CERN-TH/95-186
TIFR/TH/95-30

DISCRETE-STATE MODULI OF STRING THEORY FROM THE C=1 MATRIX MODEL

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ABSTRACT

We propose a new formulation of the space-time interpretation of the $c = 1$ matrix model. Our formulation uses the well-known leg-pole factor that relates the matrix model amplitudes to that of the 2-dimensional string theory, but includes fluctuations around the fermi vacuum on *both sides* of the inverted harmonic oscillator potential of the double-scaled model, even when the fluctuations are small and confined entirely within the asymptotes in the phase plane. We argue that including fluctuations on both sides of the potential is essential for a consistent interpretation of the leg-pole transformed theory as a theory of space-time gravity. We reproduce the known results for the string theory tree level scattering amplitudes for the tachyon in flat space and linear dilaton background as a special case. We show that the generic case corresponds to more general space-time backgrounds. In particular, we identify the parameter corresponding to the background metric perturbation in string theory (black hole mass) in terms of the matrix model variables. Possible implications of our work for a consistent nonperturbative definition of string theory as well as for quantized gravity and black-hole physics are discussed.

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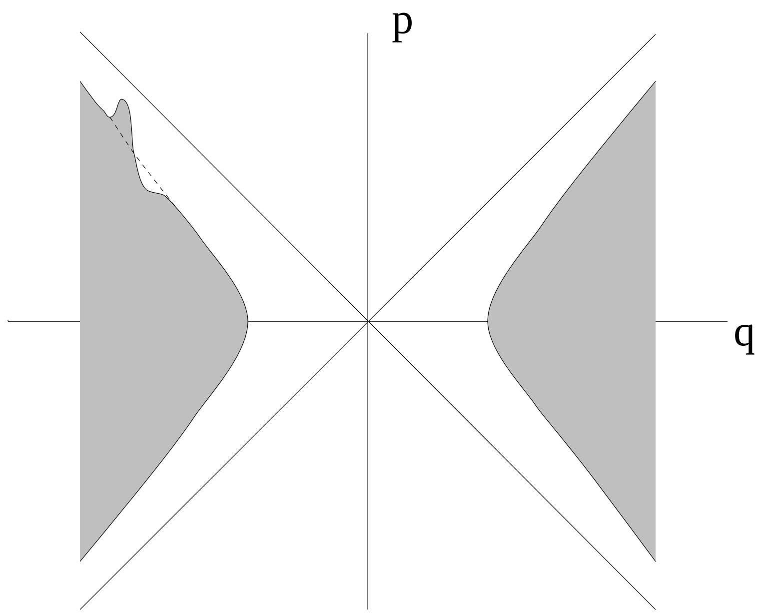

Figure 1: Small fluctuation of the fermi surface on only one side of the potential.

1 Introduction and Summary

It is well-known that the scattering amplitudes of the scalar excitation of the double-scaled $c = 1$ matrix model are not identical to the tachyon scattering amplitudes of 2-dimensional string theory, but are related to these by a 'leg-pole' factor [1]. Although this leg-pole factor is a pure phase in momentum space, it translates into a nonlocal renormalization of the wavefunctions of the scalar excitation of the matrix model, and gives rise to all of space-time gravitational physics of the string theory [2], which is otherwise absent in the matrix model.

The $c = 1$ matrix model is equivalent to a theory of nonrelativistic, noninteracting fermions in an inverted harmonic oscillator potential in the double scaling limit [3]. The semiclassical physics of the matrix model is, therefore, described by a fermi liquid theory. The existing space-time interpretation of the matrix model is based on small fluctuations of this fermi fluid, around the fermi vacuum, on only one side of the potential (Figure 1).

It is generally believed that the other side can be ignored as long as one avoids the classical configurations consisting of large fluctuations in which a part of the fluid croses the asymptotes (Figure 2), and nonperturbative tunnelling issues [4].

There is, however, an argument which suggests that one has to decide the fate of the other side of the potential even in the small-field perturbative situation when the two sides of the potential are entirely decoupled in the matrix model. The point is that string theory is a theory of gravity, and in any consistent space-time interpretation of the matrix model the space-time metric must couple to the <u>entire</u> energy-momentum tensor of the theory. Now, a generic perturbation of the

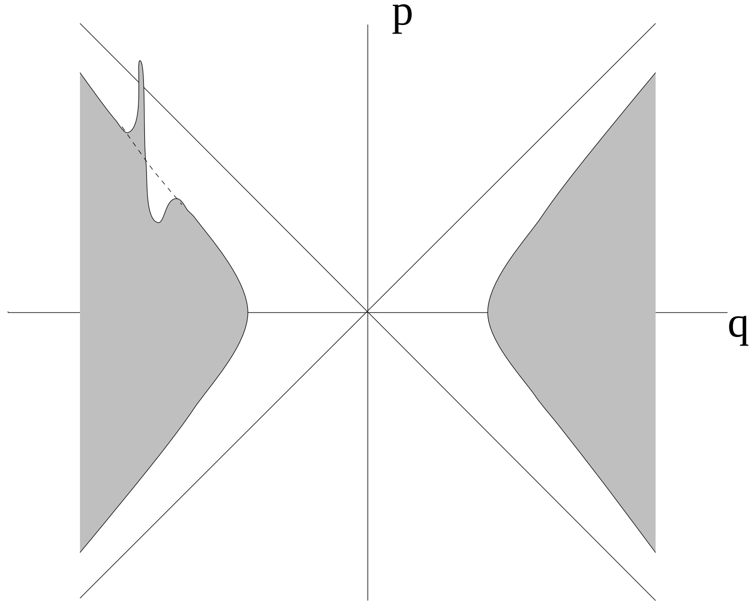

Figure 2: A part of the fluid fluctuation crosses the $p = -q$ asymptote.

fermi fluid has fluctuations on both sides of the potential, even in the small-field perturbative situation, and the total energy of this configuration has contributions from fluctuations on both sides. The string theory metric must couple to this total energy, since the hamiltonian of the string theory is identical to that of the matrix model, unless we decide to remove the other side from the start, by modifying the potential. This is presumably also the case for all the other higher conserved charges corresponding to the diagonal generators of W-infinity symmetry of the matrix model. Thus, we must decide on the fate of the other side of the potential, for any fermi fluid configuration, before arriving at a consistent space-time interpretation of the model.

