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Point Mass Geometries, Spectral Flow and $\text{AdS}_3\text{-CFT}_2$ Correspondence

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ABSTRACT

We discuss, in terms of the $\text{AdS}_3\text{-CFT}_2$ correspondence, a one-parameter family of (asymptotically AdS_3) conical geometries which are generated by point masses and interpolate between AdS_3 and BTZ spacetimes. We show that these correspond to spectral flow in $\mathcal{N} = (4, 4)$ SCFT₂ which interpolate between NS and R sectors. Our method involves representing the conical spaces as solutions of three-dimensional supergravity based on the supergroup $SU(1, 1|2) \times SU(1, 1|2)$. The boundary CFT we use is based on the D1/D5 system. The correspondence includes comparing the Euclidean free energies between supergravity and SCFT for the family of conical spaces including BTZ black holes.

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1 Introduction and Summary

The recently conjectured $\text{AdS}_{d+1}/\text{CFT}_d$ correspondence has led to a number of remarkable predictions for $\mathcal{N} = 4$ Yang-Mills theory in four dimensions. Considering the fact that such a correspondence potentially *defines* quantum gravity in these backgrounds, one should be able to describe various interesting dynamical phenomena in gravity as well, including, *e.g.* black hole formation. The case of $d = 3$ is particularly attractive because both AdS_3 and CFT_2 are more tractable than their higher dimensional counterparts and has nevertheless a rich physics content.

A particularly interesting class of solutions of three-dimensional gravity with $\Lambda < 0$ are conical spacetimes [1] which are generated by a point mass at the origin (see equation (5)). If the mass exceeds a certain critical value, the conical spacetime becomes a BTZ black hole. Indeed, it was observed by [4] Peleg and Steif that in the context of a gravitational collapse of a shell of variable rest mass, the resulting spacetime is conical unless the rest mass exceeds a critical value. At this value there is a critical phase transition and a BTZ black hole is formed.

It was pointed out by Brown and Henneaux [2] that the conical spacetimes are “asymptotically AdS_3 ” and they belong to the class of geometries whose “asymptotic isometry group” is $\text{Virasoro} \times \text{Virasoro}$. Therefore it seems reasonable to expect, in the light of AdS/CFT correspondence, that these spaces should correspond to boundary conformal field theories. It is already known [3] that the “end-points” of the one-parameter family of metrics ($\gamma = 0, 1$), namely the pure AdS_3 and the zero-mass BTZ black hole (see below equation (8)), correspond to the NS and R sectors of a $\mathcal{N} = (1, 1)$ superconformal field theory (SCFT).

If we go back to the origin of the AdS_3 - CFT_2 correspondence [6], namely to the near-horizon limit of the D1/D5 system, the natural supersymmetry of the SCFT is $\mathcal{N} = (4, 4)$. (This corresponds to a supergravity based on the group $\mathcal{G} = \text{SU}(1, 1|2) \times \text{SU}(1, 1|2)$ [9]). In terms of such a SCFT, there is a natural interpolation between the NS and R sectors called spectral flow [10]. It was conjectured some time back [12] that the conical spaces should correspond to spectral flow, the defect angle being related to the parameter of the flow (see equation (21)). One evidence was that the ADM mass in supergravity matched

exactly with $L_0 + \bar{L}_0$ in the SCFT (see equation (26)) for all values of the interpolating parameter. This connection has also been mentioned recently in [13].

In incorporating conical spaces in the ambit of $\text{AdS}_3\text{-CFT}_2$ correspondence, the first step is to realize the conical spaces as supersymmetric solutions of three-dimensional gravity, since the boundary theory mentioned above is supersymmetric. If one tries to generalize to conical spaces the work of [3] which embeds BTZ and AdS_3 in the framework of (1,1)-type¹ AdS_3 , the conical spaces turn out to be non-supersymmetric: the Killing spinors, in particular, are quasiperiodic and hence not globally defined. If, however, one tries to realize these as solutions of $SU(1, 1|2) \times SU(1, 1|2)$ supergravity, which is natural from the viewpoint of the boundary theory, the Killing spinors become globally defined. This construction also allows us to establish the main point of the bulk-to-boundary correspondence in this case: since the same supergroup is represented in the bulk theory and in the boundary theory, we can find the operator in the supergroup which deforms the value of the spectral parameter in the SCFT and the operator which changes the value of the defect angle in the supergravity. We find that the same operator causes the one-parameter flow in both cases, thus establishing the correspondence between the spectral flow and conical spaces.

As additional evidence for this correspondence we compute the Euclidean free energy of the family of spacetimes and compare with the corresponding quantity in SCFT. The BTZ free energy, which has already been discussed in [16, 17], is shown to be reproduced by SCFT based on the symmetric product $S_{Q_1 Q_5}(T^4)$ at high temperatures. The free energies of the conics are reproduced at low temperatures.

