad [\exp 2i\{p(q' - q'') + p'(q'' - q) + p''(q - q')\} - c.c.]\end{aligned} \tag{1.7}$$

Equation (1.7) is also a version of the large symmetry algebra, the W-infinity algebra [20], which is a symmetry of the matrix model [3-7]. The more standard version of the generators of this symmetry algebra is the following:

$$W_{mn} = e^{-(m-n)t} \int \frac{dpdq}{2\pi} (-p - q)^m (p - q)^n u(p, q, t), \tag{1.8}$$

where $m, n \geq 0$. One can easily check, using eqn. (1.5), that W_{mn} are conserved. They satisfy the classical algebra

$$\{W_{mn}, W_{m'n'}\} = 2(m'n - mn')W_{m+m'-1,n+n'-1}. \tag{1.9}$$

The quantum version of this is more complicated, but can be computed using eqn. (1.7).

The above phase space density formalism was first introduced in the present context in [22] and using this variable a bosonization of the model was carried out [22, 23]. A crucial ingredient in that bosonization is a quadratic constraint satisfied by $u(p, q, t)$ [22]. In the semi-classical limit this quantum constraint reduces to the simpler equation

$$u^2(p, q, t) = u(p, q, t) \ . \tag{1.10}$$

Moreover, one also has the constraint of fixed fermion number, which implies that fluctuations above the fermi surface, eqn. (1.3), satisfy

$$\int \frac{dpdq}{2\pi} \delta u(p, q, t) = 0, \qquad \delta u(p, q, t) \equiv u(p, q, t) - u_0(p, q) \tag{1.11}$$

where $u_0(p, q)$ describes the fermi vacuum. In this way we recover the Thomas-Fermi limit of an incompressible fermi fluid. The dynamics of the fluctuations $\delta u(p, q, t)$, which satisfies eqn. (1.11) and another constraint because of eqn. (1.10), resides only in the boundary of the fermi fluid (in the semi-classical limit that we are considering here) [24, 25].

In the following sections we will use the general framework developed above. It turns out that we never need to solve the constraints, eqns. (1.10) and (1.11), and introduce an explicit parametrization of the fluid boundary fluctuations $\delta u(p, q, t)$. In fact, we will be able to develop the entire formalism treating 'p' and 'q' more or less symmetrically. This is important since, as we shall see, in this way we are able to avoid spurious singularities, such as the fold singularity [21], which may be dynamically generated in an otherwise perfectly nonsingular initial parametrization of the fluid boundary fluctuation. We are able to avoid these singularities because extracting space-time physics from the matrix model requires a nonlocal transform. In a sense, therefore, this is a bonus of the necessity of a mapping from the matrix model to string theory.

Although our general discussion will never need an explicit parametrization of $\delta u(p, q, t)$, it will, nevertheless, be useful at times to express things in a familiar parametrization of the fluctuations. For this reason we now summarize, in the rest of this section, some relevant aspects of the 'quadratic profile' [24, 25] or 'collective field' [10] parametrization of the fluctuations $\delta u(p, q, t)$.

In the semiclassical limit the fermi vacuum is described by the density

$$u_0(p, q) = \theta(P_+^0(q) - p)\theta(p - P_-^0(q)), \tag{1.12}$$

where

$$P_\pm^0(q) \equiv \pm P_0(q) = \pm\sqrt{q^2 + 2\mu} \tag{1.13}$$

satisfy the equation that describes the fermi surface hyperbola, eqn. (1.3). The quadratic profile or collective field approximation corresponds to a description of small ripples on the fermi surface by a density of the form

$$u(p, q, t) = \theta(P_+(q, t) - p)\theta(p - P_-(q, t)). \tag{1.14}$$

Substituting this in eqn. (1.5), we get the equations of motion of $P_\pm$:

$$\partial_t P_\pm(q,t) = \frac{1}{2}\partial_q(q^2 - P_\pm^2(q,t)). \tag{1.15}$$

This equation is clearly solved by the fermi vacuum, eqns. (1.12) and (1.13). Fluctuations around this ground state,

$$P_\pm(q,t) - P_\pm^0(q) \equiv \eta_\pm(q,t), \tag{1.16}$$

satisfy the equations of motion

$$\partial_t \eta_\pm(q,t) = \mp\partial_q \left[P_0(q)\,\eta_\pm(q,t) \pm \frac{1}{2}\eta_\pm^2(q,t)\right]. \tag{1.17}$$

If the fluctuations are small so that they never cross the asymptotes $p = \pm q$ of the hyperbola defined by eqn. (1.3), then one can rewrite eqns. (1.17) in a form that exhibits the presence of a massless particle. This is done by introducing the time-of-flight variable τ, defined by

$$q \equiv -|2\mu|^{\frac{1}{2}}\cosh\tau, \quad P_0(q) = |2\mu|^{\frac{1}{2}}\sinh\tau, \quad 0 \le \tau < \infty, \tag{1.18}$$

where we have assumed that the fluctuations are confined to the left half of the hyperbola ($q < 0$). We now introduce the new variables $\bar\eta_\pm(\tau,t)$ defined by

$$\bar\eta_\pm(\tau,t) \equiv P_0(q(\tau))\eta_\pm(q,t). \tag{1.19}$$

They satisfy the equations of motion

$$(\partial_t \mp \partial_\tau)\bar\eta_\pm(\tau,t) = \partial_\tau\left[\bar\eta_\pm^2(\tau,t)/2P_0^2(q(\tau))\right]. \tag{1.20}$$