In this paper we will retain both sides of the potential and consider the generic situation in which there are fluctuations of the fermi surface on both sides (Figure 3). The summary of our results is as follows. The usual leg-pole transform of the special configurations in which the fluctuations of the fermi surface, lying entirely within the asymptotes, are identical on the two sides, reproduces the tachyon scattering amplitudes of 2-dimensional string theory in flat space, linear dilaton background. For the generic case, when the fluctuations of the fermi surface on the two sides of the potential are different, we get additional contributions to the tachyon scattering amplitudes. By comparing with early-time bulk scattering amplitudes of tachyon in the effective tachyon-graviton-dilaton field theory, we are able to identify two of the leading additional contributions as due to the presence of perturbations of the background tachyon and the metric. Roughly speaking, the string theory tachyon turns out to be the leg-pole transform of the 'average' ($\phi(\tau)$) of the fluid fluctuations on the two sides of the potential, while

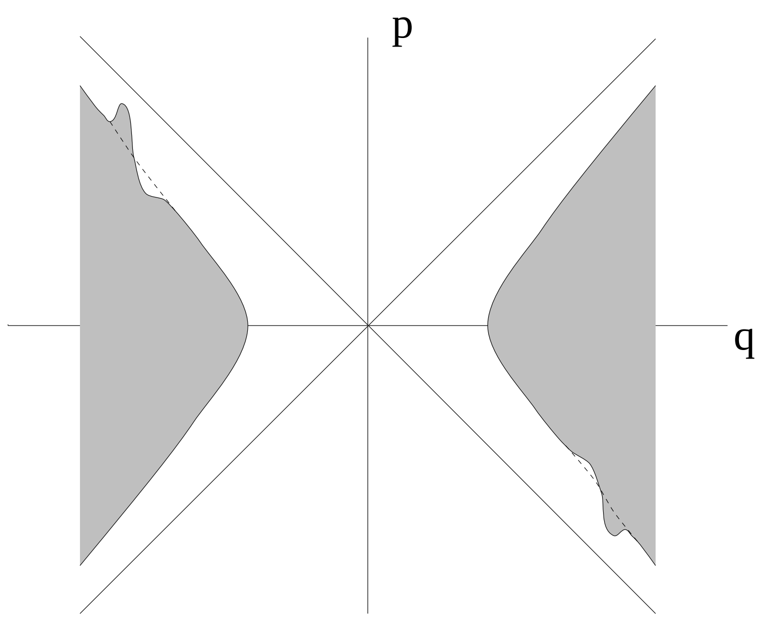

Figure 3: A generic initial small-field fluctuation of the fermi surface.

the 'difference' $\Delta(\tau)$ is now an additional variable in the problem (see Sec. 4 for details). Both the extra tachyon background and the metric perturbation vanish if $\Delta(\tau) = 0$. The metric pertubation to the leading order has the form of a linearised black hole metric of mass $M = (1/4\pi) \int d\tau\, \Delta^2(\tau)$, which is precisely the extra energy of the fermi fluid contributed by $\Delta(\tau)$. Besides the contribution attributable to additional tachyon and metric backgrounds [5] there are an infinite series of subleading contributions to the bulk scattering at early times in presence of non-zero value of $\Delta(\tau)$. Presumably these reflect the presence of perturbations in the background values of the other higher tensor fields of 2-dimensional string theory corresponding to the higher discrete states [5].

The plan of this paper is as follows. To make this paper somewhat self-contained, in the next section we will review the phase space approach [6, 7] to the double-scaled matrix model and the collective field parametrization of the fluid fluctuations. In Sec. 3 we will discuss the leg-pole transform and recall how the known tachyon string amplitudes in flat space and linear dilaton background are obtained using this transform in the existing approach which neglects the

[5]We should emphasize that any additional tachyon background of course causes curvature and therefore a perturbation to the metric(back reaction); it is easy to excite *such* metric perturbations in the conventional framework (ignoring fluid fluctuations on the other side) by simply declaring one of the pulses in a scattering experiment as 'background'. However, such a metric perturbation would not constitute independent moduli. On the other hand, the metric perturbation that we find from the scattering amplitudes is over and above the back reaction (the latter is also present in our model, but is of order Δ^4 and is therefore subleading and distictly identifiable). In the language of string field theory, in our framework we can tune vacuum expectation values of metric and tachyon independently.

other side of the potential. In Sec. 4 we will include both sides in the transform and identify the 'average' and 'difference' variables mentioned above. We will see that the flat space physics of the string theory is obtained in the limit in which the difference variable vanishes. In the generic case, additional leading contributions to early-time bulk scattering will be identified. In Sec. 5 we will show that the above results are identical to the bulk scattering amplitudes coming from the effective tachyon-graviton-dilaton action in the presence of background tachyon and metric perturbations. The parameters of these perturbations will be identified in terms of the 'difference' variable. In particular, we will see that the parameter of the background metric perturbation is precisely the contribution of the 'difference' variable to the total energy of the fermi fluid fluctuations. Sec. 6 is devoted to some concluding remarks in which we will point out the implications of our work for a consistent nonperturbative definition of string theory as well as for quantized gravity and black-hole physics.

2 Review of the Phase Space Formulation of the Matrix Model

Here we will briefly review some aspects of the phase space formulation of the $c = 1$ matrix model that will be relevant to the discussion in the following sections.

In the double-scaling limit, the $c = 1$ matrix model is mapped to a model of noninteracting, nonrelativistic fermions in an inverted harmonic oscillator potential [3] in one space dimension. The single-particle hamiltonian for this model is

$$h(p, q) = \frac{1}{2}(p^2 - q^2) \tag{2.1}$$

where (p, q) labels the single-particle phase space of the fermions. There is a convenient field theoretic description for the double-scaled model in terms of free nonrelativistic fermions [8]. The fermion field, which we denote by $\psi(q, t)$, satisfies the equation of motion

$$i\partial_t \psi(q, t) = -\frac{1}{2}(\partial_q^2 + q^2)\psi(q, t) \tag{2.2}$$

and its conjugate $\psi^\dagger(q, t)$ satisfies the complex conjugate of eqn. (2.2). The ground state of this model is the fermi vacuum obtained by filling up to the energy level μ (< 0). The semiclassical limit is obtained as $|\mu| \to \infty$, and in this limit the fermi surface is described by the hyperbola

$$\frac{1}{2}(p^2 - q^2) = \mu = -|\mu| \tag{2.3}$$

The basic building block for this work will be the phase space density of fermions, which we denote by $\mathcal{U}(p, q, t)$. In terms of the fermi field $\psi(q, t)$ it is defined as

$$\mathcal{U}(p,q,t) \equiv \int_{-\infty}^{+\infty} d\lambda \; e^{-ip\lambda} \; \psi^\dagger\left(q-\frac{\lambda}{2},t\right)\psi\left(q+\frac{\lambda}{2},t\right), \tag{2.4}$$

and it satisfies the equation of motion

$$(\partial_t + p\partial_q + q\partial_p)\,\mathcal{U}(p,q,t) = 0. \tag{2.5}$$

This equation can be obtained using eqn. (2.2), or by directly using the hamiltonian

$$H = \int \frac{dp\; dq}{2\pi}\; h(p,q)\mathcal{U}(p,q,t) \tag{2.6}$$

and the equal-time commutation relation for the phase space density $\mathcal{U}(p,q,t)$, which follows from its definition, eqn. (2.4), in terms of the underlying fermions:

$$\begin{aligned}[\mathcal{U}(p,q,t),\mathcal{U}(p',q',t)] \;&=\; -4\int \frac{dp''dq''}{2\pi}\mathcal{U}(p'',q'',t) \\ &\qquad [\exp 2i\{p(q'-q'')+p'(q''-q) \\ &\qquad\qquad +p''(q-q')\} - c.c.]\end{aligned} \tag{2.7}$$

Equation (2.7) is also a version of the large symmetry algebra, W_∞ [9], which is a symmetry of the matrix model [10]. The more standard version of the generators of this symmetry algebra is the following:

$$W_{mn} = e^{-(m-n)t}\int \frac{dp\; dq}{2\pi}(-p-q)^m(p-q)^n\mathcal{U}(p,q,t), \tag{2.8}$$

where $m,n \geq 0$. One can easily check, using eqn. (2.5), that W_{mn} are conserved. They satisfy the classical algebra

$$\{W_{mn}, W_{m'n'}\}_{P.B.} = 2(m'n - mn')W_{m+m'-1,n+n'-1}. \tag{2.9}$$

The quantum version of this is more complicated, but can be computed using eqn. (2.7).

