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## Gauge Theory on a Quantum Phase Space

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### ABSTRACT

In this note we present a operator formulation of gauge theories in a quantum phase space which is specified by a operator algebra. For simplicity we work with the Heisenberg algebra. We introduce the notion of the derivative (transport) and Wilson line (parallel transport) which enables us to construct a gauge theory in a simple way. We illustrate the formulation by a discussion of the Higgs mechanism and comment on the large N masterfield.

# 1 Introduction

Recently there has been much interest in the study of non-commutative (nc) geometry in the context of quantum field theory (including gauge theories) and string theory [1, 2, 3, 4, 5, 6, 7, 8, 9] There are several directions of exploration. To begin with let us cite some examples of nc geometry.

1. Witten's open string field theory is formulated with a associative, non-commutative product [10].
2. The BFSS matrix model formulation of M-theory. Here space coordinates are matrices and hence there is a natural associative, non-commutative product. If we express the Hamiltonian of this model in terms of variables that describe 2-branes the associative, non-commutative matrix algebra is reflected in the 2-brane world volume being a 2-dim. phase space. [11, 12].
3. World volume theories of brane systems in the presence of certain moduli fields like the Neveu-Schwarz  $B_{NS}$  (in the Seiberg-Witten limit) become field theories where ordinary multiplication of fields is replaced by the Moyal star product in a phase space. [13] .
4. In the study of the large  $k$  limit of the  $SU(2)$  WZW model describing open strings moving on a group manifold one encounters the coadjoint orbits of  $SU(2)$ . These are 2-spheres with a natural symplectic form (called fuzzy 2-spheres.) The radius of the coadjoint orbit is  $j \leq k/2$ . [14]
5. The fuzzy 2-sphere also occurs in the Polchinski-Strassler description of bulk geometry in the presence of relevant perturbations of the  $D_3$  brane system [15, 16].
6. The  $c=1$  matrix model is exactly formulated as a particular representation of the  $W_\infty$  algebra, that comes from 1-dim. non-relativistic fermions. The coadjoint representation is carried by the phase space density  $u$  of fermions constrained by the equation  $u \star u = u$ , which is a quantum statement of fermi statistics [17]. Collective field theory [18] describes the classical geometry limit where the constraint involves the ordinary product  $u^2 = u$ .
7. The gauge invariant description of 2-dim.  $U(N)$  QCD with fermions also leads to a specific representation of the  $W_\infty$  algebra. Here too the coadjoint representation is described by the equation  $M * M = M$  where  $M$  represents the gauge invariant Wilson line between two quarks. The Regge trajectory of mesons appear as solutions of the small fluctuation equation around the large  $N$  classical solution of  $M * M = M$  [19].

The examples cited above share one important feature in common: they are theories whose fields are valued in a *quantum phase space* rather than on a manifold. It is presently not clear how Witten's string field theory fits into this framework.<sup>1</sup>

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<sup>1</sup>Witten has recently made progress in this direction. See the note added at the end of this paper.

The quantum phase space is specified by a operator algebra. By virtue of this, these fields become operators and act in an appropriate Hilbert space. If one considers the coadjoint representation of the operator algebra then one can have a correspondence between the operator valued fields and classical functions on the coadjoint orbit. For the simplest example of the Heisenberg algebra the coadjoint orbit corresponds to the familiar phase space. The composition law for functions on the phase space is the Moyal star product. This point is briefly reviewed in the next section and is called the Weyl-Moyal correspondence.

Besides the fact that space-time is non-commutative at the fundamental level, as suggested by the Matrix model formulation of M-theory, (see also [20]) this new type of field theory has many interesting properties. One very significant property is the IR/UV connection [21]. Such a connection is indeed novel and seems to signal a breakdown of decoupling. As has been suggested it may be useful in understanding the cosmological constant problem [21]. Another important property of a nc theory is the role it plays in the resolution of singularities [22, 23, 24]. Another use can be in understanding the large N limit of gauge theories.