Furthermore, one can also deduce the commutation relations

$$\begin{aligned} [\bar\eta_\pm(\tau,t), \bar\eta_\pm(\tau',t)] &= \pm 2\pi i \partial_\tau \delta(\tau-\tau'), \\ [\bar\eta_+(\tau,t), \bar\eta_-(\tau',t)] &= 0, \end{aligned} \tag{1.21}$$

since we know the hamiltonian for the fluctuations

$$\begin{aligned} H_{\text{fluc.}} &= \int \frac{dp\;dq}{2\pi} h(p,q)\delta u(p,q,t) \\ &= \frac{1}{4\pi}\int_0^\infty d\tau \left[\bar\eta_+^2(\tau,t) + \bar\eta_-^2(\tau,t) + \frac{1}{3P_0^2(q(\tau))}\left(\bar\eta_+^3(\tau,t) - \bar\eta_-^3(\tau,t)\right)\right] \end{aligned} \tag{1.22}$$

Finally, there is the 'fixed area' (i.e. fixed fermion number) constraint eqn. (1.11), which reads now

$$\int_0^\infty d\tau\,(\bar\eta_+(\tau,t) - \bar\eta_-(\tau,t)) = 0. \tag{1.23}$$

Equations (1.19)-(1.23) define the massless scalar field of the $c = 1$ matrix model. One can now obtain the scattering amplitudes and discuss various other properties of this model. We refer to a recent review [21] for details and original references.

2 The Leg-Pole Connection with String Theory

It has been known for some time now that the tree-level scattering amplitudes for the matrix model scalar above do not exactly coincide with the tree-level scattering amplitudes for the tachyon in 2-dimensional string theory [16]. The difference can be understood in terms of a wave-function renormalization and is a simple momentum-dependent phase factor for real momenta. In coordinate space this renormalization factor relates the Hilbert space of the matrix model to that of the string theory by a nonlocal transform of the states [15]. Denoting the tachyon field of 2-dimensional string theory by $\mathcal{T}(x,t)$ (x,t are space-time labels), this relationship can be expressed as

$$\mathcal{T}_{\rm in}(t+x) = \int_{-\infty}^{+\infty} d\tau \; f\left(|\frac{\mu}{2}|^{\frac{1}{2}} e^{\tau-(t+x)} \right) \bar{\eta}_{+\,\rm in}(\tau), \tag{2.1}$$

$$\mathcal{T}_{\rm out}(t-x) = -\int_{-\infty}^{+\infty} d\tau \; f\left(|\frac{\mu}{2}|^{\frac{1}{2}} e^{-\tau+(t-x)} \right) \bar{\eta}_{-\,\rm out}(\tau), \tag{2.2}$$

where the 'in' and 'out' refer, as usual, to the asymptotic fields obtained in the limits $t \to -\infty$ and $t \to +\infty$ respectively. In both cases x is taken to be large and positive, keeping respectively $(t+x)$ and $(t-x)$ fixed. The function $f(\alpha)$ is given by

$$f(\alpha) \equiv \frac{1}{2\sqrt{\pi}} J_0 \left(2 \left(\frac{2}{\pi} \right)^{\frac{1}{8}} \sqrt{\alpha} \right), \quad \alpha \geq 0, \tag{2.3}$$

where J_0 is the standard Bessel function of order zero [26].

It was recently emphasized in [15] that such a nonlocal transform is necessary for extracting the space-time physics of 2-dimensional string theory from the matrix model. In the following we shall present evidence which not only reinforces this but, among other things, also seems to suggest that, in fact, it may be possible to set up a detailed operator correspondence between the matrix model and string theory by means of a nonlocal and nonlinear transform.

3 The Transform – General Considerations

Equations (2.1) and (2.2) constitute both the starting point and our motivation for the following enquiry. We wish to explore the possibility of a more detailed mapping from the matrix model to string theory, which would also be valid away from the asymptotic space-time region of eqs. (2.1) and (2.2). Moreover, we wish to develop the general framework without using any specific parametrization of the fermi fluid boundary fluctuations $\delta u(p,q,t)$, unlike the collective field $\bar{\eta}_{\pm}$ that appear in eqs. (2.1) and (2.2). As mentioned earlier, this would enable us to avoid spurious singularities that might dynamically develop in any specific (initially nonsingular) parametrization. We will see an example of how this works later. Let us for the moment go on with the building of the general framework.

Now, the most general form for a matrix model $\leftrightarrow$ string theory mapping in a perturbative framework must have the expansion

$$\mathcal{T}(x,t) \;\; \equiv \;\; \int dp \; dq \; G_1(x;p,q) \delta u(p,q,t)$$

$$+\frac{1}{2}\int dp\; dq \int dp'\; dq'\; G_2(x;p,q;p',q')\delta u(p,q,t)\delta u(p',q',t)$$
$$+\cdots \tag{3.1}$$

The dots represent higher order terms in the fluctuation δu. We shall assume that the fluctuations have support only around the left half of the fermi surface hyperbola, eqn. (1.3), i.e. only for $q \leq -|2\mu|^{\frac{1}{2}}$. This is consistent with the perturbative framework within which we are working. Also, we have chosen the kernels $G_1(x;p,q)$, $G_2(x;p,q;p',q')$, etc., to be time-independent because in the present work we would like to recover the results of perturbative string theory in flat space and linear dilaton background. In this background time evolution in string theory is controlled by the same hamiltonian as in the matrix model, namely $H_{\text{fluc.}}$ as given in eqn. (1.22).