The above phase space density formalism was first introduced in the present context in [6], and using this variable a bosonization of the double-scaled matrix model was carried out in [6, 7]. A crucial ingredient in that bosonization is a quadratic constraint satisfied by $\mathcal{U}(p,q,t)$ [6]. In the semiclassical limit this quantum constraint reduces to the simpler equation

$$\mathcal{U}^2(p,q,t) = \mathcal{U}(p,q,t). \tag{2.10}$$

Moreover, one also has the constraint of fixed fermion number, which implies that fluctuations of the fermi surface satisfy

$$\int \frac{dp\; dq}{2\pi}\delta\mathcal{U}(p,q,t) = 0, \quad \delta\mathcal{U}(p,q,t) = \mathcal{U}(p,q,t) - \mathcal{U}_0(p,q), \tag{2.11}$$

where $\mathcal{U}_0(p,q)$ describes the filled fermi vacuum. In this way we recover the Thomas-Fermi limit of an incompressible fermi fluid. The dynamics of the fluctuations $\delta\mathcal{U}(p,q,t)$, which satisfy eqn. (2.11) and another constraint because of eqn. (2.10), resides only in the boundary of the fermi fluid (in the semiclassical limit that we are considering here) [11].

Although one need not ever use an explicit parametrization of $\delta\mathcal{U}(p,q,t)$, it will, nevertheless, be useful at times to do so. For this reason we summarize in the following some relevant aspects of the 'quadratic profile' [11] or 'collective field' [12] parametrization of the fluctuations $\delta\mathcal{U}(p,q,t)$.

In the semiclassical limit, the fermi vacuum is described by the density

$$\mathcal{U}_0(p,q) = \theta\left(P^0_+(q) - p\right)\theta\left(p - P^0_-(q)\right), \tag{2.12}$$

where

$$P^0_\pm(q) \equiv \pm P_0(q) = \pm\sqrt{q^2 + 2\mu} \tag{2.13}$$

satisfy eqn. (2.3), which describes the fermi surface hyperbola. The quadratic profile or collective field description corresponds to a description of small ripples on the fermi surface by a density of the form

$$\mathcal{U}(p,q,t) = \theta\left(P_+(q,t) - p\right)\theta\left(p - P_-(q,t)\right) \tag{2.14}$$

Substituting this in eqn. (2.5), we get the equation of motion of $P_\pm$:

$$\partial_t P_\pm(q,t) = \frac{1}{2}\partial_q\left(q^2 - P^2_\pm(q,t)\right). \tag{2.15}$$

This equation is clearly solved by the fermi vacuum, eqns. (2.12) and (2.13). Fluctuations around this ground state,

$$P_\pm(q,t) - P^0_\pm(q) \equiv \eta_\pm(q,t) \tag{2.16}$$

satisfy the equations of motion

$$\partial_t \eta_\pm(q,t) = \mp\partial_q\left[P_0(q)\eta_\pm(q,t) \pm \frac{1}{2}\eta^2_\pm(q,t)\right]. \tag{2.17}$$

If the fluctuations are so small that they never cross the asymptotes, $p = \pm q$, of the hyperbola defined by eqn. (2.3), then one can rewrite eqns. (2.17) in a form that exhibits the presence of a massless particle. This is done by introducing the time-of-flight variable τ, defined by

$$q = \mp|2\mu|^{\frac{1}{2}}\cosh\tau, \quad 0 \leq \tau < \infty, \tag{2.18}$$

where the $-$ve sign is appropriate for the left side ($q < 0$) of the potential and the $+$ve sign for the right side ($q > 0$).

We now introduce the new variables $\bar\eta^\alpha_\pm(\tau,t)$, $\alpha=1,2$, defined by

$$\bar\eta^1_\pm(\tau,t) \equiv P_0(q(\tau))\eta_\pm(q,t), \quad q \ < \ 0 \tag{2.19}$$

$$\bar\eta^2_\pm(\tau,t) \equiv -P_0(q(\tau))\eta_\mp(q,t), \quad q \ > \ 0 \tag{2.20}$$

The variables $\bar\eta^\alpha_\pm(\tau,t)$ describe a generic small-field fluctuation on the two sides of the potential. They satisfy the equations of motion

$$(\partial_t \mp \partial_\tau)\bar\eta^\alpha_\pm(\tau,t) = \partial_\tau \left[(\bar\eta^\alpha_\pm(\tau,t))^2/2P_0^2(q(\tau))\right]. \tag{2.21}$$

Furthermore, one can also deduce the commutation relations

$$\begin{aligned} \left[\bar\eta^\alpha_\pm(\tau,t), \bar\eta^\beta_\pm(\tau,t)\right] &= \pm 2\pi i \delta^{\alpha\beta}\partial_\tau\delta(\tau-\tau'), \\ \left[\bar\eta^\alpha_+(\tau,t), \bar\eta^\beta_-(\tau,t)\right] &= 0, \end{aligned} \tag{2.22}$$

since we know the hamiltonian for the fluctuations

$$\begin{aligned} H_{fluc.} &= \int \frac{dp\ dq}{2\pi} h(p,q)\delta\mathcal{U}(p,q,t) \\ &= \frac{1}{4\pi}\int_0^\infty d\tau \sum_{\alpha=1,2}\Bigg[(\bar\eta^\alpha_+(\tau,t))^2 + (\bar\eta^\alpha_-(\tau,t))^2 + \frac{1}{3P_0^2(q(\tau))} \\ &\qquad \left\{\left(\bar\eta^\alpha_+(\tau,t)\right)^3 - \left(\bar\eta^\alpha_-(\tau,t)\right)^3\right\}\Bigg]. \end{aligned} \tag{2.23}$$

Finally, the 'fixed area' (i.e. fixed fermion number) constraint, eqn. (2.11), reads now

$$\int_0^\infty d\tau \sum_{\alpha=1,2}\left[\bar\eta^\alpha_+(\tau,t) - \bar\eta^\alpha_-(\tau,t)\right] = 0. \tag{2.24}$$

In the semiclassical limit, for fluctuations of the fermi surface that are entirely confined within the asymptotes, eqns. (2.19) – (2.23) define a set of two decoupled massless scalar fields. Moreover, the constraint eqn. (2.24) is also satisfied if it is satisfied for each individual value of α. Therefore, we may consistently ignore fluctuations on one side of the potential within the framework of the matrix model. The mapping to string theory is, however, a different story, as discussed in the previous section and as we shall see in detail in Sec. 4.

