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A review of the microscopic modeling of the 5-dim. black hole of IIB string theory

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Abstract. We review the theory of the microscopic modeling of the 5-dim. black hole of type IIB string theory in terms of the D_1–D_5 brane system. A detailed discussion of the low energy effective Lagrangian of the brane system is presented and the black hole micro-states are identified. These considerations are valid in the strong coupling regime of supergravity due to the non-renormalization of the low energy dynamics in this model. Using Maldacena duality and standard statistical mechanics methods one can account for black hole thermodynamics and calculate the absorption cross section and the Hawking radiation rates. Hence, at least in the case of this model black hole, since we can account for black hole properties within a unitary theory, there is no information paradox.

Keywords. Black holes; string theory; Hawking radiation.

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1. Quantum mechanics and general relativity

The application of quantum field theory to general relativity (GR) leads to some basic problems:

1. The problem of ultra-violet divergences renders GR an ill-defined quantum theory. This specifically means that if we perform a perturbation expansion around flat Minkowski space-time (our world!) then to subtract infinities from the divergent diagrams we have to add an infinite number of terms to the Einstein–Hilbert action with coefficients that are proportional to appropriate powers of the ultraviolet cutoff.

There is good reason to believe that string theory solves this ultra-violet problem because the extended nature of string interactions have an inherent ultra-violet cutoff given by the fundamental string length $\sqrt{\alpha'}$. One also knows that in string theory the Einstein–Hilbert action emerges as a low energy effective action for energy scales much larger than the string length and Newton's constant (in 10-dim.) is given by

$$G_{10} = \kappa_{10}^2 = 8\pi^6 g_s^2 \alpha'^4, \tag{1}$$

where g_s is the string coupling.

2. The solutions of general relativity can be singular. There are a variety of singularities that have been encountered. Examples are i) the singularity of the black hole solution, ii) the singularities encountered in various brane solutions of supergravity, iii) singularities of the cosmological solutions of GR etc. A quantum theory of gravity must present an understanding about what are good and bad singularities in the sense whether one can have a well defined quantum mechanics in their presence. String theory has resolved some of these singularities, but a complete understanding of the issue of singularities is still lacking.

3. While the above problem is related to the high energy (short distance) behavior of GR, there exists another problem when we quantize matter fields in the presence of a black hole which does not involve high energy processes. This problem is called the information puzzle and in the following we shall explain the issue and also summarize the attempts within string theory to resolve the puzzle in a certain class of black holes.

String theory has been proposed as a theory that describes all elementary particles and their interactions. Presently the theory is not in the stage of development where it can provide quantitative predictions in particle physics. However in case this framework resolves some logical problems that arise in the applications of quantum field theory to general relativity, then it is a step forward for string theory.

4. Finally there is the problem of the cosmological constant, which is getting renewed attention in recent times.

In this review we will discuss only point 3. We will focus on the black hole solution of IIB string theory and discuss its modeling by the D_1–D_5 system of branes. We will describe the low energy excitations of this system and learn how they couple to the bulk supergravity degrees of freedom using Maldacena duality. We will present the calculation of the Hawking rate for a class of massless particles which agrees with supergravity calculations due to the high degree of supersymmetry of the D_1–D_5 system.

2. Organization of the notes

- Sections 3 and 4 present a general description of black hole thermodynamics and the information puzzle.
- Section 5 presents the string theory framework for black holes.
- Section 6 presents various supergravity solutions of relevance to our discussion: The BPS and the non-BPS black hole solution, the Maldacena limit and Ad$S_3 \times S^3$, and the solution with a non-zero value of the Neveu–Schwarz B-field in the 4 compact dimensions. We also discuss the semi-classical derivation of Hawking radiation.
- Section 8 presents the D_1–D_5 system and the $N = 4, U(Q_1) \times U(Q_5)$ gauge theory in 2-dimensions. We discuss low energy degrees of freedom and the conformally invariant sigma model at the infrared fixed point.
- Section 9 presents the discussion of D_1 branes as solitonic strings of the D_5 gauge theory. We discuss the moduli space of instantons which forms the target space of the solitonic strings.
- Sections 10–12 discuss the $\mathcal{N} = 4$ super conformal algebra, the $\mathcal{N} = (4,4)$ SCFT on the orbifold $(\tilde{T}^4)^{Q_1 Q_5}/S(Q_1 Q_5)$, and the classification of states of the SCFT in