With these and other applications in mind it is important to understand these theories in various ways. In this note we formulate a gauge theory on a quantum phase space. Our main point is to give a formulation directly in the language of operators, using simple rules that physicists are familiar with. For simplicity we discuss the Heisenberg algebra and show that the operator formulation of the gauge theory corresponds to a formulation on the coadjoint orbit (phase space) in terms of the Moyal star product.

In section 2. we introduce the basic notion of a derivative operator which can translate operators. We also introduce the notion of operator valued forms, the exterior derivative, the wedge product and the Dirac operator. In section 3. we discuss the Weyl-Moyal correspondence. In section 4. we introduce the notion of parallel transport of operators using the Wilson line. We also introduce the corresponding gauge field , gauge transformations, field strength, action and instanton number. In section 5. we define the Wilson loop. In section 6. comment on the quantum theory. In section 7. we discuss the Higgs mechanism and in section 8. we make some remarks on the large N master field.

## 2 The Derivative Operator on a Quantum Phase Space

A quantum phase space is specified by an operator algebra. This algebra can have finite or infinite number of generators. Examples of finite number of generators are the Heisenberg algebra and Lie algebras of compact and non-compact groups. The algebras with infinite number of generators are eg. the Kac-Moody algebra and the Virasoro algebra. One can and should include super-algebras to this list.

To simplify matters we restrict ourselves to the Heisenberg algebra.

$$[X_i, X_j] = iI\theta_{i,j} \tag{1}$$

where ,  $I$  is the identity operator,  $i, j = 1, 2, \dots, 2d$ , and  $\theta_{i,j}$  is a real anti-symmetric, invertible

matrix, with inverse  $\theta_{i,j}^{-1}$ . We have to specify the Hilbert space on which these operators act. For our present purposes we can take this to be the space of delta-function normalizable functions in d-dimensions.

Using the basic operators (1) we can construct other operators in terms of polynomials of  $X_i$  over the complex numbers. Instead of a polynomial basis we can more fruitfully use the Weyl basis defined by the exponential operators

$$g(\alpha) = \exp i\alpha_i X_i \quad (2)$$

In this way we can introduce complex, self-adjoint and unitary operators in the Hilbert space.

Since we would like to develop an operator calculus, the first thing that we should do is to define the derivative operator and the notion of translations in the space of operators. We define the derivative by,

$$\bar{\partial}_i = \theta_{i,j}^{-1} ad X_j \quad (3)$$

where the adjoint action is defined by  $(adA)B = [A, B]$ . This derivative operator has a number of important properties which we list:

1.  $\bar{\partial}_i$  is anti-hermitian and linear,

$$\begin{aligned} \bar{\partial}_i^\dagger &= -\bar{\partial}_i \\ \bar{\partial}_i(a_1 \mathcal{O}_1 + a_2 \mathcal{O}_2) &= a_1 \bar{\partial}_i \mathcal{O}_1 + a_2 \bar{\partial}_i \mathcal{O}_2 \end{aligned} \quad (4)$$

2.  $\bar{\partial}_i$  satisfies the Leibniz rule,

$$\bar{\partial}_i(\mathcal{O}_1 \mathcal{O}_2) = (\bar{\partial}_i \mathcal{O}_1) \mathcal{O}_2 + \mathcal{O}_1 (\bar{\partial}_i \mathcal{O}_2) \quad (5)$$

3.  $\bar{\partial}_i$  commute amongst themselves,

$$[\bar{\partial}_i, \bar{\partial}_j] = 0 \quad (6)$$

4. The commutative property of the derivative enables us to introduce the notion of an exterior derivative that acts on operator valued forms. The operator valued n-form and its exterior derivative are defined by,

$$\begin{aligned} O^{(n)} &= \sum \mathcal{O}_{i_1, \dots, i_n} dy_{i_1} \wedge \dots \wedge dy_{i_n} \\ dO^{(n)} &= \sum \bar{\partial}_{i_1} \mathcal{O}_{i_2, \dots, i_n} dy_{i_1} \wedge \dots \wedge dy_{i_{n+1}} \end{aligned} \quad (7)$$

In the above  $dy_i$  are real 1-forms. The commutative property of the derivative clearly implies that  $d^2 = 0$ .