Before closing this section we mention that in terms of the incoming fields,

$$\bar\eta^\alpha_{+in}(t+\tau) \equiv \underset{t\to-\infty}{Lt.}\ \bar\eta^\alpha_+(\tau,t), \tag{2.25}$$

the hamiltonian, $H_{fluc.}$, and the constraint in eqn. (2.24) have the following simple expressions

$$H_{fluc.} = \frac{1}{4\pi}\int_{-\infty}^{+\infty} d\tau \sum_{\alpha=1,2}\left(\bar\eta^\alpha_{+in}(\tau)\right)^2, \tag{2.26}$$

$$\int_{-\infty}^{+\infty} d\tau \sum_{\alpha=1,2} \bar{\eta}^{\alpha}_{+in}(\tau) = 0. \tag{2.27}$$

We will use these expressions in Sec. 4.

3 The Leg-pole Transformation to String Theory

In this section we will discuss the leg-pole transformation of the matrix model to string theory within the framework that completely ignores one side of the potential for fluctuations which are entirely confined within the asymptotes. This will set the stage for the considerations of the next section.

It has been known for some time now that the tree-level scattering amplitudes for the matrix model scalar (discussed in the last section) do not exactly coincide with the tree-level scattering amplitudes for the tachyon in 2-dimensional string theory [1]. The difference can be understood in terms of a wavefunction renormalization and is a simple momentum-dependent phase factor for real momenta. In coordinate space this renormalization factor relates the Hilbert space of the matrix model to that of the string theory by a nonlocal transform of the states [2]. Denoting the tachyon field of 2-dimensional string theory by $\mathcal{T}(x,t)$, (x,t) being space-time labels, this relationship can be expressed as

$$\mathcal{T}_{in}(x^+) = \int_{-\infty}^{+\infty} d\tau \; f\left(\left|\frac{\mu}{2}\right|^{\frac{1}{2}} e^{\tau - x^+} \right) \bar{\eta}_{+in}(\tau), \tag{3.1}$$

$$\mathcal{T}_{out}(x^-) = -\int_{-\infty}^{+\infty} d\tau \; f\left(\left|\frac{\mu}{2}\right|^{\frac{1}{2}} e^{-\tau + x^-} \right) \bar{\eta}_{-out}(\tau), \tag{3.2}$$

where $x^\pm \equiv (t \pm x)$ and the '*in*' and '*out*' refer, as usual, to the asymptotic fields obtained in the limits $t \to -\infty$ and $t \to +\infty$ respectively. In both cases x is taken to be large positive, keeping respectively x^+ and x^- fixed. The function f is given by

$$f(\sigma) \equiv \frac{1}{2\sqrt{\pi}} J_0 \left(2 \left(\frac{2}{\pi}\right)^{\frac{1}{8}} \sqrt{\sigma} \right), \quad \sigma \geq 0 \tag{3.3}$$

where J_0 is the standard Bessel function of order zero [13].

That the above nonlocal transformation is essential for extracting the space-time physics of 2-dimensional string theory from the matrix model has only recently become clear [2, 4]. It contains all of the space-time gravitational physics of the string theory, which is absent in the matrix model. Moreover, a detailed operator mapping from the latter to the former emerges only after the above nonlocal, and in general a nonlinear (in matrix model variables), transformation.

In general, the transform, which is valid even away from the asymptotic space-time region of eqns. (3.1) and (3.2), is a complicated nonlinear and nonlocal combination of the matrix model scalar [14]. The general mapping may be written as

$$\begin{aligned}\mathcal{T}(x,t) \;\equiv\; & \int dp\ dq\ G_1(x;p,q)\delta\mathcal{U}(p,q,t) \\ & +\frac{1}{2}\int dp\ dq \int dp'\ dq'\ G_2(x;p,q;p',q')\delta\mathcal{U}(p,q,t)\delta\mathcal{U}(p',q',t) \\ & +\cdots \end{aligned} \tag{3.4}$$

where the dots stand for terms of higher order in $\delta\mathcal{U}$. The kernels G_1, G_2, etc. that are necessary for $\mathcal{T}(x,t)$ to satisfy the tachyon β-function equation of string theory and the canonical commutation relations of a scalar field in flat space, linear dilaton background are known [14] (upto corrections of order xe^{-4x}).

For the purposes of computing tree level tachyon scattering amplitudes only the asymptotic form of the kernels is relevant, since the corrections drop out at large +ve values of x, as argued in [2, 14]. Since in this section we are ignoring one side of the potential, we shall assume that $\delta\mathcal{U}(p,q,t) = 0$ for $q > 0$ and restrict our attention to only -ve values of q. The asymptotic form of the mapping is, then, given by

$$\mathcal{T}(x,t) = \int dp\ dq\ f(-qe^{-x})\delta\mathcal{U}(p,q,t) + O(xe^{-2x}). \tag{3.5}$$

Although this equation reproduces eqns. (3.1) and (3.2) for quadratic profiles, in this form it is valid for any arbitrary fluctuation, not necessarily of the quadratic profile form.

The scattering amplitudes may now be computed by shifting the entire t-dependence of $\delta\mathcal{U}(p,q,t)$ into the argument of the function f, using the equation of motion

$$(\partial_t + p\partial_q + q\partial_p)\delta\mathcal{U}(p,q,t) = 0. \tag{3.6}$$

According to this equation $\delta\mathcal{U}(p,q,t) = \delta\mathcal{U}(p',q',t')$, where $(p'\pm q')e^{\mp t'} = (p\pm q)^{\mp t}$. We use this in the integral in eqn. (3.5) to change variables from (p,q) to (p',q'), with t and t' as fixed parameters for the purposes of this change of variables. Under this change of variables the measure $(dp\ dq)$ and the fermi surface, defined by eqn. (2.3), are invariant. Therefore, making this change of variables in eqn. (3.5), we get

$$\mathcal{T}(x,t) = \int dp\ dq\ f(-Q(t)e^{-x})\delta\mathcal{U}(p,q,t_0) + O(xe^{-2x}), \tag{3.7}$$

where

$$Q(t) \equiv q\cosh(t-t_0) + p\sinh(t-t_0) \tag{3.8}$$

The right hand side of eqn. (3.7) can be proved to be independent of the parameter t_0, using eqn. (3.6), and shows that the fermi fluid fluctuation enters eqn. (3.5) only as a boundary condition.