We would now like to ask whether there is a choice of the kernels $G_1(x;p,q)$, $G_2(x;p,q;p',q')$, etc., which will allow us to identify the scalar field $\mathcal{T}(x,t)$, defined by eqn. (3.1), with the tachyon of 2-dimensional string theory. One criterion for this identification is that $\mathcal{T}(x,t)$ should satisfy the σ-model tachyon β-function equation in flat space and linear dilaton background [17, 15]:

$$(\partial_t^2 - \partial_x^2)\mathcal{T}(x,t) = -4g_s^{-1}(2x+c)e^{-2x}\mathcal{T}(x,t) + 2\sqrt{2}e^{-2x}\mathcal{T}^2(x,t) + \cdots \tag{3.2}$$

where the dots represent possible terms of higher order in e^{-2x} as well as higher order in $\mathcal{T}(x,t)$. The string coupling g_s and the constant c are defined by

$$g_s^{-1} \equiv \frac{|\mu|}{\sqrt{2\pi}}, \quad c \equiv 1 + 4\Gamma'(1) + \ln g_s. \tag{3.3}$$

The linear term in $\mathcal{T}(x,t)$ of the specific form appearing on the r.h.s. of eqn. (3.2) is known from string scattering amplitudes and reflects the linear dilaton background [18].

In addition to eqn. (3.2), the operator $\mathcal{T}(x,t)$ defined by eqn. (3.1) must satisfy another crucial criterion for it to be identified with the tachyon of 2-dimensional string theory. We must ensure that the choice of the kernels G_1, G_2, etc., that satisfies eqn. (3.2) is consistent with the choice that makes $\mathcal{T}(x,t)$ and its conjugate $\Pi_{\mathcal{T}}(x,t)$, defined by

$$\Pi_{\mathcal{T}}(x,t) \equiv -i\left[\mathcal{T}(x,t), H_{\text{fluc.}}\right], \tag{3.4}$$

satisfy the canonical commutation relations:

$$[\mathcal{T}(x,t), \mathcal{T}(y,t)] = 0, \tag{3.5}$$

$$[\mathcal{T}(x,t), \Pi_{\mathcal{T}}(y,t)] = i\delta(x-y). \tag{3.6}$$

Note that the conjugate of $\mathcal{T}(x,t)$ is defined by eqn. (3.4) because, as we mentioned earlier, the generator of time-translations in the string theory in flat background is identical to that in the matrix model, namely $H_{\text{fluc.}}$ as given in eqn. (1.22).

We emphasize that in the present framework eqns. (3.5) and (3.6) are not automatically satisfied even if eqn. (3.2) is arranged. The operator $\mathcal{T}(x,t)$ inherits a certain quantization from the matrix model via the r.h.s. of eqn. (3.1). It is not a priori clear that the same choice of the kernels G_1, G_2, etc., that satisfies the classical equation (3.2) also necessarily satisfies the canonical commutation relations in eqns. (3.5) and (3.6).

If there exists a choice of the kernels G_1, G_2, etc., order by order in perturbation theory, such that eqns. (3.2), (3.5) and (3.6) are satisfied, then we may identify $\mathcal{T}(x,t)$ with the tachyon of 2-dimensional string theory.

Before closing this section, we mention that a choice of the kernels satisfying the above criteria must necessarily reduce to the asymptotic form implied in eqns. (2.1) and (2.2), at asymptotic space-time. Together with eqn. (3.2), this means that corrections to the kernels, away from the asymptotic region, must be of the general form

$$G_1(x;p,q) \overset{x\to\infty}{\sim} f(-qe^{-x}) + O(xe^{-2x}) \tag{3.7}$$

$$G_2,\ G_3,\ \text{etc.} \overset{x\to\infty}{\sim} O(e^{-2x}) \tag{3.8}$$

The function f above is the same as that given in eqn. (2.3). The precise fall-off of the correction terms as we approach the asymptotic region in eqns. (3.7) and (3.8) is dictated by eqn. (3.2).

There is an immediate consequence of eqns. (3.7) and (3.8). Together with the general form of the matrix model $\leftrightarrow$ string theory mapping in eqn. (3.1), they imply that the tree-level scattering amplitudes of $\mathcal{T}(x,t)$ are <u>entirely</u> determined by the asymptotic kernel function f. To appreciate this point fully we will give below a short derivation of scattering amplitudes from eqn. (3.1), before we proceed to seek a solution of eqns. (3.2), (3.5) and (3.6). It is useful to do this in any case, since our derivation of the amplitudes will also illustrate our earlier assertion that a specific choice of parametrization of the fermi fluid boundary fluctuations is not required to derive physical properties of the field $\mathcal{T}(x,t)$ from the matrix model.

4 Scattering Amplitudes

We will compute the scattering amplitudes at tree-level as usual by resolving the 'out' field $\mathcal{T}_{\text{out}}(t-x)$ in terms of the 'in' field $\mathcal{T}_{\text{in}}(t+x)$. Since these are defined at asymptotic times $t \to \pm\infty$, together with $x \to \infty$, where the interactions vanish, it is immediately clear that the detailed form of the correction terms away from the asymptotic region in eqns. (3.7) and (3.8) will not enter in the construction of the 'out' and 'in' fields from eqn. (3.1). It is, therefore, sufficient for us to write

$$\mathcal{T}(x,t) = \int dp\ dq\ f(-qe^{-x})\delta u(p,q,t) + O(xe^{-2x}) \tag{4.1}$$

for the purposes of computing the tree-level scattering amplitudes.