Using the above definition of the operator valued n-form we can introduce the notion of a non-commutative wedge product

$$O^{(n)} \bar{\wedge} O^{(m)} = C_{n,m} \sum \mathcal{O}_{i_1, \dots, i_n} \mathcal{O}_{i_{n+1}, \dots, i_{n+m}} dy_{i_1} \wedge \dots \wedge dy_{i_{n+m}} \quad (8)$$

$C_{n,m}$  is a normalization constant. The above definition of the wedge product is associative. One can also define  ${}^*O^{(n)}$  the Poincare dual of  $O^{(n)}$  in the standard fashion using the totally antisymmetric  $\epsilon$  tensor in  $2d$  dimensions.

5.  $\bar{\partial}_i$  is the generator of translations in the following sense,

$$\exp ia_i \bar{\partial}_i \mathcal{O}(X) = \mathcal{O}(X + Ia) \quad (9)$$

6. The integral of an operator is defined by its trace in the corresponding Hilbert space. Then using the trace formula  $TrA[B, C] = Tr[A, B]C$  we have the formula for integration by parts,

$$Tr\mathcal{O}_1(\bar{\partial}_i \mathcal{O}_2) = -Tr(\bar{\partial}_i \mathcal{O}_1) \mathcal{O}_2 \quad (10)$$

We note that there is no ‘surface term’.

7. We can introduce the Dirac operator  $\bar{\partial} = \gamma_i \bar{\partial}_i$ , where  $\gamma_i$  are the standard Dirac gamma matrices. Note that  $(\bar{\partial})^2 = \bar{\partial}_i \bar{\partial}_i$ .

### The Landau Condition

The defining equation for the derivative operator can be understood in a more physical setting by introducing additional momentum operators  $P_i$  and extending the algebra,

$$[P_i, P_j] = 0 \quad (11)$$

$$[P_i, X_j] = -i\delta_{ij}I \quad (12)$$

and introducing the constraint,

$$adP_i = \bar{\partial}_i = \theta_{i,j}^{-1} adX_j \quad (13)$$

On states in the Hilbert space this equation becomes the Landau constraint [25, 26, 27, 28],

$$(P_i - \theta_{i,j}^{-1} X_j) |\Psi\rangle = 0 \quad (14)$$

which implies for such states the uncertainty principle,

$$\delta X_i \theta_{i,j}^{-1} \delta X_j \geq 1 \quad (15)$$

## 3 Weyl-Moyal Correspondence

The Weyl-Moyal (WM) correspondence is best understood in terms of the operators  $g(\alpha) = \exp i\alpha_i X_i$ . Using (1) these operators satisfy the Heisenberg-Weyl algebra,

$$g(\alpha)g(\beta) = \exp\left(\frac{1}{2}\theta_{i,j}\alpha_i\beta_j\right)g(\alpha + \beta) \quad (16)$$

The WM correspondence is given by

$$\mathcal{O}(X) = \int d^{2d} \alpha g(\alpha, X) \tilde{O}(\alpha) \quad (17)$$

$\tilde{O}$  is the Fourier transform of  $O$ . Using (16) we can derive correspondence between the operator product and the star product,

$$\begin{aligned}\mathcal{O}_1 \mathcal{O}_2 &= \int d^{2d} \alpha g(\alpha, X) \tilde{O}_{12}(\alpha) \\ O_{12} &= O_1 \star O_2 \\ O_1 \star O_2 &= \exp\left(i \frac{1}{2} \theta_{i,j} \frac{\partial}{\partial \xi_i} \frac{\partial}{\partial \eta_j}\right) O_1(x + \xi) O_2(x + \eta) \Big|_{\xi=\eta=0}\end{aligned}\quad (18)$$

Using the WM correspondence, we can easily prove the correspondence,

$$\begin{aligned}\bar{\partial}_i \mathcal{O}(X) &\implies \partial_i O(x) \\ \mathcal{O}(X)_1 \mathcal{O}(X + Ia)_2 &\implies O(x)_1 \star O(x + a)_2\end{aligned}\quad (19)$$

## 4 Parallel Transport of Operators, Wilson Lines and Gauge Fields

In section 2. we introduced the notion of translating operators. In this section we introduce the notion of parallel transport and connection. For convenience of presentation we will suppress the  $U(N)$  indices carried by the various operators.