Let us now assume that $\delta\mathcal{U}(p,q,t_0)$ is of the quadratic profile form and let us further use the t_0-independence of the r.h.s. of eqn. (3.7) to take $t_0 \to -\infty$. Under these assumptions $\delta\mathcal{U}(p,q,t_0)$ is parametrized by the single field $\bar{\eta}^1_{+in}(\tau) \equiv \eta(\tau)$ discussed in Sec. 2 (since $\delta\mathcal{U} = 0$ for $q > 0$). Using this and the formalism developed in Sec. 2 in eqn. (3.7) in the limit $t_0 \to -\infty$, we get

$$\mathcal{T}(x,t) = \int_{-\infty}^{+\infty} d\tau \int_0^{\eta(\tau)} d\epsilon \; f\left(\left|\frac{\mu}{2}\right|^{\frac{1}{2}} \left[e^{\tau - x^+} + \left(1 - \frac{\epsilon}{|\mu|}\right) e^{-\tau + x^-}\right]\right) + O(xe^{-2x}). \tag{3.9}$$

It is now trivial to write down expressions for the '*in*' and '*out*' fields from this equation. We get

$$\mathcal{T}_{in}(x^+) = \int_{-\infty}^{+\infty} d\tau \; \eta(\tau) f\left(\left|\frac{\mu}{2}\right|^{\frac{1}{2}} e^{\tau - x^+}\right), \tag{3.10}$$

$$\mathcal{T}_{out}(x^-) = \int_{-\infty}^{+\infty} d\tau \int_0^{\eta(\tau)} d\epsilon \; f\left(\left|\frac{\mu}{2}\right|^{\frac{1}{2}} \left(1 - \frac{\epsilon}{|\mu|}\right) e^{-\tau + x^-}\right). \tag{3.11}$$

The tree level scattering amplitudes may now easily be obtained by inverting eqn. (3.10) for $\eta(\tau)$,

$$\eta(\tau) = -4\pi \int_{-\infty}^{+\infty} dx^+ \; \mathcal{T}_{in}(x^+) \partial_\tau^2 f\left(\left|\frac{\mu}{2}\right|^{\frac{1}{2}} e^{\tau - x^+}\right), \tag{3.12}$$

and substituting in $\mathcal{T}_{out}$ after expanding it in a power series in $\eta(\tau)$. It is a simple exercise to check that the correct string tachyon scattering amplitudes in flat space, linear dilaton background are obtained in this way.

4 Transforming fluctuations on both sides of the potential

In the previous section we applied the leg-pole transformation to fermi surface fluctuations only on one side of the potential, ignoring the other side. As we have argued in Sec. 1, this procedure is inconsistent with the space-time gravitational physics that one wants to extract from the matrix model, unless one considers a modified potential in which one side is absent right from the start. Here we will consider the case in which the potential is unmodified. In that case, therefore, we must include fluctuations on both sides for a consistent space-time gravitational physics to emerge after the transformation.

Let us, then, consider the generic case of fluctuations on both sides of the potential (Fig. 3). As before, we will assume that the fluctuations are small and entirely confined within the asymptotes.

We must now decide as to what generalization of the leg-pole transform considered in the previous section (in which q was always -ve) should be taken for q

+ve. The symmetry between the two sides of the potential suggests that we try the following symmetrical leg-pole transform

$$\mathcal{T}(x,t) = \frac{1}{\sqrt{2}} \int dpdq \; f(2^{\frac{1}{4}}|q|e^{-x}) \delta\mathcal{U}(p,q,t) + O(xe^{-2x}), \tag{4.1}$$

where now $\delta u \neq 0$ for both $q < 0$ and $q > 0$. The overall factor of $\dfrac{1}{\sqrt{2}}$ and the extra factor of $2^{\frac{1}{4}}$ in the argument of the function f relative to the expression in eqn. (3.5) have been put there for convenience only. The appearance of $|q|$ in the argument of f is suggested by the symmetry of the potential.

We would now like to ask the question as to whether the field $\mathcal{T}(x,t)$ defined in eqn. (4.1) could reproduce string theory tachyon scattering amplitudes. To answer this question we need the analogues of eqns. (3.10) and (3.11) of the previous section in the present case. To derive these we proceed exactly as before. The analogue of eqn. (3.7) is

$$\mathcal{T}(x,t) = \frac{1}{\sqrt{2}} \int dpdq \; f(2^{\frac{1}{4}}|Q(t)|e^{-x}) \delta\mathcal{U}(p,q,t_0) + O(x\bar{e}^{2x}), \tag{4.2}$$

where $Q(t)$ is given, as before, by eqn. (3.8). Using the t_0-independence of eqn. (4.2) to take the limit $t_0 \to -\infty$ and assuming once again that $\delta\mathcal{U}(p,q,t_0)$ has the quadratic profile form on both sides of the potential, we get

$$\begin{aligned} \mathcal{T}(x,t) &= \frac{1}{2} \int_{-\infty}^{+\infty} d\tau \sum_{\alpha=1,2} \int_0^{\sqrt{2}\bar{\eta}^{\alpha}_{+in}(\tau)} d\epsilon \\ & \quad f\left(\left|\frac{\mu'}{2}\right|^{\frac{1}{2}} \left[e^{\tau - x^+} + \left(1 - \frac{\epsilon}{|\mu'|}\right) e^{-\tau + x^-} \right] \right) + O(x\bar{e}^{2x}) \end{aligned} \tag{4.3}$$

where $x^{\pm} \equiv t \pm x$ as before and $\mu' \equiv \sqrt{2}\mu$. We can now write down expressions for the '*in*' and '*out*' fields. They are

$$\mathcal{T}_{in}(x^+) = \int_{-\infty}^{+\infty} d\tau \left(\frac{1}{\sqrt{2}} \sum_{\alpha=1,2} \bar{\eta}^{\alpha}_{+in}(\tau) \right) f\left(\left|\frac{\mu'}{2}\right|^{\frac{1}{2}} e^{\tau - x^+} \right) \tag{4.4}$$

and

$$\mathcal{T}_{out}(x^-) = \frac{1}{2} \int_{-\infty}^{+\infty} d\tau \sum_{\alpha=1,2} \int_0^{\sqrt{2}\bar{\eta}^{\alpha}_{+in}(\tau)} d\epsilon f\left(\left|\frac{\mu'}{2}\right|^{\frac{1}{2}} \left(1 - \frac{\epsilon}{|\mu'|}\right) e^{-\tau + x^-} \right). \tag{4.5}$$

The form of eqns. (4.4) and (4.5) suggests that the matrix model variable whose leg-pole transform has the potential of reproducing the string theory physics be identified with the combination appearing in eqn. (4.4), namely,

$$\frac{1}{\sqrt{2}} \sum_{\alpha=1,2} \bar{\eta}^{\alpha}_{+in}(\tau) \equiv \phi(\tau). \tag{4.6}$$

However, this immediately poses a problem, since the fields $\bar{\eta}^1_{+in}(\tau)$ and $\bar{\eta}^2_{+in}(\tau)$ appear individually in eqn. (4.5) and not as the sum $\phi(\tau)$. To see the implication of this, let us introduce the other combination of $\bar{\eta}^1_{+in}(\tau)$ and $\bar{\eta}^2_{+in}(\tau)$, namely,

$$\frac{1}{\sqrt{2}}\left(\bar{\eta}^1_{+in}(\tau)-\bar{\eta}^2_{+in}(\tau)\right)\equiv\Delta(\tau). \tag{4.7}$$

In terms of the variables $\phi(\tau)$ and $\Delta(\tau)$, the Hamiltonian, eqn. (2.26), and the 'fixed area' constraint, eqn. (2.27), are

$$H_{fluc.}=\frac{1}{4\pi}\int_{-\infty}^{+\infty}d\tau\left(\phi^2(\tau)+\Delta^2(\tau)\right) \tag{4.8}$$

and

$$\int_{-\infty}^{+\infty}d\tau\ \phi(\tau)=0. \tag{4.9}$$

Note that there is no constraint on $\Delta(\tau)$.