We will now use a simple trick to shift the time-dependence from the fluctuation $\delta u(p,q,t)$ to the function f. This is done by noting that (i) the measure $(dp\,dq)$ for integration over phase space is invariant under area-preserving diffeomorphisms and (ii) time-evolution of $\delta u(p,q,t)$ is equivalent to an area-preserving diffeomorphism on it by the hamiltonian $h(p,q)$ in eqn. (1.1). In other words, we make the following change of variables

$$(p \pm q)e^{\mp t} = (p' \pm q')e^{\mp t'} \tag{4.2}$$

from (p,q) to (p',q') in the integral in eqn. (4.2), with t and t' appearing as fixed parameters in the change of variables. Under this change of variables, the measure

($dp\ dq$) and the fermi surface, eqn. (1.3), are invariant. Moreover, using the equation of motion of $\delta u(p,q,t)$

$$(\partial_t + p\partial_q + q\partial_p)\delta u(p,q,t) = 0, \tag{4.3}$$

we deduce that

$$\delta u(p,q,t) = \delta u(p',q',t'). \tag{4.4}$$

Therefore, we get from eqn. (4.1)

$$\mathcal{T}(x,t) = \int dp'\ dq'\ f(-e^{-x}\left[q'\cosh(t-t') + p'\sinh(t-t')\right])\,\delta u(p',q',t') + O(xe^{-2x}). \tag{4.5}$$

The r.h.s. of eqn. (4.5) is actually independent of t', as can be easily varified by using eqn. (4.3). Thus, the choice $t' = t$ gives back eqn. (4.1). What we have achieved by rewriting eqn. (4.1) in the form of eqn. (4.5) is to shift the entire t-dependence into the argument of the kernel. Most importantly, however, we have in this way introduced a parameter t', which we may regard as some initial value of time. The fermi fluid boundary fluctuation then enters eqn. (4.5) only as a boundary condition. In a more standard notation, writing $t' = t_0$ and dropping the 'primes' from p and q, we get

$$\mathcal{T}(x,t) = \int dp\ dq\ f(-e^{-x}\left[q\cosh(t-t_0) + p\sinh(t-t_0)\right])\,\delta u(p,q,t_0) + O(xe^{-2x}). \tag{4.6}$$

Equation (4.6) proves our assertion that $\mathcal{T}(x,t)$ is insensitive to any singularities that specific parametrizations of the fluctuation $\delta u(p,q,t)$ might dynamically generate even if the parametrization was perfectly nonsingular to begin with.

To proceed further, we will now use the t_0-independence of the r.h.s. of eqn. (4.6) to make the choice $t_0 \to -\infty$. It is then convenient to use the quadratic profile parametrization for $\delta u(p,q,t_0)$, which is essentially given by the fields $\bar{\eta}_\pm(\tau,t_0)$ discussed in Sec. 1. We choose the profile such that in the limit $t_0 \to -\infty$, $\bar{\eta}_-$ vanishes. Also, in this limit $\bar{\eta}_+(\tau,t_0) = \bar{\eta}^0_+(t_0+\tau)$ is a function of $(\tau + t_0)$ only. Using this, and the formalism developed in eqs. (1.12) - (1.23), in eqn. (4.6) in the limit $t_0 \to -\infty$, we get

$$\mathcal{T}(x,t) = \int_{-\infty}^{+\infty} d\tau \int_0^{\bar{\eta}^0_+(\tau)} d\epsilon\ f\left(\left|\frac{\mu}{2}\right|^{\frac{1}{2}}\left[e^{\tau-(t+x)} + \left(1 - \frac{\epsilon}{|\mu|}\right)e^{-\tau+(t-x)}\right]\right) + O(xe^{-2x}). \tag{4.7}$$

It is now trivial to write down expressions for the 'in' and 'out' fields from eqn. (4.7),

$$\mathcal{T}_{\rm in}(t+x) = \int_{-\infty}^{+\infty} d\tau\ f\left(\left|\frac{\mu}{2}\right|^{\frac{1}{2}} e^{\tau-(t+x)}\right)\bar{\eta}^0_+(\tau), \tag{4.8}$$

$$\mathcal{T}_{\rm out}(t-x) = \int_{-\infty}^{+\infty} d\tau \int_0^{\bar{\eta}^0_+(\tau)} d\epsilon\ f\left(\left|\frac{\mu}{2}\right|^{\frac{1}{2}}\left(1 - \frac{\epsilon}{|\mu|}\right)e^{-\tau+(t-x)}\right), \tag{4.9}$$

and using these the tree-level amplitudes are easily obtained by eliminating $\bar{\eta}^0_+$ and expressing $\mathcal{T}_{\rm out}$ as a power series in $\mathcal{T}_{\rm in}$. It is a simple exercise to check that the correct string amplitudes are obtained in this way [27]. We see that the amplitudes are determined entirely by the asymptotic kernel function f.

5 The Transform – Specific Form

We now return to the question of whether there exists an explicit choice of the kernels G_1, G_2, etc., that satisfies eqns. (3.2), (3.5) and (3.6).