As an off-shoot of our construction we find a vertex operator in the SCFT which corresponds to creation of a point mass geometry in an asymptotically AdS_3 space. The scattering of such point masses (including the cases when they form a black hole [5]) is naturally represented in terms of correlation function of such vertex operators.

This paper is organized as follows. In section 2, we discuss the conical spaces as solutions of three-dimensional supergravity (with $\Lambda < 0$) based on $\mathcal{G} = SU(1, 1|2) \times SU(1, 1|2)$. In section 3 we make the identification with spectral flow in $\mathcal{N} = (4, 4)$ SCFT.

¹For the definition of (p, q) -type AdS_3 supergravity, please see [21]. Note the difference with the notation $\mathcal{N} = (p, q)$ which represents the number of supersymmetries in the left- and right-moving sectors respectively of the SCFT.

In section 4 we compute the Euclidean free energies of these spaces from supergravity. In section 5 we compute the free energies from the SCFT viewpoint and compare with the results of section 4. In the concluding section (Sec. 6) we discuss the issue of scattering of the point masses and black hole formation in terms of correlation functions in the SCFT.

2 Conical spaces as solutions of supergravity based on \mathcal{G}

The action for three-dimensional supergravity (with cosmological constant $\Lambda < 0$) based on $\mathcal{G} \equiv SU(1, 1|2) \times SU(1, 1|2)$ is as follows [14, 15]:

$$\begin{aligned}
S = & \frac{1}{16\pi G_N^{(3)}} \int d^3x \left[eR + \frac{2}{l^2} e \right. \\
& - \epsilon^{\mu\nu\rho} \bar{\psi}_\mu \mathcal{D}_\nu \psi_\rho - 8l\epsilon^{\mu\nu\rho} (A_\mu^i \partial_\nu A_\rho^i - \frac{4i\epsilon_{ijk}}{3} A_\mu^i A_\nu^j A_\rho^k) \\
& \left. - \epsilon^{\mu\nu\rho} \bar{\psi}'_\mu \mathcal{D}'_\nu \psi'_\rho + 8l\epsilon^{\mu\nu\rho} (A_\mu^i \partial_\nu A_\rho^i - \frac{4i\epsilon_{ijk}}{3} A_\mu^i A_\nu^j A_\rho^k) \right]
\end{aligned} \tag{1}$$

where $l^2 = -1/\Lambda$, $G_N^{(3)}$ is the three-dimensional Newton's constant, and $\mathcal{D}_\nu = \partial_\nu + \omega_{ab\nu}\gamma^{ab}/4 - e_{a\nu}\gamma^a/(2l) - 2A_\nu^i\sigma^i$ and $\mathcal{D}'_\nu = \partial_\nu + \omega_{ab\nu}\gamma^{ab}/4 + e_{a\nu}\gamma^a/(2l) - 2A_\nu^i\sigma^i$. The basic fields appearing in the lagrangian are the vierbein e_μ^a , ψ_ρ , A_μ^i , ψ'_μ and A_μ^i .

The same three-dimensional supergravity can be obtained from type IIB string theory compactified on $K = T^4 \times S^3$ (with constant flux on S^3). Recall that the near-horizon geometry of the D1/D5 system involves $K \times \text{AdS}_3$ whose (super)isometries are \mathcal{G} . Similarly the near-horizon geometry of the five-dimensional black hole is $K \times \text{BTZ}$ [16]. This implies in an obvious fashion that AdS_3 and BTZ are solutions of (1). Furthermore, the three-dimensional Newton's constant $G_N^{(3)}$ and the AdS radius l are given by the following string theoretic expressions:

$$\begin{aligned}
G_N^{(3)} &= \frac{4\pi^4 g_s^2}{V_4 l^3} \\
l^4 &= \frac{16\pi^4 g_s^2 Q_1 Q_5}{V_4}
\end{aligned} \tag{2}$$

where V_4 is the volume of T^4 and g_s is the string coupling (we are working in the units $\alpha' = 1$).