Equations (4.4) and (4.5) may now be recast into the following form

$$\mathcal{T}_{in}(x^+)=\int_{-\infty}^{+\infty}d\tau\ \phi(\tau)f\left(\left|\frac{\mu'}{2}\right|^{\frac{1}{2}}e^{\tau-x^+}\right), \tag{4.10}$$

$$\begin{aligned}\mathcal{T}_{out}(x^-) \;=\; & \frac{1}{2}\int_{-\infty}^{+\infty}d\tau\Bigg[\int_0^{\phi(\tau)+\Delta(\tau)}d\epsilon f\left(\left|\frac{\mu'}{2}\right|^{\frac{1}{2}}\left(1-\frac{\epsilon}{|\mu'|}\right)e^{-\tau+x^-}\right)\\ & +\int_0^{\phi(\tau)-\Delta(\tau)}d\epsilon f\left(\left|\frac{\mu'}{2}\right|^{\frac{1}{2}}\left(1-\frac{\epsilon}{|\mu'|}\right)e^{-\tau+x^-}\right)\Bigg].\end{aligned} \tag{4.11}$$

Comparing these two equations with eqns. (3.10) and (3.11) of the previous section, we see immediately that in the simple case of $\Delta(\tau)=0$ we recover the tree level tachyon scattering amplitudes of string theory in the background of flat space, linear dilaton (except for a rescaling of the string coupling, $\mu\to\sqrt{2}\mu\equiv\mu'$). Thus, <u>in the present framework these results of flat background emerge only when the fluctuations of the fermi surface on the two sides of the potential are identical</u>.

Equations (4.10) and (4.11), of course, describe the generic situation of different fluctuations on the two sides and it is natural to ask whether these equations have a sensible space-time interpretation for $\Delta(\tau)\neq 0$. To investigate this question, let us for now assume that $\Delta(\tau)$ is small, so that we may Taylor expand eqn. (4.11) around $\Delta(\tau)=0$. Retaining only upto the first nontrivial term in $\Delta(\tau)$, we get

$$\mathcal{T}_{out}(x^-)=\mathcal{T}^{(0)}_{out}(x^-)+\mathcal{T}^{(1)}_{out}(x^-)+\cdots \tag{4.12}$$

where

$$\mathcal{T}^{(0)}_{out}(x^-) \equiv \int_{-\infty}^{+\infty} d\tau \int_0^{\phi(\tau)} d\epsilon\ f\left(\left|\frac{\mu'}{2}\right|^{\frac{1}{2}}\left(1-\frac{\epsilon}{|\mu'|}\right)e^{-\tau+x^-}\right), \tag{4.13}$$

$$\mathcal{T}^{(1)}_{out}(x^-) \equiv -\frac{1}{2|2\mu'|^{\frac{1}{2}}}\int_{-\infty}^{+\infty} d\tau\ \Delta^2(\tau)\ e^{-\tau+x^-}\ \ f'\left(\left|\frac{\mu'}{2}\right|^{\frac{1}{2}}\left(1-\frac{\phi(\tau)}{|\mu'|}\right)e^{-\tau+x^-}\right) \tag{4.14}$$

and the dots in eqn. (4.12) indicate $O(\Delta^4)$ terms. The 'prime' on the function f in eqn. (4.14) denotes a derivative with respect to its argument. The first term in the expansion is the one that gives the flat space results. It is the second term that we would now like to focus on.

Notice that the second term in (4.12) is not zero <u>even when $\phi(\tau)$ is set to zero</u>. Since we are already committed to interpreting the transform of $\phi(\tau)$ as the string theory tachyon, we are forced to interpret this extra contribution at $\phi = 0$ to $\mathcal{T}_{out}$ coming from the second term in eqn. (4.12), as a new contribution to the tachyon background, dynamically generated by a nonzero value of Δ. For such an interpretation to be self-consistent, however, we should find additional contributions to the amplitudes for the bulk scattering of tachyons coming from this extra contribution to the tachyon background. In eqn. (4.12) there are indeed extra contributions to tachyon scattering amplitudes, since the additional Δ-dependent term depends on ϕ also. It is these extra contributions to the tachyon scattering amplitudes that we would now like to study in detail.

Let us Taylor expand (4.14) in powers of ϕ and retain terms upto linear order only. This is because we want to focus here only on $1 \to 1$ scattering off the backgrounds.

We get,

$$\begin{aligned}\mathcal{T}^{(1)}_{out}(x^-) &= -\frac{1}{2|2\mu'|^{\frac{1}{2}}}\int_{-\infty}^{+\infty} d\tau\ \Delta^2(\tau)e^{-\tau+x^-} \\ &\left[f'\left(\left|\frac{\mu'}{2}\right|^{\frac{1}{2}}e^{-\tau+x^-}\right) - \frac{\phi(\tau)e^{-\tau+x^-}}{|2\mu'|^{\frac{1}{2}}}f''\left(\left|\frac{\mu'}{2}\right|^{\frac{1}{2}}e^{-\tau+x^-}\right)\right] \\ &+O(\phi^2).\end{aligned} \tag{4.15}$$

In what follows we will focus on the $1 \to 1$ bulk scattering amplitudes at early times ($x^- \to -\infty$). We will also assume that the incoming tachyon wavefunction $\mathcal{T}_{in}(x^+)$ is a sufficiently localized (e.g. gaussian) wavepacket concentrated at a very large positive value of x^+. This is because it is only under this condition that the background created by a nonzero value of Δ will have a chance to develop before the tachyon scatters off it. Finally, we will also assume that $\Delta(\tau)$ is sufficiently localized for integrals like $\int_{-\infty}^{+\infty} d\tau\ \Delta^2(\tau)e^{-n\tau}$ to be finite.

Now, at early times ($x^- \to -\infty$), the first two leading contributions to $\mathcal{T}_{out}^{(1)}$ are given by

$$
\begin{aligned}
\mathcal{T}_{out}^{(1)}(x^-) \overset{x^- \to -\infty}{\sim} & - \frac{f'(0)}{2|2\mu'|^{\frac{1}{2}}} e^{x^-} \int_{-\infty}^{+\infty} d\tau\ \Delta^2(\tau) e^{-\tau} \\
& + \frac{f''(0)}{4|\mu'|} e^{2x^-} \int_{-\infty}^{+\infty} d\tau\ \Delta^2(\tau) e^{-2\tau} \phi(\tau) \\
& + O(\phi^2)
\end{aligned}
\tag{4.16}
$$

It is the first term in eqn. (4.16) that we would like to interpret as a new contribution to the tachyon background and it is the second term in eqn. (4.16) that we would like to show contains the amplitude for scattering off this background.