terms of the supergroup $SU(1,1|2)$. We identify the maximally twisted sector of the SCFT with the set of states that constitute the black hole.

- Section 13 compares the near horizon supergravity moduli and correspondence with SCFT operators.
- Section 14 discusses the microscopic derivation of Hawking radiation.
- Section 15 discusses some future directions.

In sections 10–12 there is overlap with Justin's PhD thesis [10]. The details of the construction of all the chiral primaries have been presented there.

3. The classical black hole horizon

Classically a black hole is a solution of the GR equations, and it is characterized by an event horizon, which is a null surface. The horizon is a one way gate, in the sense that once we are inside it we cannot get out because of the causal structure of the black hole spacetime. Physically one can imagine the formation of an event horizon due to the bending property of light by the matter that makes up the black hole.

Let us list a few properties of classical black holes: (see e.g. the text book by Wald [1]).

Firstly the event horizon has an area and there is a area law which states that in any adiabatic process involving black holes the final area of the event horizons is never less than the initial area(s):

$$A_{12} \geq A_1 + A_2. \tag{2}$$

The 'no hair theorems', tell us that the state of a classical black hole is completely characterized by its mass, angular momentum and global gauge charges. In particular the area of the event horizon depends only on these quantities. If we perturb a black hole then the perturbation decays in Planck time, and the new state of the black hole is again characterized by a event horizon whose area has increased and is characterized by the changed mass, angular momentum or charge of the final state.

The area law (2) prompted Bekenstein [2] to associate a entropy with the black hole that is proportional to its area:

$$S = aA, \tag{3}$$

where 'a' is a universal constant. The area law (2) then resembles the second law of thermodynamics where the black hole is treated as a macroscopic object,

$$S_{12} \geq S_1 + S_2. \tag{4}$$

From the viewpoint of classical general relativity there is no information puzzle because the stuff that went inside a black hole stays inside because the horizon is a one way gate.

4. Quantum mechanics and the information puzzle of black holes

In the quantum theory since the absorption process is described by the matrix element of a hermitian hamiltonian, the emission amplitude is necessarily non-zero. Black holes radiate.

Application of quantum field theory to matter propagating in a black hole background leads to the following results which we briefly summarize:

Black holes behave like black bodies. They emit thermal radiation and they are characterized by a temperature which depends only on the mass, angular momentum and the global charges of the black hole. The fundamental formula for the temperature, due to Hawking [3] is given by

$$T = \frac{\hbar\kappa}{2\pi}, \tag{5}$$

κ is surface gravity (acceleration due to gravity felt by a static observer) at the horizon of the black hole. For a Schwarzschild black hole:

$$\kappa = \frac{1}{4G_N M}. \tag{6}$$

The constant of proportionality in (3) is determined using the first law of thermodynamics and the temperature formula: $TdS = dM$,

$$\mathcal{S}_{bh} = aA_h, \qquad a = \frac{c^3}{4G_N\hbar}. \tag{7}$$

Using this we can now interpret (4) as the second law of black hole thermodynamics.

Formula (7), called the Bekenstein–Hawking formula is very fundamental because it is a formula that counts the effective degrees of freedom of the black-hole. a^{-1} is a basic unit of area and it has all the 3 fundamental constants in it.

The Hawking radiation as calculated in semi-classical GR is a mixed state. It turns out to be difficult to calculate the correlations between the ingoing and outgoing Hawking particles in the standard framework of general relativity. Such a calculation would require a good quantum theory of gravity where controlled approximations are possible.