Conical spaces

As mentioned in the introduction, there is a one-parameter family of classical solutions of three-dimensional gravity (with $\Lambda < 0$) discovered by Deser and Jackiw which are asymptotically AdS₃ and interpolate between the AdS₃ and zero-mass BTZ solutions. These represent geometries around a point mass m , $0 < m < 1/(4G_N^{(3)})$ which create a conical singularity at the origin. The metric is given by

$$ds^2 = -dt^2\left(\gamma + \frac{r^2}{l^2}\right) + dr^2\left(\gamma + \frac{r^2}{l^2}\right)^{-1} + r^2 d\phi^2, \quad (3)$$

It is easy to see that there is a conical singularity at the origin $r = 0$ with defect angle

$$\Delta\phi = 2\pi(1 - \sqrt{\gamma}) \quad (4)$$

The parameter γ varies from 0 to 1, and is related to the point mass m at the origin:

$$m = \frac{1 - \sqrt{\gamma}}{4G_N^{(3)}} \quad (5)$$

We will denote these spaces by X_γ . Note that the mass m is the DeserJackiw mass which is different from the BTZ definition of mass, denoted by M , which is given by

$$M = -\gamma/(8G_N^{(3)}) \quad (6)$$

BTZ and AdS₃ as end-points:

Recall that the metric of pure AdS₃ is given by

$$ds^2 = -dt^2\left(1 + \frac{r^2}{l^2}\right) + dr^2\left(1 + \frac{r^2}{l^2}\right)^{-1} + r^2 d\phi^2 \quad (7)$$

while the metric for BTZ black holes is given by

$$ds^2 = -\left[\frac{r^2}{l^2} - M + \left(\frac{J}{2r}\right)^2\right] dt^2 + \left[\frac{r^2}{l^2} - M + \left(\frac{J}{2r}\right)^2\right]^{-1} dr^2 + r^2\left(\frac{-J}{2r^2} dt + d\phi\right)^2 \quad (8)$$

where M and J refer to the mass and the angular momentum of the BTZ black hole ($J \leq M$).

It is clear that the space X_γ becomes AdS₃ for $\gamma = 1$, whereas for $\gamma = 0$ it becomes BTZ with $M = 0$ ($M = 0 \Rightarrow J = 0$).

We have already remarked that AdS₃ and BTZ are supersymmetric solutions of (1). We will now show that the entire family of spaces X_γ can be obtained as supersymmetric solutions of (1).

It is known [19]² how to embed the conical spaces (3) as supersymmetric solutions in (2, 0)-type AdS₃ supergravity. The solution looks like:

$$A_\mu dx^\mu = -\frac{l}{2}\gamma d\phi \quad (9)$$

where A_μ is the $U(1)$ gauge field appearing in the graviton supermultiplet.

The embedding into $\mathcal{N} = (4, 4)$ is a straightforward extension of the above:

$$A_\mu^3 dx^\mu = -\frac{l}{2}\gamma d\phi, \quad A_\mu^\pm dx^\mu = 0 \quad (10)$$

where the superscripts 3, \pm refer to the R-parity group $SU(2)$.

It can be explicitly checked that the equations of motion following from (1), as well as the Killing spinor equations ensuring supersymmetry are satisfied by the above solution. The equations of motion reduce to those of the $U(1)$ problem [19] with the ansatz (10). As far as the Killing spinors are concerned, the solution in our case is a doublet constructed out of the solution of the $U(1)$ problem. The Killing spinor equation in the present case is

$$\mathcal{D}_\nu \epsilon = \left[\partial_\nu + \frac{\omega_{ab\nu}\gamma^{ab}}{4} - \frac{1}{2l}e_{a\nu}\gamma^a + \frac{i}{l}A_\nu^3 \right] \epsilon = 0 \quad (11)$$

The solution is given by

$$\epsilon = \begin{pmatrix} \epsilon_1 \\ \epsilon_1^* \end{pmatrix}, \quad (12)$$

where ϵ_1^* is the complex conjugate of ϵ_1 , the latter being the Killing spinor in the $U(1)$ problem [19].

Connection with (1, 1)-type AdS₃ supergravity:

We emphasize that extended supersymmetry is quite essential to construct the conical spaces as supersymmetric solutions. The Killing spinors have holonomies under both the spin connection and the gauge connection. Under either one of them the spinors are

²We thank P.Townsend for pointing out reference [19] to us.

quasiperiodic, corresponding to the fact that the (1, 1)-type Killing spinors constructed using the formulae of [3], which see only the spin connection, are quasiperiodic. With extended supersymmetry (*i.e.*, (p, q) -type AdS₃ supergravity, with either p or q or both greater than 1) there is a $U(1)$ gauge field. The holonomy under this gauge connection cancels that under the spin connection, making the Killing spinors periodic and hence globally defined. The NS or R boundary conditions of the spinors of [3] for $\gamma = 1, 0$ now refer to the condition that the holonomy under the gauge connection, represented by the Wilson line

$$\mathcal{W} = \text{Tr} \exp\left[\frac{i}{\gamma} \int A_\mu dx^\mu\right], \quad (13)$$

is $-\mathbf{1}$ or $\mathbf{1}$ respectively.