Let us begin with eqn. (3.2). Requiring that $\mathcal{T}(x,t)$ defined in eqn. (3.1) satisfy it gives, after a tedious but straightforward calculation, certain differential equations for the kernels. In deriving these differential equations one uses the equation of motion of the fluctuations, eqn. (4.3). It turns out that these differential equations can be explicitly solved and explicit expressions can be obtained for the kernels G_1 and G_2, these being the only two kernels relevant to the accuracy of the present calculations. Since the calculations are rather straightforward, we will not give the details here but will merely list the results which are conveniently summarized in the following parametrization of the kernels:

$$G_1(x;p,q) \equiv f(\alpha) + \left|\frac{\mu}{2}\right| e^{-2x}\left(2xG(\alpha)+K(\alpha)\right) + O(xe^{-4x}), \tag{5.1}$$

$$G_2(x;p,q;p',q') \equiv e^{-2x}\left[\left(1+\frac{pp'}{qq'}\right)F_+(\alpha,\alpha') + \left(1-\frac{pp'}{qq'}\right)F_-(\alpha,\alpha')\right] + O(xe^{-4x}), \tag{5.2}$$

where $\alpha \equiv -qe^{-x}$ and $\alpha' \equiv -q'e^{-x}$. The function $f(\alpha)$ is given by eqn. (2.3) and

$$G(\alpha) \equiv -\left(\frac{2}{\pi}\right)^{\frac{1}{4}} f'(\alpha), \tag{5.3}$$

$$K(\alpha) \equiv -\left(\frac{2}{\pi}\right)^{\frac{1}{4}}\left[(c+1)f'(\alpha)+\frac{f(\alpha)}{\alpha}\right] - \frac{f'(\alpha)}{\alpha}, \tag{5.4}$$

$$F_+(\alpha,\alpha') \equiv \frac{1}{4}f'(\alpha)\delta(\alpha-\alpha') + \frac{1}{\sqrt{2}}\left(\frac{2}{\pi}\right)^{-\frac{1}{4}}(\alpha-\alpha')^{-1} \left(\alpha\partial_\alpha - \alpha'\partial_{\alpha'}\right)\left(f(\alpha)f(\alpha')\right), \tag{5.5}$$

$$F_-(\alpha,\alpha') \equiv -\frac{\sqrt{\pi}}{2}f'(\alpha)f'(\alpha'). \tag{5.6}$$

Here $f'(\alpha) \equiv \frac{d}{d\alpha}f(\alpha)$. In obtaining (5.3)-(5.6) we have made extensive use of the differential equation satisfied by $f(\alpha)$,

$$(\alpha f'(\alpha))' = -\left(\frac{2}{\pi}\right)^{\frac{1}{4}} f(\alpha), \tag{5.7}$$

and the properties of Bessel functions of integer order [26].

One remarkable thing about the solutions in eqns. (5.3)-(5.6) is that the kernels G_1 and G_2 are determined entirely in terms of the asymptotic kernel function f. This is, however, not really surprising. The reason for this is that, as we have seen in the previous section, the information about string scattering amplitudes is entirely encoded in the function f and that eqn. (3.2), which was used to fix G_1 and G_2, reproduces these amplitudes [15]. The last statement of course presumes that the canonical commutation relations, eqs. (3.5) and (3.6), are satisfied. So let us now turn to these.

We need to check that the operator $\mathcal{T}(x,t)$ defined by eqns. (3.1) and (5.1)-(5.6) and its conjugate $\Pi_{\mathcal{T}}(x,t)$, obtained using eqn. (3.4), satisfy eqns. (3.5) and (3.6). In order to carry out this check it is convenient, though by no means necessary, to rewrite $\mathcal{T}(x,t)$ in terms of the collective field parametrization of the fluctuation $\delta u(p,q,t)$. In terms of the collective field variables $\bar{\eta}_{\pm}(\tau,t)$ of Sec. 2, $\mathcal{T}(x,t)$ is given by

$$
\begin{aligned}
\mathcal{T}(x,t) \quad = \quad & \int_0^\infty d\tau(\bar{\eta}_+ - \bar{\eta}_-) f(v) \\
& + e^{-2x}\Bigg[-\left|\frac{\mu}{2}\right| \left(\frac{2}{\pi}\right)^{\frac{1}{4}} \int_0^\infty d\tau(\bar{\eta}_+ - \bar{\eta}_-)\left\{(2x+c+1)f'(v) + \frac{f(v)}{v}\right\} \\
& \qquad + \frac{1}{4}\int_0^\infty d\tau(\bar{\eta}_+^2 + \bar{\eta}_-^2)\frac{f'(v)}{v} - \frac{\sqrt{\pi}}{4}\int_0^\infty d\tau \int_0^\infty d\tau' \\
& \qquad \{(\bar{\eta}_+ - \bar{\eta}_-)(\bar{\eta}'_+ - \bar{\eta}'_-) - (\bar{\eta}_+ + \bar{\eta}_-)(\bar{\eta}'_+ + \bar{\eta}'_-)\} f'(v) f'(v') \\
& \qquad + \frac{1}{2\sqrt{2}}\left(\frac{\pi}{2}\right)^{\frac{1}{4}} \int_0^\infty d\tau \int_0^\infty d\tau' \\
& \qquad \{(\bar{\eta}_+ - \bar{\eta}_-)(\bar{\eta}'_+ - \bar{\eta}'_-) + (\bar{\eta}_+ + \bar{\eta}_-)(\bar{\eta}'_+ + \bar{\eta}'_-)\} \times \\
& \qquad\quad \times (v-v')^{-1}(v\partial_v - v'\partial_{v'})\left(f(v)f(v')\right)\Bigg] \\
& + O(xe^{-4x}, \bar{\eta}_{\pm}^3)
\end{aligned}
\tag{5.8}
$$

where $v \equiv \left|\frac{\mu}{2}\right|^{\frac{1}{2}} e^{\tau - x}$, $v' \equiv \left|\frac{\mu}{2}\right|^{\frac{1}{2}} e^{\tau' - x}$. We have also used the short-hand notation, $\bar{\eta}_{\pm} \equiv \bar{\eta}_{\pm}(\tau,t)$, $\bar{\eta}'_{\pm} \equiv \bar{\eta}'_{\pm}(\tau',t)$, etc. On the other hand, $f'(v) \equiv \frac{d}{dv} f(v)$.