To cast the second term in eqn. (4.16) into a form from which we can simply read off the scattering amplitude, we need to invert eqn. (4.10), using eqn. (4.9), to express $\phi(\tau)$ in terms of $\mathcal{T}_{in}(x^+)$. The expression for $\phi(\tau)$ obtained in this way is identical to eqn. (3.12),

$$
\phi(\tau) = -4\pi \int_{-\infty}^{+\infty} dx^+\ \mathcal{T}_{in}(x^+) \partial_\tau^2\ f\left(\left| \frac{\mu'}{2} \right|^{\frac{1}{2}} e^{\tau - x^+} \right)
\tag{4.17}
$$

Using this, we get

$$
\begin{aligned}
\int_{-\infty}^{+\infty} & d\tau \quad \Delta^2(\tau) e^{-2\tau} \phi(\tau) \\
& = \quad -4\pi \int_{-\infty}^{+\infty} dx^+ \mathcal{T}_{in}(x^+) \int_{-\infty}^{+\infty} d\tau\ \Delta^2(\tau) e^{-2\tau} \partial_\tau^2 f \left(\left| \frac{\mu'}{2} \right|^{\frac{1}{2}} e^{\tau - x^+} \right) \\
& = \quad -4\pi f'(0) \left| \frac{\mu'}{2} \right|^{\frac{1}{2}} \int_{-\infty}^{+\infty} d\tau\ \Delta^2(\tau) e^{-\tau} \int_{-\infty}^{+\infty} dx^+\ e^{-x^+} \mathcal{T}_{in}(x^+) \\
& \qquad -4\pi f''(0) |\mu'| \int_{-\infty}^{+\infty} d\tau\ \Delta^2(\tau) \int_{-\infty}^{+\infty} dx^+\ e^{-2x^+} \mathcal{T}_{in}(x^+) \\
& \qquad + \cdots
\end{aligned}
$$

where in writing the last step we have used the assumption that $\mathcal{T}_{in}(x^+)$ is localized at a large +ve value of x^+ to expand the function f in a Taylor series, and retained only the first two terms. Putting all this in (4.16) we finally get

$$
\begin{aligned}
\mathcal{T}^{(1)}_{out}(x^-) \overset{x^- \to -\infty}{\sim} & - \frac{f'(0)}{2|2\mu'|^{\frac{1}{2}}} e^{x^-} \int_{-\infty}^{+\infty} d\tau \; \Delta^2(\tau) e^{-\tau} \\
& - \frac{f'(0)}{2\sqrt{2}|2\mu'|^{\frac{1}{2}}} e^{2x^-} \int_{-\infty}^{+\infty} d\tau \; \Delta^2(\tau) e^{-\tau} \\
& \qquad \times \int_{-\infty}^{+\infty} dx^+ \; e^{-x^+} \mathcal{T}_{in}(x^+) \\
& - \frac{1}{8\pi} e^{2x^-} \int_{-\infty}^{+\infty} d\tau \; \Delta^2(\tau) \int_{-\infty}^{+\infty} dx^+ \; e^{-2x^+} \mathcal{T}_{in}(x^+) \quad (4.18)
\end{aligned}
$$

In the above we have used that $f''(0) = \dfrac{1}{2\sqrt{2}\pi}$.

In the next section we will show that the second term in eqn. (4.18) arises as a result of tachyon scattering in the presence of a tachyon background given by the first term in eqn. (4.18). We will also show that the last term in eqn. (4.18) arises from tachyon scattering off a background metric, perturbed around flat space, and of the form given by the line element

$$
(ds)^2 = (1 - Me^{-4x})dt^2 - (1 + Me^{-4x})dx^2, \tag{4.19}
$$

where the parameter M of perturbation is given by

$$
M \equiv \frac{1}{4\pi} \int_{-\infty}^{+\infty} d\tau \; \Delta^2(\tau). \tag{4.20}
$$

This is identical to the contribution of $\Delta(\tau)$ to the total energy of the fluctuations, $H_{fluc.}$, eqn. (4.8).

5 Effective Low-energy String Theory

Let us now verify in detail the claims made above by a calculation in the known tachyon-dilaton-graviton effective field theory limit of 2-dimensional string theory [15].

The field equations of this effective field theory are

$$
\begin{aligned}
R_{\mu\nu} + 2\nabla_\mu \nabla_\nu \Phi - 2\partial_\mu T \partial_\nu T &= 0, \\
R + 4\nabla^2 \Phi - 4(\nabla\Phi)^2 - 2(\nabla T)^2 - 8T^2 &= 16, \\
\nabla^2 T - 2\nabla\Phi\nabla T - 4T - 2\sqrt{2}T^2 &= 0.
\end{aligned} \tag{5.1}
$$

We will work in the gauge in which the dilaton is always given by its linear form (at least in a local neighbourhood), namely,

$$
\Phi = -2x,
$$

and the metric has the diagonal form

$$g_{\mu\nu} = diag(g_{00}, g_{11}).$$

Perturbing around the flat background

$$g_{00} = 1 - g_0, \quad g_{11} = -(1 + g_1), \tag{5.2}$$

and retaining only upto linear order in tachyon fluctuations and upto the first nontrivial order in the tachyon backgrounds, we get the following equations for the metric and tachyon fluctuations:

$$\begin{aligned} \partial_x g_0 + 4g_1 &= 0, \\ (\partial_x + 4)g_1 &= 0, \\ \partial_t g_1 &= 0, \end{aligned} \tag{5.3}$$

and

$$\partial_+\partial_- S = -\frac{1}{4}g_0\partial_t^2 S - \frac{1}{4}g_1(\partial_x^2 + 4\partial_x - 12)S + \sqrt{2}e^{-2x}S_0 S, \tag{5.4}$$

where we have set $T = e^{-2x}S$ and S_0 is the tachyon background.

The eqns. (5.3) have the black-hole solution [16]

$$g_0 = g_1 = \tilde{M}e^{-4x}, \tag{5.5}$$

where $\tilde{M}$ is a constant of integration. Using this solution in eqn. (5.4) and integrating it to first order in the background S_0, with the boundary condition

$$S(x,t) \stackrel{t\to-\infty}{\longrightarrow} S_{in}(x^+),$$

we get

$$\begin{aligned} S_{out}(x^-) &= \frac{1}{\sqrt{2}}e^{2x^-}\int_{-\infty}^{+\infty} dx^+ \; e^{-x^+} \; \tilde{S}_0(x^-, x^+)S_{in}(x^+) \\ &\quad -\frac{\tilde{M}}{2}e^{2x^-}\int_{-\infty}^{+\infty} dx^+ e^{-2x^+} S_{in}(x^+). \end{aligned} \tag{5.6}$$

In this equation $S_{out}(x^-) = \underset{t\to+\infty}{Lt.} S(x,t)$ and

$$\tilde{S}_0(x^-, x^+) \equiv 2e^{-2x^-}\int_{-\infty}^{x^-} du^- \; e^{u^-} \; S_0(u^-, x^+).$$

For S_0 given by the first term in eqn. (4.18), we get

$$\tilde{S}_0(x^-, x^+) = -\frac{f'(0)}{2|2\mu'|^{\frac{1}{2}}}\int_{-\infty}^{+\infty} d\tau \; \Delta^2(\tau)e^{-\tau}$$

We thus see that eqn. (5.6) is identical to the last two terms of eqn. (4.18), provided we identify S with $\mathcal{T}$, S_0 with the first term in eqn. (4.18) and $\tilde{M}$ with M given in eqn. (4.20). This proves the claims made in the previous section.