Conics as one-parameter family of gauge transforms

It is clear from the above discussion that the family of solutions can be parameterized uniquely by the value of the Wilson line \mathcal{W} (the metric is fixed once we specify this). We can, therefore, change from the classical solution $X_{\gamma=0}$ to X_γ by making a “gauge transformation”

$$A_\phi \rightarrow U^{-1}A_\phi U + iU^{-1}\partial_\phi U, \quad U(t, r, \phi) = \exp[i\zeta(\phi)T^3], \quad \zeta(\phi) = \phi\sqrt{\gamma} \quad (14)$$

The reason why the Wilson line \mathcal{W} changes is that U is not single-valued. In other words, (14) describes an improper gauge transformation.

The equations of motion demand that the metric changes appropriately from its value at $\gamma = 0$ to the value at γ . It is interesting to note that this change in the metric can also be understood as an (improper) $SL(2, R) \times SL(2, R)$ gauge transformation. Following [20, 21] we combine the dreibein e_μ^M and the spin connection $\omega_{N\mu}^M$ into an $SL(2, R) \times SL(2, R)$ vector potential \mathcal{A}_\pm . For the metric (3), it turns out to be

$$\mathcal{A}_\pm \equiv \mathcal{A}_{\mu\pm}^a T^a dx^\mu = \frac{1}{2} \begin{pmatrix} \pm d\rho & -\sqrt{\gamma}e^{\mp\rho} dt_\pm \\ -\sqrt{\gamma}e^{\pm\rho} dt_\pm & \mp d\rho \end{pmatrix} \quad (15)$$

where $r = \gamma \sinh \rho$ and $t_\pm = t \pm \phi$. One can show that this can be transformed to the value at $\gamma = 0$ by the following $SL(2, R) \times SL(2, R)$ gauge transformation

$$\mathcal{A}_\pm \rightarrow V_\pm^{-1} \mathcal{A}_\pm V_\pm + V_\pm^{-1} dV_\pm \quad (16)$$

$$\begin{aligned}
V_{\pm} &= V_{1\pm}V_{2\pm} \\
V_{1\pm} &\equiv \begin{pmatrix} e^{\mp\rho/2} & 0 \\ 0 & e^{\pm\rho/2} \end{pmatrix} \\
V_{2\pm} &\equiv \exp[(1/2)\zeta(\phi)\sigma_1]
\end{aligned}$$

$\zeta(\phi)$ being the same function as in (14). Here σ_1 is the Pauli matrix.

As emphasized before, the gauge transformation of the metric can be regarded as a consequence of the gauge transformation (14) and the equations of motion. For comparison with the boundary CFT, we note that the gauge transformation (14) is implemented in the quantum theory by the operator

$$\hat{U} = \exp[i\sqrt{\gamma}\phi\widehat{J}_3] \quad (17)$$

where \widehat{J}_a , $a = 1, 2, 3$ are generators of the $SU(2)$ part of $SU(1, 1|2)$.

3 Correspondence with spectral flow

We have already remarked that the Brown-Henneaux Virasoro algebra can be supersymmetrized [3] by embedding the AdS_3 or BTZ solutions in $\mathcal{N} = 1$ supergravity, and that the realizations of the superconformal algebra corresponding to the AdS_3 and BTZ solutions respectively map to the NS and R sector of the boundary conformal field theory. In the context of extended supersymmetry, as we have remarked above, the AdS_3 and BTZ solutions correspond to a Wilson line \mathcal{W} equal to $-\mathbf{1}$ or $\mathbf{1}$. In order to facilitate comparison, let us define the gauge-invariantized fermion

$$\tilde{\psi} \equiv U[A]\psi, \quad U[A] = \exp\left[\frac{i}{l} \int_{P_0}^P A\right] \quad (18)$$

The AdS_3 and BTZ solution correspond to antiperiodic or periodic $\tilde{\psi}$ respectively. It is this $\tilde{\psi}$ of supergravity that maps to the fermion ψ of CFT.

Now, we know that there is a one-parameter flow, called spectral flow, between periodic and antiperiodic boundary conditions in the case of $\mathcal{N} = 4$ superconformal theory [10]. This corresponds to quasiperiodic boundary condition on ψ :

$$\psi(ze^{2\pi i}) = \exp(i\pi\eta)\psi(z) \quad (19)$$

Let us compare this flow with the one-parameter flow caused by the Wilson line (Eqs. (14) and (17)). We note that the gauge-invariantized fermion $\tilde{\psi}$ satisfies, under $\phi \rightarrow \phi + 2\pi$

$$\tilde{\psi}(ze^{2\pi i}, r \rightarrow \infty) = \exp(i\pi\sqrt{\gamma})\tilde{\psi}(z) \quad (20)$$

where $z = e^{i(t+\phi)}$. This suggests that the Hilbert space corresponding to spectral flow is the realization of boundary SCFT for the conical spaces, with the identification

$$\eta = \sqrt{\gamma} \quad (21)$$

The fact that this is the right correspondence follows by noting that spectral flow in the SCFT is defined in terms of the generator [10, 11]

$$\hat{U} = \exp[i\sqrt{\gamma}\phi\hat{J}_3] \quad (22)$$

where \hat{J}_3 is the $SU(2)$ R-parity current. Thus the $SU(2)$ generator for spectral flow (22) is the same as the $SU(2)$ generator (17) in the bulk that changes the conical defect angle. This is exactly as it should be for the proposed AdS/CFT correspondence to work.