Commuting this with the hamiltonian $H_{\text{fluc.}}$ given in eqn. (1.22) using the commutation relations in eqn. (1.21), we get the following expression for the conjugate of $\mathcal{T}(x,t)$:

$$
\begin{aligned}
\Pi_{\mathcal{T}}(x,t) &\equiv -i\,[\mathcal{T}(x,t), H_{\text{fluc.}}] \\
&= -\int_0^\infty d\tau(\bar\eta_+ + \bar\eta_-)\partial_\tau f(v) \\
&\quad + e^{-2x}\Bigg[\left|\frac{\mu}{2}\right|\left(\frac{2}{\pi}\right)^{\frac14}\int_0^\infty d\tau(\bar\eta_+ + \bar\eta_-) \\
&\qquad \left\{(2x+c+1)\partial_\tau f'(v) + \partial_\tau\left(\frac{f(v)}{v}\right)\right\} \\
&\qquad + \frac14\left(\frac{2}{\pi}\right)^{\frac14}\int_0^\infty d\tau\left(\bar\eta_+^2 - \bar\eta_-^2\right)\frac{f(v)}{v} \\
&\qquad + \frac{\sqrt{\pi}}{2}\int_0^\infty d\tau\int_0^\infty d\tau'(\bar\eta_+ + \bar\eta_-)(\bar\eta'_+ - \bar\eta'_-)\times \\
&\qquad \times(v\partial_v - v'\partial_{v'})(f'(v)f'(v')) \\
&\qquad - \frac{1}{\sqrt2}\left(\frac{\pi}{2}\right)^{\frac14}\int_0^\infty d\tau\int_0^\infty d\tau'(\bar\eta_+ + \bar\eta_-)(\bar\eta'_+ - \bar\eta'_-)\times \\
&\qquad \times(v\partial_v + v'\partial_{v'})(v-v')^{-1}(v\partial_v - v'\partial_{v'})(f(v)f(v'))\Bigg] \\
&\quad + O(xe^{-4x}, \bar\eta_\pm^3). \qquad (5.9)
\end{aligned}
$$

Notice that $\mathcal{T}(x,t)$ (as well as $\Pi_{\mathcal{T}}(x,t)$) contains both combinations $(\eta_+ - \eta_-)$ and $(\eta_+ + \eta_-)$. It is, therefore, not obvious that it would satisfy eqn. (3.5). A straightforward calculation using the commutation relations, eqn. (1.21), however, shows that eqn. (3.5) is indeed satisfied by the combination of terms in eqn. (5.8). This is a rather nontrivial check on our construction. We made extensive use of integrals of two Bessel functions of integer order [28] in carrying out this calculation. Actually, in the quoted reference the lower limit on the relevant integrals is zero while we get the lower limit as $e^{-x/2}$ (or $e^{-y/2}$). The difference, however, is at least of order $e^{-(x+y)}$, which can safely be ignored to the accuracy of our present considerations (because there is already a factor of e^{-2x} (or e^{-2y}) present outside the square brackets in eqn. (5.8)).

The commutator in eqn. (3.6) is somewhat more nontrivial to verify. The subtlety comes from the fact that the first term in eqn. (5.8) already gives the required answer for the commutator with the first term in eqn. (5.9). So the contribution of the rest of the terms to the commutator is required to vanish, but it is not clear how the contribution of terms linear in x (or y) would cancel since, on the face of it, there is only one such contribution. Explicit calculation, however, shows that the last term in the curly brackets in eqn. (5.8) (and eqn. (5.9)) which is linear in $\bar\eta_\pm$ gives a contribution that is proportional to the integral

$$
\int_{e^{-x/2}}^\infty \frac{d\nu}{\nu} J_0(\nu) J_0\left(\nu e^{\frac{x-y}{2}}\right).
$$

Because of a logarithmic divergence we can now not naively set the lower limit of integration to zero, as was done previously. The divergence can be extracted by rewriting this integral as

$$\int_{e^{-x/2}}^{\infty} \frac{d\nu}{\nu} J_0(\nu) \left[J_0 \left(\nu e^{\frac{x-y}{2}} \right) - 1 \right] + \int_{e^{-x/2}}^{\infty} \frac{d\nu}{\nu} J_0(\nu).$$

We may now safely set the lower limit in the first term to zero. The second term can be evaluated explicitly in the limit $x \to \infty$. The logarithmic divergence appears as a linear term in x. This precisely cancels the contribution to the commutator coming from terms in eqns. (5.8) and (5.9) which are manifestly linear in x. It is remarkable that eqn. (3.6) is satisfied in this rather nontrivial fashion.