If we accept the semi-classical result that black holes emit radiation that is EXACTLY thermal then it leads to the information puzzle:

Initially the matter that formed the black hole is in a pure quantum mechanical state. Here in principle we know all the quantum mechanical correlations between the degrees of freedom of the system. In case the black hole evaporates completely then the final state of the system is purely thermal and hence it is a mixed state. This evolution of a pure state to a mixed state is in conflict with the standard laws of quantum mechanics which involve unitary time evolution of pure states into pure states.

Hence we either have to modify quantum mechanics, as was advocated by Hawking [4], or as we shall argue, the other possibility is to replace the paradigm of quantum field theory by that of string theory. In string theory we retain quantum mechanics and resolve the information puzzle (for a certain class of black holes) by discovering the microscopic degrees of freedom of the black hole. In string theory the Hawking radiation is NOT thermal and in principle we can reconstruct the initial state of the system from the final state.

In standard statistical mechanics, for a system with a large number of degrees of freedom we introduce a density matrix to derive the thermodynamic description. The same thing can be done for black holes in string theory. In this way the thermodynamic formulas for black hole entropy and decay rates of Hawking radiation can be derived from string theory. In particular the Bekenstein–Hawking formula is derived from Boltzmann's law:

$$S = \ln \Omega, \tag{8}$$

where Ω is the number of micro-states of the system.

The possible connection of the degeneracy of the fundamental string spectrum with the black hole entropy has been speculated by many authors and more recently by Susskind [5]. The approximate verity of this suggestion was demonstrated for the first time by Sen [6] for supersymmetric extremal black holes with a small horizon area. Subsequently Strominger and Vafa [7] gave a brane construction for a extremal black hole of IIB string theory and exactly verified the Bekenstein–Hawking formula using the Boltzmann's formula. This paper led to a lot of activity in the microscopic modeling of black holes and the description and derivation of Hawking radiation from near extremal black holes to which this review is devoted.

It is well worth pointing out that the existence of black holes in nature (for which there is mounting evidence) compels us to resolve the conundrums that black holes present. One may take recourse to the fact that for a black hole whose mass is a few solar masses the Hawking temperature is very tiny ($\sim 10^{-8}$ degs. Kelvin), and not of any observable consequence. However the logical problem that we have described above cannot be wished away and its resolution makes a definitive case for the string paradigm as a correct framework for fundamental physics as opposed to standard local quantum field theory.

5. The string theory framework for black holes

The basic point in the string theory description is that a black hole is described by a density matrix:

$$\begin{aligned} \rho &= \frac{1}{\Omega} \sum_i |i\rangle\langle i|, \\ S &= \ln \Omega, \end{aligned} \tag{9}$$

where $|i\rangle$ is a micro-state.

Given this we can calculate formulas of black hole thermodynamics just like we calculate the thermodynamic properties of macroscopic objects using standard methods of statistical mechanics. Here the quantum correlations that existed in the initial state of the system are in principle all present and are only erased by our procedure of defining the black hole state in terms of a density matrix. In this way one can account for not only the entropy of the system which is a counting problem but also the rate of Hawking radiation which depends on interactions.

Let us recall the treatment of radiation coming from a star, or a lump of hot coal. The 'thermal' description of the radiation coming is the result of averaging over a large number of quantum states of the coal. In principle by making detailed measurements on the wave function of the emitted radiation we can infer the precise quantum state of the emitting body. For black holes the reasoning is similar.

Hence in the string theory formulation the black hole can exist as a pure state: one among the highly degenerate set of states that are characterized by a small number of parameters. Let us also note that in Hawking's semi-classical analysis, which uses quantum field theory in a given black-hole space-time, there is no possibility of a microscopic construction of the black hole wave functions.