The equality of (17) and (22) under the AdS/CFT correspondence proves our assertion.

A rather important consequence of the above correspondence is that we know exactly what state in the CFT corresponds to a point mass in the bulk that creates the conical singularity, namely it is the state

$$|\gamma\rangle \equiv \hat{U}|0\rangle, \quad (23)$$

This suggests a boundary representation in terms of CFT vertex operators of a point mass in the bulk and consequently a representation of their scattering in terms of CFT correlation in the sense of [7, 8]. We will make some more remarks on this in the concluding section. Related remarks also appear in [24] in a somewhat different context.

We note that we are parameterizing the flow in supergravity by the Wilson line only, by adopting the attitude that the metric gets fixed as a consequence of the equation of motion. For example, the ADM mass for the metric corresponding to the Wilson line (10) is

$$M = -\gamma/8G_N^{(3)} \quad (24)$$

The counterpart of this statement in CFT is that by performing a spectral flow along J^3 we automatically change the value of L_0 on the ground state from 0 to

$$L_0 = -\frac{c}{24}\eta^2 \quad (25)$$

Similar remarks apply to \bar{L}_0 . Using (2) and the fact that $c = 6Q_1Q_5$ (see Section 5) for the AdS₃ and BTZ points, it is easy to see that

$$M = \frac{L_0 + \bar{L}_0}{l}. \quad (26)$$

We see that (24) and (25) agree with (26) if we use $\gamma = \eta^2$, which is the same condition as in Eqn. (21). This provides additional support to our proposed correspondence. This matching has recently also been mentioned in [13].

We will continue to explore this correspondence in the rest of the paper in the context of the Euclidean free energy.

4 The Euclidean free energy of asymptotically AdS₃ solutions

In this section we will compute the Euclidean free energy of conical spaces (also of AdS₃ and BTZ) following the method of Gibbons and Hawking [22]. Let us recall that the free energy is given by

$$Z \equiv \exp[-\beta F(\alpha)] = \int \mathcal{D}_\alpha[\text{fields}] \exp[-S] \quad (27)$$

where S is now the Euclidean version of the action written in (1).

Boundary Conditions:

In the above equation, α denotes boundary conditions on the fields. As noted in [16], the boundary (corresponding to $r \rightarrow \infty$) is T^2 , coordinatized by ϕ and the Euclidean time $\tau \equiv -it$, with appropriate identifications. For the AdS₃ solution, the identifications are

$$(\tau, \phi) \equiv (\tau + \beta, \phi + 2\pi) \quad (28)$$

where β denotes inverse temperature and is arbitrary. For the BTZ solution (with mass M and Euclidean angular momentum $J_E = iJ$) the identifications are (dictated by smoothness of metric)

$$(\tau, \phi) \equiv (\tau + \beta_0, \phi + \Phi) \quad (29)$$

where β_0 and ϕ are given by

$$\beta_0 = \frac{2\pi r_+ l^2}{r_+^2 - r_-^2} \quad (30)$$

$$\begin{aligned}
\Phi &= \frac{2\pi|r_-|l^2}{r_+^2 - r_-^2} \\
r_+ &= \left[\frac{l^2 M}{2} \left(1 + \sqrt{1 + \frac{J_E^2}{M^2 l^2}} \right) \right]^{1/2} \\
r_- &= -i \left[\frac{l^2 M}{2} \left(\sqrt{1 + \frac{J_E^2}{M^2 l^2}} - 1 \right) \right]^{1/2}
\end{aligned}$$

In anticipation we note that implication of the identifications is that the boundary geometry is that of a torus with the modular parameter proportional to $\beta_0 + i\Phi$. Therefore the partition function of the dual CFT need to be evaluated on the torus with the modular parameter proportional to $\beta_0 + i\Phi$. The boundary condition on the fields is that the bosonic fields must have the same values at identified points whereas the fermion fields must have the same value upto a sign (periodic (P) or antiperiodic (A)). Since we are talking about identifications on a two-torus, the fermion boundary conditions can be

$$\alpha = (P, P), (P, A), (A, P), (A, A) \quad (31)$$

where the first entry denotes boundary condition in the ϕ direction and the second entry denotes boundary condition along the τ direction.

For the conical spaces, the identification is the same as for AdS_3 , except that the specification of the functional integral includes the Wilson line (equation (10)) as a part of the boundary condition.