To end this section we mention that the full contribution (at order Δ^2, but to all orders in e^{x^-}) to what we would like to identify as a perturbation to the tachyon background, given by the first term in eqn. (4.15), is

$$\mathcal{T}_{out}^{bgd.}(x^-) \equiv -\frac{1}{2|2\mu'|^{\frac{1}{2}}} \int_{-\infty}^{+\infty} d\tau \; \Delta^2(\tau) e^{-\tau+x^-} f'\left(\left|\frac{\mu'}{2}\right|^{\frac{1}{2}} e^{-\tau+x^-} \right) + O(\Delta^4) \quad (5.7)$$

One can easily check that it is a consistent interpretation of this contribution to $\mathcal{T}_{out}$, since the term linear in $\phi(\tau)$ in eqn. (4.15) contains a contribution that equals the amplitude of the tachyon to scatter off this background, as calculated in the effective field theory above.

We also mention here that the full contribution to the $1 \to 1$ background scattering of the tachyon (including all the nonleading terms in e^{x^-} as well as e^{-x^+} for $x^- \to -\infty$ and $x^+ \to +\infty$ respectively), as given by the second term in eqn. (4.15), is

$$- \frac{\pi}{|\mu'|} \int_{-\infty}^{+\infty} dx^+ \; \mathcal{T}_{in}(x^+) \int_{-\infty}^{+\infty} d\tau \; \Delta^2(\tau) e^{-2\tau+2x^-}$$
$$\partial_\tau^2 f \left(\left|\frac{\mu'}{2}\right|^{\frac{1}{2}} e^{\tau-x^+} \right) f'' \left(\left|\frac{\mu'}{2}\right|^{\frac{1}{2}} e^{-\tau+x^-} \right) .$$

We have used eqn. (4.18) in arriving at this expression. Assuming, as before, that $\mathcal{T}_{in}(x^+)$ is a localized wavepacket concentrated around a large positive value of x^+, and taking the early time limit, $x^- \to -\infty$, we can rewrite the above as a double series in e^{-x^+} and e^{x^-} by expanding both the Bessel functions around the origin. The final result can be written in the compact form

$$\sum_{m,n=1}^{\infty} C_{mn} e^{(n+1)x^-} \int_{-\infty}^{+\infty} dx^+ e^{-(m+1)x^+} \mathcal{T}_{in}(x^+) \quad (5.8)$$

where

$$\begin{aligned} C_{mn} \;\; &\equiv \;\; \frac{(-)^{m+n+1}}{4\pi} \left| \frac{\mu'}{\sqrt{2\pi}} \right|^{\frac{m+n}{2}-1} \frac{1}{(m!)^2 (n+1)!(n-1)!} \times \\ &\quad \times \int_{-\infty}^{+\infty} d\tau \; \Delta^2(\tau) e^{(m-n)\tau} \end{aligned} \quad (5.9)$$

We have omitted the $m = 0$ (arbitrary n) term from the sum in eqn. (5.8) which, as we have already seen, corresponds to a perturbation of the tachyon background. The $m = n = 1$ term is due to the metric perturbation while the other terms have a structure that is consistent with their interpretation as scattering off backgrounds corresponding to the higher discrete states of 2-dimensional string theory. Unfortunately we do not have an effective theory to guide us here. But perhaps one could turn this situation around and use eqn. (5.8) to learn something about such a theory. However, we shall not pursue this interesting problem here any further.

6 Quantum gravity, Black hole physics and Nonperturbative String Theory

Our discussion has so far been classical. By quantizing $\phi(\tau)$ in the standard manner, and treating $\Delta(\tau)$ as a classical variable, we can convert our classical statements into statements about tree level scattering amplitudes in string theory in the presence of space-time background fields. As long as the fermi surface fluctuations are confined within the asymptotes and the string coupling is small (so that tunnelling amplitudes are supressed), the two sides of the matrix model potential are decoupled from each other and so are the fields $\phi(\tau)$ and $\Delta(\tau)$. It is then consistent to quantize $\phi(\tau)$ while treating $\Delta(\tau)$ as a classical variable. If the string coupling is large, however, there will be appreciable tunnelling from one side of the potential to the other. In that case we can no longer treat the two sides of the potential as decoupled from each other and so we must quantize the entire system. Thus, the backgrounds are dynamical fields and there are really no parameters in the theory, as is expected in a string theory. (Note that even the string coupling, $|\mu|^{-1}$, is not a parameter in this model since it can be absorbed into the constant part of the dilaton by shifting x.) In particular, this means that we must quantize the background geometry. It is satisfying that this scenario emerges in the present framework since one does indeed expect the classical picture of space-time geometry to break down in strong coupling string theory. It is to be hoped that the results of this work, together with the full power of the nonperturbative solution of the $c = 1$ matrix model, will give us a concrete handle on this question.

The picture of two decoupled sides of the potential breaks down even for small values of the string coupling, when one is considering certain special fermi fluid configurations. These are the configurations in which part of the fluid crosses the asymptotes (Figure 2). This is because for such configurations some of the fluid eventually goes across to the other side. In the framework in which one side of the potential is completely ignored in the leg-pole transform, such configurations lead to a nonperturbative consistency problem [17], basically because some of the fluid is lost to the other side. In the framework considered in the present paper, in which both sides of the potential are always taken into account, no fluid can ever get lost. Thus the problem as posed in [17] does not occur here. Nevertheless, it does not follow that there is no problem for such configurations, and we need to establish the consistency of our framework for such configurations. This situation is made complicated by the fact that there is a dynamical rearrangement of the fluid between the two sides, and so it is not clear that it is consistent to treat $\phi(\tau)$ quantum mechanically while $\Delta(\tau)$ is treated classically. It is, in fact, quite conceivable that such configurations are accompanied by a change of the background fields and, in particular, of the background geometry, from the given initial state to some other final state. If this is true, then we would have discovered a way of dynamically changing backgrounds in our framework. It is possible, then, to eventually hope to get a handle on the formation and evaporation of a black-hole in this exactly solvable model. Although we are far from realizing this at the moment, we are encouraged by the fact that our framework contains both the background metric perturbation as well as some as yet little understood large-field classical situations.

Finally, we would like to end with the following remark. The presence of background perturbations corresponding to the discrete states of 2-dimensional string theory, in the framework which retains both sides of the potential and treats them consistently, is a strong indication that a consistent nonperturbative definition of 2-dimensional string theory must include both sides of the potential. To use the full nonperturbative power of the matrix model to address some of the basic questions of string theory and quantum gravity, it is clearly of utmost importance to discover this nonperturbative formulation of the string theory in terms of the matrix model.

Acknowledgement: S.W. and G.M. would like to acknowledge the hospitality of the theory division of CERN where most of this work was done.

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