We summarize the four basic ingredients we need in string theory to calculate Hawking radiation from low temperature near extremal black holes:

1. The microscopic constituents of the black hole. In the case of the 5-dim. black hole of type IIB string theory the microscopic modeling is in terms of a system of D_1–D_5 branes wrapped on $S^1 \times M_4$, where M_4 is a 4-dim. compact manifold, which can be either T^4 or K_3. Here we will consider T^4.

2. The spectrum of the low energy degrees of freedom of the bound state of the D_1–D_5 system. Usually these are arrived at weak coupling and we need to know if the spectrum survives at strong coupling.

3. The coupling of the low energy degrees of freedom to supergravity modes.

4. The description of the black hole as a density matrix. This implies expressions for decay and absorption probabilities which are related to S-matrix elements between initial and final states of the black hole.

The decay probability from a state $|i\rangle$ to a state $|f\rangle$ is given by

$$P_{\text{decay}}(i \to f) = \sum_{i,f} \frac{1}{\Omega_f} |\langle f|S|i\rangle|^2. \tag{10}$$

The absorption probability from a state $|i\rangle$ to a state $|f\rangle$ is given by

$$P_{\text{abs}}(i \to f) = \sum_{i,f} \frac{1}{\Omega_i} |\langle f|S|i\rangle|^2. \tag{11}$$

In the above formulae Ω_f and Ω_i refer to the number of final and initial states respectively.

One of the important issues in this subject is that 1 and 2 are usually known in the the case when the effective open string coupling is small. In this case the Schwarzschild radius R_{sch} of the black hole is smaller than the string length l_{s} and we have a complicated string state. As the coupling is scaled up we go over to the supergravity description where $R_{\text{sch}} \gg l_{\text{s}}$ and we have a black hole. Now it is an issue of dynamics whether the spectrum of the theory undergoes a drastic enough change, so that the description of states in weak coupling which enabled a thermodynamic description is still valid. In the model we consider we will see that the description of the weak coupling effective lagrangian goes over to strong coupling because of supersymmetry. It is an outstanding challenge to understand this problem when the weak coupling theory has little or no supersymmetry [8,9].

6. Black holes of IIB string theory: Supergravity solutions

We will now present a summary of the SUGRA solutions of relevance to the D_1–D_5 system. This will include the BPS and near BPS black hole solutions, and the near horizon geometry of the D_1–D_5 system. We also discuss the geometry in the presence of the vev of the Neveu–Schwarz $B_{\rm NS}$ with components along the directions of the internal space T^4. There is a huge literature on this subject and we refer the reader to the review by Mandal [11]. The material relevant to our discussion can be found in [7,12–16].

6.1 *The BPS black hole*

Let us begin by describing type IIB string theory in 10-dimensions. This string theory has 32 real sypersymmetries. Its massless bosonic content in the NS-NS sector consists of the metric $G_{a,b}$, the dilaton ϕ and the 2-form $B^{(2)}_{\rm NS}$. The R–R sector consists of the gauge potentials C^n, $n=0,2,4$. The low energy effective action is given by

$$S_{\rm IIB} = \frac{1}{2\kappa^2}\int d^{10}x\sqrt{-G}\bigg\{ {\rm e}^{-2\phi}\left(R+4(\nabla\phi)^2 - \frac{1}{2.3!}(H^{(3)})^2\right) \\ -\frac{1}{2.3!}(F^{(3)})^2\frac{1}{4.5!}(F^{(5)})^2\bigg\} + \frac{1}{4\kappa^2}\int C^{(4)}\wedge F^{(3)}\wedge H^{(3)}, \tag{12}$$

where $(H^{(3)})^2 = H^{(3)}_{\rm MNP}H^{(3){\rm MNP}}$, $(F^{(n)})^2 = F^{(n)}_{M_1\cdots M_n}F^{(n)M_1\cdots M_n}$ and, using the standard form notation,

$$H^{(3)} = dB^{(2)}_{\rm NS}, \\ F^{(3)} = dC^{(2)},\ \ F^{(5)} = dC^{(4)} - \frac{1}{2}C^{(2)}\wedge H^{(3)} + \frac{1}{2}B^{(2)}\wedge F^{(3)}. \tag{13}$$

The self-duality constraint, $*F^{(5)} = F^{(5)}$, is imposed at the level of the equations of motion. Also, $\kappa^2 = 8\pi G_{10}$, where $G_{10} = 8\pi^6 g_s^2\alpha'^4$ is the 10-dimensional Newton's constant (in the convention that the dilaton, ϕ, vanishes asymptotically).