Consider the set of operators  $\Phi(X)$ , which transform under the right action of the group of unitary operators  $\{\Omega(X)\}$

$$\Phi(X) \rightarrow \Phi(X)\Omega(X) \quad (20)$$

Then clearly

$$\Phi(X)\Phi(X + Ia)^\dagger \quad (21)$$

is not gauge invariant under the gauge transformations

$$\Phi(X) \rightarrow \Phi(X)\Omega(X) \quad (22)$$

$$\Phi(X + Ia) \rightarrow \Phi(X + Ia)\Omega(X + Ia) \quad (23)$$

The standard way to form a gauge invariant operator is to introduce the Wilson line  $U(X, X + Ia)$ , with gauge transformation

$$U(X, X + Ia) \rightarrow \Omega(X)^\dagger U(X, X + Ia)\Omega(X + Ia) \quad (24)$$

and the property that  $U(X, X) = I$  and  $U(X, X + Ia)^\dagger = U(X + Ia, X)$ . The operator

$$\Phi(X)U(X, X + Ia)\Phi(X + Ia)^\dagger \quad (25)$$

is gauge invariant.

Now a similar construction is possible for those operators  $\Psi(X)$  which transform under the left action of the group of unitary operators,

$$\Psi(X) \rightarrow \Omega(X)\Psi(X) \quad (26)$$

In this case the gauge invariant operators are given by,

$$\Psi(X)U(X, X + Ia)^\dagger \Psi(X + Ia)^\dagger \quad (27)$$

### The Connection:

Using the Wilson line for infinitesimal  $a_i = \epsilon_i$  we can introduce the definition of the operator valued connection,

$$U(X, X + I\epsilon) = \exp(i\mathcal{A}_i(X)\epsilon_i) \quad (28)$$

The operator  $\mathcal{A}_i(X)$  is Hermitian and the gauge transformation of the Wilson line implies the gauge transformation of the operator valued gauge field.

$$\mathcal{A}_i(X) \rightarrow \Omega(X)^\dagger(\mathcal{A}_i(X) - i\bar{\partial}_i)\Omega(X) \quad (29)$$

### Covariant Derivative and Field Strength:

The operator covariant derivative and the field strength are defined by,

$$\mathcal{D}_i = -i\bar{\partial}_i + \mathcal{A}_i(X) \quad (30)$$

$$\mathcal{F}_{ij}(X) = i[\mathcal{D}_i, \mathcal{D}_j] \quad (31)$$

Both the covariant derivative and the field strength are gauge covariant and the Jacobi identity for the covariant derivative implies the Bianchi identity for the field strength,

$$[\mathcal{D}_i, \mathcal{F}_{jk}] + [\mathcal{D}_j, \mathcal{F}_{ki}] + [\mathcal{D}_k, \mathcal{F}_{ij}] = 0 \quad (32)$$

### The Action and Equations of Motion:

The gauge invariant action is given by

$$\begin{aligned} S &= \frac{1}{4g^2} \text{Tr}(\mathcal{F}(X) \bar{\wedge}^\star \mathcal{F}) \\ &= \frac{1}{4g^2} \text{Tr}(\mathcal{F}_{ij}(X) \mathcal{F}_{ij}(X)) \end{aligned} \quad (33)$$

In the above we have chosen the euclidean metric. From the above action we can easily derive the equations of motion by requiring the action to be stationary w.r.t the variation  $\mathcal{A}_i(X) \rightarrow \mathcal{A}_i(X) + \delta\mathcal{A}_i(X)$ ,

$$\mathcal{D}_i \mathcal{F}_{ij}(X) = 0 \quad (34)$$

We can also define the nc version of the instanton number

$$\begin{aligned} I &= \text{Tr}(\mathcal{F}(X)\bar{\wedge}\mathcal{F}) \\ &= \text{Tr}(\mathcal{F}_{i,j}(X)\mathcal{F}_{k,l}(X)\epsilon_{ijkl}) \end{aligned} \quad (35)$$

(In the above formulas the trace can also includes a trace over  $U(N)$ .)