Saddle points

We will evaluate (27) by finding saddle points of the action subject to specific boundary conditions. These are Euclidean versions of the classical solutions that we have described in the previous section. By virtue of the equation of motion $R = -6/l^2$, the Euclidean action S of a classical spacetime X is simply its volume times a constant. To be precise,

$$\begin{aligned}
S(X) &= \frac{1}{4\pi l^2 G_N^{(3)}} \text{Vol}(X) \\
\text{Vol}(X) &= \int_0^\beta d\tau \int_{r_0}^R dr \int_0^{2\pi} d\phi \sqrt{g}
\end{aligned} \quad (32)$$

The ranges of ϕ, τ follow from the identifications mentioned above. The lower limit r_0 of the r -integral is identically zero for AdS_3 and the conical spaces, whereas for BTZ it denotes the location of the horizon (the Euclidean section is defined only upto the horizon). The upper limit R is kept as an infrared regulator to make the volume finite. We will in practice only be interested in free energies relative to AdS_3 and the R -dependent divergent term will disappear from that calculation.

Free energy of BTZ:

We will now compute the free energy of BTZ relative to AdS_3 . The AdS_3 solution can be at any temperature β while the temperature of the black hole is fixed to be β_0 . To compare with the AdS_3 background one must adjust β so that the geometries of the two manifolds match at the hypersurface of radius R (in other words, we must use the *same* infrared regulator on all saddle points of the functional integral). This gives the following relation

$$\beta_0 = \beta \sqrt{\frac{1 + l^2/R^2}{1 - M^2 l^2/R^2}} \quad (33)$$

$$\begin{aligned} S(BTZ) - S(AdS_3) &= \frac{1}{4\pi l^2 G_N^{(3)}} \left[\int_{BTZ} d^3x \sqrt{g} - \int_{AdS_3} d^3x \sqrt{g} \right] \quad (34) \\ &= \frac{1}{4\pi l^2 G_N^{(3)}} \left[\pi \beta_0 (R^2 - r_+^2) - \pi \beta R^2 \right] \end{aligned}$$

Substituting the value of β_0 in terms of β and taking the limit $R \rightarrow \infty$ we obtain

$$S(BTZ) - S(AdS_3) = \frac{1}{4\pi l^2 G_N^{(3)}} \left[\pi \beta_0 l^2 - \pi^2 r_+ l^2 \right] \quad (35)$$

For convenience let us define the left and right temperatures [16] as

$$\begin{aligned} \beta_+ &= \frac{2\pi l^2}{r_+ + r_-} \quad (36) \\ \beta_- = \bar{\beta}_+ &= \frac{2\pi l^2}{r_+ - r_-} \end{aligned}$$

Using these variables the difference in the action becomes

$$S(BTZ) - S(AdS_3) = \frac{1}{4\pi l^2 G_N^{(3)}} \left[\frac{\pi}{2} (\beta_+ + \beta_-) l^2 - \pi^3 l^4 \left(\frac{1}{\beta_+} + \frac{1}{\beta_-} \right) \right] \quad (37)$$

It is easily seen that the BTZ black hole dominates when $(1/\beta_+ + 1/\beta_-)$ is much larger than $(\beta_+ + \beta_-)$ and vice versa. Thus at high temperatures we can ignore the first term in the above equation.

$$S(BTZ) - S(AdS_3) = -\frac{\pi^2 l^2}{4G_N^{(3)}} \left(\frac{1}{\beta_+} + \frac{1}{\beta_-} \right) \quad (38)$$

Using equation (2), the high temperature partition function of BTZ is, therefore, given by

$$-\ln Z = \pi^2 Q_1 Q_5 l \left(\frac{1}{\beta_+} + \frac{1}{\beta_-} \right) \quad (39)$$

Free energy of conical spaces

We denote the conical spaces by X_γ . The volume of X_γ is given by

$$\text{Vol}(X_\gamma) = \int_0^R r dr \int_0^{\beta_\gamma} d\tau \int_0^{2\pi} d\phi = \pi \beta_\gamma R^2 \quad (40)$$

Once again, β_γ is determined in terms of β by the requirement that the hypersurface $r = R$ that acts as an infrared regulator has the same 2-geometry as the corresponding surface in AdS_3 . This gives:

$$\beta_g \sqrt{R^2/l^2 + \gamma} = \beta \sqrt{R^2/l^2 + 1} \quad (41)$$

Using this, it is easy to find

$$\text{Vol}(X_\gamma) - \text{Vol}(AdS_3) = \pi \beta \frac{l^2}{2} (1 - \gamma) \quad (42)$$