Let us now present the supergravity solution that preserves 4 out of the 32 SUSYs of the original theory. A simple ansatz is to consider all the bosonic fields in (12) to be zero except the metric $G_{a,b}$, the dilaton ϕ and the Ramond 2-form C^2. We compactify the $6,7,8,9$ directions on a torus T^4 of volume V_4 and the x_5 direction on a circle of radius R_5. We then wrap Q_5 D_5-branes along the directions $5,6,7,8,9$ and Q_1 D_1-branes along the x_5 direction. We introduce N units of momentum along the x_5 direction in order to obtain a black hole of finite horizon area. The supergravity solution with these boundary conditions is given by:

$$ds^2 = f_1^{-\frac{1}{2}}f_5^{-\frac{1}{2}}(-dt^2 + dx_5^2 + k(dt-dx_5)^2) \\ + f_1^{\frac{1}{2}}f_5^{\frac{1}{2}}(dx_1^2+\cdots+dx_4^2) + f_1^{\frac{1}{2}}f_5^{-\frac{1}{2}}(dx_6^2+\cdots+dx_9^2), \\ {\rm e}^{-2\phi} = \frac{1}{g_s^2}f_5 f_1^{-1}, \\ C^{(2)}_{05} = \frac{1}{2}(f_1^{-1}-1), \\ F^{(3)}_{abc} = (dC^{(2)})_{abc} = \frac{1}{2}\epsilon_{abcd}\partial_d f_5, \quad a,b,c,d = 1,2,3,4 \tag{14}$$

where f_1, f_5 and k are given by

$$f_1 = 1 + \frac{16\pi^4 g_s \alpha'^3 Q_1}{V_4 r^2},\ f_5 = 1 + \frac{g_s \alpha' Q_5}{r^2},\ k = \frac{16\pi^4 g_s^2 \alpha'^3 N}{V_4 R_5^2 r^2}. \tag{15}$$

Here $r^2 = x_1^2 + x_2^2 + x_3^2 + x_4^2$ denotes the distance measured in the transverse direction to all the D-branes.

The horizon in the above solution occurs at $r = 0$ and we can read off the horizon area and hence the entropy which is given by,

$$S = 2\pi\sqrt{Q_1 Q_5 N}. \tag{16}$$

The mass of the black hole for the above solution turns out to be a linear combination of its 3 charges,

$$M = \frac{1}{g_s^2}(a_1 g_s Q_1 + a_2 g_s Q_5 + a_3 g_s^2 N), \tag{17}$$

where

$$a_1 = \frac{R}{\alpha'}, \quad a_2 = \frac{RV_4}{16\pi^4\alpha'^3}, \quad a_3 = \frac{1}{R}. \tag{18}$$

Anticipating the microscopic modeling, this means that we have a marginal bound state with zero binding energy. Also, since this is an extremal black hole its Hawking temperature is zero, a fact which will have an obvious explanation in the microscopic theory.