Now that our construction of $\mathcal{T}(x,t)$ has passed all the consistency checks, we may identify it with the tachyon of 2-dimensional string theory.

6 Discrete States and Other Backgrounds

One of the aspects of 2-dimensional string theory most difficult to understand in the matrix model has been the apparent absence of any trace of the so-called W^- discrete states [6]. This issue is rather crucial since, among other things, these states contain the dilaton black hole. The first hint that a nonlocal transform is necessary to extract the physics of discrete states from the matrix model was seen in the gravitational scattering calculation of [15]. Now that we have seen strong evidence that it might, in fact, even be possible to set up a detailed operator correspondence between string theory and matrix model by a nonlocal and nonlinear transform, it has become easy to at least begin to imagine how the W^- discrete states might make their appearance in this setting. It seems natural to think that these states correspond to other nonlocal and nonlinear combinations of the fluid boundary fluctuation $\delta u(p,q,t)$, which satisfy properties expected of the W^- discrete operators. Included among these properties must be the following [6]. These operators must commute with each other and with the tachyon $\mathcal{T}(x,t)$ and they must transform as a representation of the W-infinity algebra generated by the operators given in eqn. (1.8). (We will have more to say about this last point in the next section.)

A motivation for believing that discrete states, like the tachyon, correspond to nonlocal and nonlinear combinations of the density fluctuations comes from the full set of string σ-model β-function equations:

$$\begin{aligned} R_{\mu\nu} + 2\nabla_\mu \nabla_\nu \Phi - 2\partial_\mu T \partial_\nu T &= 0, \\ R + 4\nabla^2\Phi - 4(\nabla\Phi)^2 - 2(\nabla T)^2 - 8T^2 &= 16, \\ \nabla^2 T - 2\nabla\Phi\nabla T - 4T - 2\sqrt{2}T^2 &= 0. \end{aligned} \tag{6.1}$$

Let us fix the gauge in which the dilaton is always given (at least in a local neighbourhood) by its linear form, namely,

$$\Phi = -2x,$$

and the metric has the diagonal form

$$g_{\mu\nu} = \mathrm{diag}\,(g_{00}, g_{11}).$$

Perturbing around the flat background

$$g_{00} = 1 + g_0, \quad g_{11} = -1 + g_1,$$

gives the following equations for the metric fluctuations:

$$\begin{aligned} \partial_x g_0 &= \frac{1}{2}\left[(\partial_t T)^2 + (\partial_x T)^2 + 4T^2\right] - 4g_1, \\ (\partial_x + 4)\, g_1 &= -\frac{1}{2}\left[(\partial_t T)^2 + (\partial_x T)^2 - 4T^2\right] \\ \partial_t g_1 &= -(\partial_t T)(\partial_x T). \end{aligned} \tag{6.2}$$

In the absence of the tachyon field, eqns. (6.2) lead to the famous black-hole solution [8] for the metric perturbations

$$g_0 = g_1 \propto e^{-4x}.$$

With the tachyon field present, eqns. (6.2) are the analogues of eqn. (3.2) for the present case of the metric perturbations. Knowing $T(x,t)$ $\left(= e^{-2x}\{\mathcal{T}(x,t) - \frac{1}{\sqrt{2}g_s}(x+c)\}\right)$ in terms of the matrix model variables, for example as given in (5.8), allows us to rewrite these equations in terms of the density fluctuations of the matrix model. In this way, one obtains <u>new</u> nonlocal and nonlinear combinations of the matrix model variables which represent the metric fluctuations.

It seems reasonable to assume that this holds true for the higher tensor modes of the string as well. If that is so, then we may be able to construct all the discrete state operator combinations, even without knowing their equations. This is because all the W^- discrete state operators must together form a representation of the W-infinity algebra [6]. It seems clear to us that it is the techniques of W-infinity representation theory that will then prove to be more powerful and more useful (than any higher order σ-model calculations to incorporate the higher tensor modes of the string theory) for discovering these operators in the matrix model.

Connected with the question of discrete states is the issue of backgrounds other than the flat space and linear dilaton background we have considered here. These backgrounds can be created by using coherent state techniques once we know the discrete state operators constructed along the above lines. The tachyon operator in eqn. (5.8) would, in general, have a nonzero value in such coherent states. One effect of this would be to change the asymptotic kernel in the transform to string theory. It would, therefore, seem that different backgrounds should correspond to different choices of the asymptotic kernel. This line of enquiry, as well as the construction of discrete state operators outlined above, is under active investigation.

7 Realization of W_∞-Symmetry

Given that it is the nonlocal operator $\mathcal{T}(x,t)$ in eqn. (5.8) (and not $\bar{\eta}_\pm$ themselves) that has physical (space-time) significance, it is of interest to ask how the generators of W-infinity symmetry act on it. This question is actually also of interest in view of the discussion in the previous section. Moreover, the answer to this question should, in particular, also give us an infinite number of symmetries of the equation of motion (3.2).