This leads to, by (32), the following expression for the Euclidean action

$$S(X_\gamma) - S(AdS_3) = \frac{\beta}{8G_N^{(3)}} (1 - \gamma) \quad (43)$$

In the next section we will compare with the free energy computed from the boundary CFT. For that comparison it will turn out to be more appropriate to consider as reference spacetime the BTZ black hole with $J = M = 0$ (which is simply the space X_0 , also denoted BTZ_0):

$$S(X_\gamma) - S(BTZ_0) = -\ln Z(X_\gamma) - (-\ln Z(X_0)) = -\frac{\beta}{8G_N^{(3)}} \gamma \quad (44)$$

5 The partition function from the CFT

The aim of this section is to calculate the partition function of the (4, 4) CFT on the orbifold $T^{4Q_1Q_5}/S(Q_1Q_5)$. The partition function will depend of the boundary conditions of the fermions of the CFT. Different bulk geometries will induce different boundary conditions for the fermions of the CFT. We will first calculate the partition function when the bulk geometry is that of the BTZ black hole.

CFT partition function corresponding to BTZ

The fermions of the CFT are periodic along the angular coordinate of the cylinder if the bulk geometry is that of the BTZ black hole. This can be seen by observing that the zero mass BTZ black hole admits killing vectors which are periodic along the angular coordinate [3]. Therefore the zero mass BTZ black hole correspond to the Ramond sector of the CFT. The general case of the BTZ black hole with mass and angular momentum correspond to excited states of the CFT over the Ramond vacuum with

$$\begin{aligned} L_0 + \bar{L}_0 &= Ml \\ L_0 - \bar{L}_0 &= J_E \end{aligned} \tag{45}$$

where M and J_E are the mass and the (Euclidean) angular momentum of the BTZ black hole. Therefore the partition function of the BTZ black hole should correspond to

$$Z = \text{Tr}_R(e^{2\pi i\tau L_0} e^{2\pi i\bar{\tau}\bar{L}_0}) \tag{46}$$

The Hilbert space of the CFT on the orbifold $T^{4Q_1Q_5}/S(Q_1Q_5)$ can be decomposed into twisted sectors labeled by the conjugacy classes of the permutation group $S(Q_1Q_5)$. The conjugacy classes of the permutation group consists of cyclic groups of various lengths. The various conjugacy classes and the multiplicity in which they occur in $S(Q_1Q_5)$ can be found from the solutions of the equation

$$\sum_{n=0}^{Q_1Q_5} nN_n = Q_1Q_5 \tag{47}$$

where n is the length of the cycle and N_n is the multiplicity of the cycle. The Hilbert

space is given by

$$\mathcal{H} = \bigoplus_{\sum n N_n = Q_1 Q_5} \bigotimes_{n > 0} S^{N_n} \mathcal{H}_{(n)}^{P_n} \quad (48)$$

$S^N \mathcal{H}$ denotes the symmetrized product of the Hilbert space \mathcal{H} , N times. By the symbol $\mathcal{H}_{(n)}^{P_n}$ we mean the Hilbert space of the twisted sector with a cycle of length n in which only states which are invariant under the projection operator

$$P_n = \frac{1}{n} \sum_{k=1}^n e^{2\pi i k (L_0 - \bar{L}_0)} \quad (49)$$

are retained. The values of L_0 or \bar{L}_0 in the twisted sector of length n is of the form p/n where p is positive integer. This projection forces the value of $L_0 - \bar{L}_0$ to be an integer on the twisted sector. It arises because the black hole can exchange only integer valued Kaluza-Klein momentum with the bulk [18].

The dominant contribution to the partition function arises from the maximally twisted sector. That is, from the longest single cycle of length $Q_1 Q_5$. It is given by

$$Z = \sum_{m,n} d(Q_1 Q_5 n + m) d(m) e^{2\pi i n \tau} e^{2\pi i m \tau / Q_1 Q_5} e^{-2\pi i m \bar{\tau} / Q_1 Q_5} \quad (50)$$

Where d 's are the coefficients defined by the expansion

$$Z_{T^4} = \left[\frac{\Theta_2(0|\tau)}{\eta^3(\tau)} \right]^2 = \sum_{n \geq 0} d(n) e^{2\pi i \tau n} \quad (51)$$

In the above equation Z_{T^4} is the partition function of the holomorphic sector of the CFT on T^4 . We will first evaluate the sum

$$P(m, \tau) = \sum_{n=0}^{\infty} d(Q_1 Q_5 n + m) e^{2\pi i n \tau} \quad (52)$$

For large values of $Q_1 Q_5$ we can use the asymptotic form of $d(Q_1 Q_5 n + m)$

$$d(Q_1 Q_5 n + m) \sim \exp \left(2\pi \sqrt{Q_1 Q_5 n + m} \right) \quad (53)$$

Substituting the above value of $d(Q_1 Q_5 n + m)$ in $P(m, \tau)$ we obtain a sum which has an integral representation as shown below.

$$\begin{aligned} P(m, \tau) &= \sum_{n=1}^{\infty} e^{2\pi \sqrt{Q_1 Q_5 n + m} + 2\pi i n \tau} + d(m) \\ &= \mathcal{P} \frac{i}{2} \int_{-\infty}^{\infty} dw \coth \pi \omega e^{2\pi \sqrt{i Q_1 Q_5 \omega + m} - 2\pi \omega \tau} + d(m) - \frac{e^{2\pi \sqrt{m}}}{2} \end{aligned} \quad (54)$$

where \mathcal{P} denotes “principal value” of the integral.