6.2 *The near extremal limit and non-zero Hawking temperature*

In order to have a black hole with a non-zero temperature we have to consider a non-BPS and non-extremal black hole solution. This solution preserves none of the original 32 supersymmetries of the type IIB theory and can be obtained by allowing the total momentum N to be distributed in both directions around the x_5 direction. The solution in 10-dims. is given by:

$$\begin{aligned}
e^{-2\phi} &= \frac{1}{g_s^2}\left(1 + \frac{r_5^2}{r^2}\right)\left(1 + \frac{r_1^2}{r^2}\right)^{-1}, \\
F^{(3)} &= \frac{2r_5^2}{g_s}\epsilon_3 + 2g_s e^{-2\phi} r_1^2 *_6 \epsilon_3, \\
ds^2 &= \left(1 + \frac{r_1^2}{r^2}\right)^{-1/2}\left(1 + \frac{r_5^2}{r^2}\right)^{-1/2}\Bigg[-dt^2 + dx_5^2 \\
&\quad + \frac{r_0^2}{r^2}(\cosh\sigma dt + \sinh\sigma dx_5)^2 + \left(1 + \frac{r_1^2}{r^2}\right) g_s Q_5 (dx_6^2 + \cdots + dx_9^2)\Bigg] \\
&\quad + \left(1 + \frac{r_1^2}{r^2}\right)^{1/2}\left(1 + \frac{r_5^2}{r^2}\right)^{1/2}\left[\left(1 - \frac{r_0^2}{r^2}\right)^{-1} dr^2 + r^2 d\Omega_3^2\right],
\end{aligned} \tag{19}$$

where $*_6$ is the Hodge dual in the six dimensions $x_0, \ldots, x_5$ and ϵ_3 is the volume form on the unit three-sphere. x_5 is periodically identified with period $2\pi R_5$ and directions

$x_6, \ldots, x_9$ are compactified on a torus T^4 of volume V_4. Ω_3 is the volume of the unit three-sphere in the transverse directions. This solution is parameterized by six independent quantities: $r_1, r_5, r_0, \sigma, R_5$ and V_4. These are related to the number of D_1-branes, D_5-branes and Kaluza–Klein momentum on x_5 as follows,

$$\begin{aligned} Q_1 &= \frac{V_4}{64\pi^6 g_s^2 \alpha'^3} \int \mathrm{e}^{2\phi} *_6 F^{(3)} = \frac{V_4 r_1^2}{16\pi^4 \alpha'^3 g_s}, \\ Q_5 &= \frac{1}{4\pi^2 \alpha'} \int F^{(3)} = \frac{r_5^2}{g_s \alpha'}, \\ N &= \frac{R_5^2 V_4 r_0^2}{32\pi^4 \alpha'^4 g_s^2} \sinh 2\sigma. \end{aligned} \tag{20}$$

r_0 is the non-extremality parameter. At $r_0 = 0$, the two classical horizons coincide. On compactifying this solution to five dimensions using the Kaluza–Klein ansatz one obtains a five-dimensional black hole with a horizon at $r = r_0$. The entropy and the mass of this black hole is given by

$$\begin{aligned} S &= \frac{A}{4G_5} = \frac{2\pi^2 r_1 r_5 r_0 \cosh 2\sigma}{4G_5}, \\ M &= \frac{\pi}{4G_5} \left(r_1^2 + r_5^2 + \frac{r_0^2 \cosh 2\sigma}{2} \right), \end{aligned} \tag{21}$$

where the five-dimensional Newton's constant is

$$G_5 = \frac{4\pi^5 \alpha'^4 g_s^2}{V_4 R_5}. \tag{22}$$

Let us now discuss the restrictions on the various parameters which result from the requirement that the above solution makes sense in the quantum theory and that we are actually describing a macroscopic black hole whose horizon is much larger than the string length $l_s = \sqrt{\alpha'}$. The above classical solution has a quantum significance only if the string coupling $g_s \to 0$. This implies that the Newton coupling $G_5 \to 0$, and hence the entropy formula (16) implies that we have a finite horizon area only if

$$\begin{aligned} &g_s \to 0, \\ &\text{with } g_s Q_1,\ g_s Q_5,\ g_s^2 N \text{ fixed.} \end{aligned} \tag{23}$$