It turns out that the action of W-infinity symmetry on $\mathcal{T}(x,t)$ is nonlocal, as one might have expected. Since this is easiest to see in the collective field parametrization, let us consider an example in this parametrization to illustrate the above statement. But first note that the W-infinity generators are represented on the fluctuations by

$$\omega_{mn} = e^{-(m-n)t} \int \frac{dp\, dq}{2\pi} (-p-q)^m (p-q)^n \delta u(p,q,t). \tag{7.1}$$

The example we will consider is that of the 'half' of Virasoro, which is generated by

$$V_n = \frac{i}{2^{n+1}} \left(\frac{\pi}{2}\right)^{-\frac{n}{4}} \left[\omega_{n+1,1} + 2\mu\, \omega_{n,0}\right], \quad n \geq 0. \tag{7.2}$$

The V_n's have the following expressions in terms of the collective variables $\bar{\eta}_\pm$:

$$V_n = \frac{g_s^{-n/2}}{4\pi i} \int_0^\infty d\tau \left[e^{-n(t+\tau)} \bar{\eta}_+^2 + e^{-n(t-\tau)} \bar{\eta}_-^2\right] + O(\bar{\eta}_\pm^3) \tag{7.3}$$

where g_s is the string coupling given by eqn. (3.3). Using the commutation relations in eqn. (1.21) one can easily check that

$$[V_n, V_m] = (n-m) V_{n+m}. \tag{7.4}$$

In writing eqn. (7.4) we have ignored the $O(\bar{\eta}_\pm^3)$ terms in eqn. (7.3). These will not affect the linearized variations of the tachyon considered below. These latter can be obtained from the linearized variations of the collective variables:

$$[V_n, \bar{\eta}_\pm(\tau,t)] = \mp g_s^{-n/2} \partial_\tau \left(e^{-n(t\pm\tau)} \bar{\eta}_\pm(\tau,t)\right) + O(\bar{\eta}_\pm^2). \tag{7.5}$$

Using this in eqn. (5.8), we get

$$[V_n, (\partial_t - \partial_x)\mathcal{T}(x,t)] = \int_{-\infty}^{+\infty} dy\; \Delta_n^-(t-x; t-y)(\partial_t - \partial_y)\mathcal{T}(y,t) + O(\mathcal{T}^2, \text{higher order in } \mathrm{e}^{-\mathrm{x}}) \tag{7.6}$$

where

$$\Delta_n^-(x;y) = -(-)^n \int \frac{dk}{2\pi} e^{ikx} e^{-(ik+n)y} \frac{\Gamma(1+ik+n)\Gamma(ik+n)}{\Gamma(ik)\Gamma(ik)}. \tag{7.7}$$

For the other branch, we get

$$[V_n, (\partial_t + \partial_x)\mathcal{T}(x,t)] = \int_{-\infty}^{+\infty} dy\; \Delta_n^+(t+x; t+y)(\partial_t + \partial_y)\mathcal{T}(y,t) + O(\mathcal{T}^2, \text{higher order in } \mathrm{e}^{-\mathrm{x}}) \tag{7.8}$$

where

$$\Delta_n^+(x;y) = (-)^n g_s^{-n} \int \frac{dk}{2\pi} e^{-ikx} e^{(ik-n)y} \frac{\Gamma(1+ik-n)\Gamma(ik-n)}{\Gamma(ik)\Gamma(ik)}. \tag{7.9}$$

The tachyon transformation law derived in eqns. (7.6) and (7.7) above is precisely the one obtained earlier in [7] using the techniques of BRST cohomolgy in liouville string theory (see eqn. (5.21) of this reference[6]). The more standard form for the

[6]Note that to get their result from (7.6) and (7.7) we need to use euclidean momenta.

Virasoro transformation obtained in this reference (in eqn. (5.23)) for a redefined 'tachyon' is nothing but the transformation in eqn. (7.5) for the collective field, which we now know is not the tachyon of 2-dimensional string theory. The string theory tachyon, as we have seen, has a nonlocal transformation under the Virasoro, and indeed under the full set of W-infinity transformations in eqn. (7.1).

There are perhaps at least two reasons for this nonlocal realization of the W-infinity symmetry of the matrix model in string theory. One is the fact that in the present framework there are no fields explicitly present which represent the higher tensor modes of the string (these have implicitly been integrated out). The other reason probably is that the W-infinity symmetry is the 'global' remnant of a large local symmetry after gauge-fixing. It is clearly very important to have a deeper understanding of both these aspects in the present framework in order to get a better handle on the underlying structure of the string theory.

8 Concluding Remarks

To sum up, we have derived the tachyon σ-model β-function equation of 2-dimensional string theory in flat space and linear dilaton background, working entirely within the $c = 1$ matrix model. This equation is derived for a nonlocal and nonlinear combination of the matrix model variables. We have also seen that the W-infinity symmetry of the matrix model has a nonlocal action on the tachyon field defined in this way, a result which was known earlier in liouville string theory. These results, among other things, present strong evidence for the viewpoint that the space-time properties of 2-dimensional string theory can only be extracted from the $c = 1$ matrix model by means of a nonlocal and nonlinear mapping. We have argued that this viewpoint also has the potential to accommodate discrete states corresponding to the higher tensor modes of the string. Some evidence for this already exists [15], but one can now hope to even explicitly construct the corresponding operators.

Throughout this work our considerations have been perturbative. It is clear that we need a nonperturbative understanding of the issues discussed here, if we are to eventually use the full nonperturbative power of the matrix model to understand some of the stringy issues, such as the nature of quantum gravity in the strong coupling regime. Progress on the nonperturbative aspects of the present work is, therefore, of urgent interest. Even at the perturbative level, however, the picture is not completely clear yet. It is likely that a better understanding of the emergence of discrete states and the role of W-infinity symmetry at the perturbative level will give us a better handle on the underlying structure of the space-time theory and, therefore, probably also on its nonperturbative aspects. These issues are under active investigation.

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