It is also possible to introduce the following operator current,

$$\mathcal{J}_i = \epsilon_{ijkl}(\mathcal{A}_j\bar{\partial}_k\mathcal{A}_l + i\frac{2}{3}\mathcal{A}_j\mathcal{A}_k\mathcal{A}_l) \quad (36)$$

so that (35) can be written as,

$$I = 4\text{Tr}(\bar{\partial}_i\mathcal{J}_i) \quad (37)$$

Using the WM correspondence it is easy to prove that the operator formulation given above goes over into a gauge theory formulated in terms of a real connection, and the star product, e.g.

$$\begin{aligned} \mathcal{A}_i(X) &\longrightarrow A_i(x) \\ \mathcal{F}_{ij}(X) &\longrightarrow \partial_i A_j - \partial_j A_i + i(A_i \star A_j - A_j \star A_i) \end{aligned} \quad (38)$$

In the operator formulation the gauge group is generated by all unitary operators. The generators of this gauge group are the set of all Hermitian operators:  $\{H(X)\}$ , and  $\{\Omega(X) = \exp i\mathcal{H}(X)\}$ . Expressing the exponential as a series we can easily obtain the correspondence for the gauge group,

$$\exp i\mathcal{H}(X) \longrightarrow (1 + H(x) + \frac{1}{2}H(x) \star H(x) + \frac{1}{6}H(x) \star H(x) \star H(x) \dots) \quad (39)$$

## 5 The Wilson Loop

Let us now present the expression for the Wilson loop operator. Consider a curve  $\Gamma$  in  $R^{2d}$  and divide it into  $n \rightarrow \infty$  infinitesimal segments each denoted by a tangent vector  $\epsilon_i^m$ .  $i = 0, 1, 2..n$  and  $\epsilon_i^0 = 0$ . Since the loop is closed we have  $\sum i = 1^n \epsilon_i^m = 0$ . The Wilson loop is composed of a product of Wilson lines around the curve,

$$\begin{aligned} W(\Gamma) &= \text{Tr} \prod_{m=0}^n U(X + I\epsilon^m, X + I\epsilon^{m+1}) \\ &= \text{Tr} \prod_{m=0}^n \exp i(\mathcal{A}_i(X + I\epsilon^{m+1})(\epsilon_i^m - \epsilon_i^{m+1})) \end{aligned} \quad (40)$$

$W(\Gamma)$  is gauge invariant. It would be useful to see the connection of this formulation with the reduced model of lattice gauge theory [30, 31].

## 6 The Quantum Theory

Until now we have been dealing with a classical theory whose fields are defined on a ‘quantum phase space’ specified by the matrix  $\theta_{i,j}$ . Quantization of this theory consists of studying fluctuations whose strength is controlled by the gauge coupling. One quantization procedure appeals to the WM correspondence and gives a path integral prescription in which one integrates over the histories of the gauge field  $A_i(x)$ . Another approach to quantization is to quantize the matrix elements of  $\mathcal{A}_i(X)$  in a coherent state basis [29].

## 7 The Higgs Mechanism

We now discuss ‘matter fields’ in the nc gauge theory. For simplicity we discuss matter fields  $\Psi(X)$  with gauge transformation (only left action)  $\Psi(X) \rightarrow \Omega(X)\Psi(X)$ . The gauge invariant action is given by

$$S = \frac{1}{4g^2} \text{Tr}(\mathcal{F}_{i,j}(X)\mathcal{F}_{i,j}(X)) + \frac{1}{2} \text{Tr}(\mathcal{D}_i\Psi)^\dagger(\mathcal{D}_i\Psi) + \frac{1}{4} \text{Tr}(\Psi^\dagger\Psi - a^2 I)^2 \quad (41)$$