The formulae in (20) indicate that this is also equivalent to

$$\begin{aligned} &g_s \to 0, \\ &\text{with } r_1,\ r_5,\ r_n \text{ fixed.} \end{aligned} \tag{24}$$

where $r_N = r_0 \sinh \sigma$. For a macroscopic black hole we require that the string length is much smaller than the horizon area, or equivalently from (21) we conclude that $r_1, r_5, r_N \gg l_s$. This implies

$$g_s Q_1 \gg 1,\ g_s Q_5 \gg 1,\ g_s^2 N \gg 1. \tag{25}$$

Since $g_s Q_1$, $g_s Q_5$ correspond to the effective open string coupling constants, the macroscopic black hole exists at strong coupling!

The non-BPS black hole has a small Hawking temperature given by

$$T_H = \frac{r_0}{2\pi r_1 r_5 \cosh\sigma} \sim \frac{r_0 \exp -\sigma}{\pi r_1 r_5} \ll 1. \tag{26}$$

In the near extremal limit, when for large σ, $r_0 \sim \exp -\sigma$, we see that $T_{\mathrm{H}} \sim 0(r_0^2)$.

We also note that the black hole has a positive specific heat $\Delta M = cT_{\mathrm{H}}^2 > 0$. This is unlike the case of the Schwarzschild bh : $\Delta M < 0$.

6.3 *The near horizon limit of Maldacena*

In this section we will exhibit the form of the classical solution in the so called near horizon limit of Maldacena [17]. To explain the basic point let us study the metric of the black hole with the KK charge $N = 0$. In this case the horizon area shrinks to zero, but that is not relevant to the physical point we want to make. The metric then takes the form,

$$\begin{aligned}
ds^2 &= f_1^{-\frac{1}{2}} f_5^{-\frac{1}{2}} (-dt^2 + dx_5^2) + f_1^{\frac{1}{2}} f_5^{\frac{1}{2}} (dx_1^2 + \cdots + dx_4^2) \\
&\quad + f_1^{\frac{1}{2}} f_5^{-\frac{1}{2}} (dx_6^2 + \cdots + dx_9^2), \\
\mathrm{e}^{-2\phi} &= \frac{1}{g_s^2} f_5 f_1^{-1}, \\
C_{05}^2 &= \frac{1}{2}(f_1^{-1} - 1), \\
F_{abc}^{(3)} &= (dC^{(2)})_{abc} = \frac{1}{2}\epsilon_{abcd}\partial_d f_5, \quad a,b,c,d = 1,2,3,4
\end{aligned} \tag{27}$$

where f_1 and f_5 are given by

$$f_1 = \frac{16\pi^4 g_s \alpha'^3 Q_1}{V_4 r^2}, \quad f_5 = \frac{g_s \alpha' Q_5}{r^2}, \tag{28}$$

here $r^2 = x_1^2 + x_2^2 + x_3^2 + x_4^2$ denotes the distance measured in the transverse direction to all the D-branes.

The basic idea of the near horizon limit is that, near the horizon of a black hole, the energies of particles as seen by the asymptotic observer get red-shifted:

$$E_\infty = \sqrt{G^{00}} E. \tag{29}$$

In the metric at hand the red-shift factor is

$$\sqrt{G}^{00} = (f_1 f_5)^{-1/4}. \tag{30}$$

Clearly as $r \to \infty$ the red shift factor is unity. However near the horizon we get the equation

$$E_\infty = \frac{r}{R} E, \tag{31}$$

where $R^2 \sim \alpha' \sqrt{g_s^2 Q_1 Q_5}$ is the typical length scale that characterizes the geometry. For $r \ll R$ we see that the energy observed by the asymptotic observer goes to zero for finite values of E. This means that near the horizon (characterized by large R) an excitation of arbitrary energy looks massless. For massless modes this means that they have almost infinitely long wavelengths and for massive modes they appear as long wavelength massless excitations. If one examines the potential energy of a particle in the above geometry then in the near horizon limit the potential barrier becomes very high so that the modes near the horizon cannot get out. In the exact limit of Q_1 and Q_5 going to infinity the horizon degrees of freedom become exactly massless and decouple from the bulk degrees of freedom. As we shall see later it is in this limit that the bulk string theory is dual to a SCFT which also exhibits massless behavior in the infrared.