We are interested in the high temperature limit of the partition function. The leading contribution to the integral in the limit $\tau \rightarrow 0$ is

$$P(m, \tau) \sim \sqrt{i\pi Q_1 Q_5 / \tau} e^{i\pi Q_1 Q_5 / 2\tau - i2\pi m\tau / Q_1 Q_5} \quad (55)$$

Substituting the above value of $P(m, \tau)$ the partition function becomes

$$Z = \sqrt{i\pi Q_1 Q_5 / \tau} \sum_{m=0}^{\infty} d(m) e^{-2\pi i m \bar{\tau} / Q_1 Q_5} \sim \exp(i\pi Q_1 Q_5 (1/2\tau - 1/2\bar{\tau})) \quad (56)$$

Thus the free energy at high temperatures is given by

$$-\ln Z = \frac{-i\pi Q_1 Q_5}{2} \left(\frac{1}{\tau} - \frac{1}{\bar{\tau}} \right) \quad (57)$$

This exactly agrees with (39) with the identification $\tau = i\beta_+ / (2\pi l)$.

CFT partition function corresponding to conical spaces

We will focus on the low temperature (large β) behaviour. At low temperature, using $\tau = i\beta / (2\pi l)$ (β real in this case), the partition function is given by

$$Z \equiv \text{Tr} \exp[2\pi i \tau L_0 - 2\pi i \bar{\tau} \bar{L}_0] = \exp[-\beta(E_0 + \bar{E}_0)](1 + O(\exp[-\beta\Delta])) \quad (58)$$

Where E_0, \bar{E}_0 represent the ground state values of L_0, \bar{L}_0 , and Δ represents the excitation energy of the first excited level. According to our proposal, the boundary CFT for conical spaces corresponds to the Hilbert space of spectral flow satisfying the relation (21). By using (25) and (26), we find that

$$-\ln Z = \frac{\beta(E_0 + \bar{E}_0)}{l} = \beta M = \frac{\beta}{8G_N^{(3)}}(-\gamma) \quad (59)$$

which agrees exactly with (44) (note that for the zero-mass BTZ solution the above expression vanishes).

6 Concluding remarks

We have argued that point mass geometries in three dimensional quantum gravity with negative cosmological constant are part of the phenomenon of AdS/CFT correspondence.

The boundary CFT corresponds to spectral flow with the spectral flow parameter identified appropriately with the point mass, or equivalently ((21)) with the defect angle of the conical singularity.

The above identification also leads to a correspondence between vertex operators of the boundary CFT with states in the bulk that corresponds to the point mass, thus leading to a possible description of scattering of such point masses in terms of CFT correlations. In this context it would be very useful to connect with the work of Steif [5] where multi-point-mass classical solutions were discussed. In particular, the spatial geometry of such solutions were obtained by quotienting two dimensional hyperbolic space by appropriate sub-groups of $SL(2, R)$. For example, quotienting by sub-groups that generate elliptic isometries gives rise to point particles, while those with hyperbolic isometries lead to black holes.

The interesting thing about these multi-body solutions is that they are generally not static, and therefore can be used to study black hole formation by collision of point particles, or more complicated processes like collisions of black holes. In the light of the AdS/CFT correspondence, it would be interesting to know how such processes manifest themselves in the CFT picture. More precisely, what effect does quotienting the spatial slice have on the CFT? We have found a hint to this answer in this paper, for the simple case of a single point particle: it corresponds to a certain vertex operator in the SCFT (representation of a point mass in the bulk as vertex operator in a boundary Liouville theory has been discussed recently in [23]). The understanding of more general quotients holds the promise of providing insights into some very interesting processes in (2+1) dimensional gravity.

Another interesting point to emerge in this paper is that the partition function of the dual CFT agrees exactly with that of the supergravity at high temperatures. This is remarkable as the partition function of the CFT was computed in weak coupling and the supergravity result is expected to be a prediction for the CFT result at strong coupling. Note that this agreement works for arbitrary angular momentum, in particular for $J = 0$ which is far from extremality and hence is far from the BPS limit.

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