To discuss the Higgs mechanism, we write  $\Psi = U\mathcal{H}$ , where  $U$  is unitary and  $\mathcal{H}$  is Hermitian, and perform the gauge transformation to the unitary gauge  $\Psi \rightarrow U^\dagger\Psi$ . In this gauge the potential term becomes  $V = \frac{1}{4} \text{Tr}(\mathcal{H}^2 - a^2 I)^2$  and the ground state is a solution of the operator equation,

$$\mathcal{H}^3 = a^2 \mathcal{H} \quad (42)$$

or equivalently

$$H \star H \star H = a^2 H \quad (43)$$

If we assume that  $\mathcal{H}^{-1}$  exists then (42) reduces to

$$\mathcal{H}^2 = a^2 I \quad (44)$$

Such operator equations were originally discussed in [17, 19]. They have also been recently discussed in the context of soliton solutions of nc field theories [32]. See also [33, 34] for subsequent applications. The minimum energy solution is given by  $\mathcal{H} = |a|I$ , where we have absorbed a possible sign ambiguity by a gauge transformation. This leads to a ‘mass term’ for the gauge field  $|a|^2 \text{Tr}(\mathcal{A}_i\mathcal{A}_i)$ . It would be interesting to look for vortex like solutions in these models.

## 8 The large N Master Field

Let us now consider the nc gauge theory with additional color gauge group  $U(N)$ . In ref. [21, 35] the perturbation expansion in the maximally non-commutative regime ( $\theta \rightarrow \infty$ ) was

discussed. In this limit the leading term in the perturbation expansion consists of planar graphs which have no theta or  $N$  dependence except for an overall phase involving  $\theta$  and a multiplicative factor of  $N^2$ . Hence the problem of summing planar diagrams of the nc gauge theory is mapped onto the problem of solving the nc gauge theory in the  $\theta \rightarrow \infty$  limit.

Let us choose  $\theta_{ij} = \theta \begin{pmatrix} 0 & -1_d \\ 1_d & 0 \end{pmatrix}$  and write the gauge theory in terms of the scaled operators  $\frac{X_i}{\sqrt{\theta}}$  or after the WM correspondence, in terms of the scaled co-ordinates  $\frac{x_i}{\sqrt{\theta}}$ . If we require that the gauge field has the transformation  $A_i(\frac{x_i}{\sqrt{\theta}}) = \sqrt{\theta}A_i(x)$ , then the action becomes

$$S = \frac{\theta^{\frac{2d-4}{2}}}{4g^2} \int d^{2d}x \text{tr}(F_{ij} \star F_{ij}) \quad (45)$$

and the path integral is given by

$$Z = \int \mathcal{D}A_i(x) e^{-S} \quad (46)$$

For  $2d > 4$  and the path integral is evaluated in the  $\theta \rightarrow \infty$  limit by the saddle point  $\frac{\delta S}{\delta A_i(x)} = \nabla_i F_{ij} = 0$ . Reverting back to the operator formalism, for  $2d > 4$ , is given by four  $\theta \rightarrow \infty$  operators  $\mathcal{A}_i(x)$  which satisfy the n.c YM equations

$$\mathcal{D}_i \mathcal{F}_{ij} = 0 \quad (47)$$

It is reasonable that (38) are equations for the  $U(\infty)$  master field for  $2d > 4$ .

The situation in the most interesting dimension is certainly more complicated and the Schwinger-Dyson equations approach [36] may help. Finally it would be interesting to make a connection of the large  $N$  limit of nc gauge theory with non-commutative probability theory as applied to the problem of the large  $N$  limit by Gopakumar and Gross [37].

### Note added:

After this work was completed we received [38] where an approach similar to ours has been discussed. However the derivative operator discussed in this paper (equation 2.7) does not necessarily commute in different directions. Gross and Nekrasov [39] have also presented the basic ingredients of the nc gauge theory in the operator formulation. Their derivative operator is identical to ours. Recently, Witten [40] has shown the emergence of the Moyal product in the large B-field limit of string field theory.

It turns out that the geometrical construction of the non-commutative gauge theory presented here can be shown to be equivalent to the perturbation of the IKKT model [41] around the brane configuration defined by (1). This point has been made in [30, 31]. We would like to thank R. Szabo and A. Dhar for pointing this out to us.

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