A more precise scaling limit of the geometry is given by

$$\alpha' \to 0, \quad \frac{r}{\alpha'} \equiv U = \text{fixed}$$
$$v \equiv \frac{V_4}{16\pi^4 \alpha'^2} = \text{fixed}, \quad g_6 = \frac{g_s}{\sqrt{v}} = \text{fixed}. \tag{32}$$

In this limit the metric in (27) becomes

$$ds^2 = \alpha' \left[\frac{U^2}{g_6 \sqrt{Q_1 Q_5}} (-dx_0^2 + dx_5^2) + g_6 \sqrt{Q_1 Q_5} \frac{dU^2}{U^2} + g_6 \sqrt{Q_1 Q_5} d\Omega_3^2 \right]$$
$$+ \sqrt{\frac{Q_1}{v Q_5}} (dx_6^2 + \ldots + dx_9^2). \tag{33}$$

Thus the near horizon geometry is that of Ad$S_3 \times S^3 \times T^4$. Our notation for coordinates here is as follows: AdS_3 : (x_0, x_5, r); S^3 : (χ, θ, ϕ); T^4 : (x_6, x_7, x_8, x_9). r, χ, θ, ϕ are spherical polar coordinates for the directions x_1, x_2, x_3, x_4. The radius of S^3 and the anti-deSitter space is $R = \sqrt{\alpha'} (g_6^2 Q_1 Q_5)^{1/4}$.

Note that the effective string coupling in the near horizon limit is given by

$$g_{\text{eff}} = g_6 \sqrt{Q_1 / Q_5}. \tag{34}$$

The formulas for the black hole entropy and temperature, which depend only on the near horizon properties of the geometry, do not change in the near horizon limit.

It is important to mention the symmetries of the near horizon geometry. The bosonic symmetries arise from the isometries of Ad$S_3 \times S^3$. The isometries of the AdS_3 space form the non-compact group $SO(2,2)$, while the isometries of S^3 form the group $SO(4)_E = SU(2)_E \times \widetilde{SU(2)}_E$. The supergroups that contain this bosonic subgroup $SO(2,2) \times SO(4)_E = (SL(2,R) \times SU(2)) \times (SL(2,R) \times SU(2))$ are either $O\mathrm{sp}(3|2,R) \times O\mathrm{sp}(3|2,R)$ and $SU(1,1|2) \times SU(1,1|2)$. It is the latter that corresponds to the symmetry of the D_1–D_5 system because the supercharges in this case transform as a spinor of $SO(4)_E$. We shall see that the identification of the near horizon symmetry group $SU(1,1|2) \times SU(1,1|2)$ plays a crucial role in matching SCFT operators with the (dual) supergravity modes.

6.4 *Supergravity solution with non-zero vev of* $B_{\rm NS}$

Our discussion so far has been devoted to SUGRA solutions in which the values of all the moduli fields were set to zero. Such solutions have the characteristic that the mass of the D_1–D_5 system is a sum of the charges that characterize the system. Such bound states are marginal, without any binding energy, and can fragment into clusters of D_1–D_5 branes. The corresponding CFT has singularities. In order to obtain a stable bound state and a non-singular CFT we have to turn on certain moduli fields. We will consider the case when $B_{\rm NS}$ is non-zero.

The construction of the supergravity solution that corresponds to a $\frac{1}{4}$ BPS configuration, with a non-zero $B_{\rm NS}$ was presented in [16]. See also [18]. $B_{\rm NS}$ has non-zero components only along the directions 6,7,8,9 of the internal torus. From the view point of open string theory this is then a non-commutative torus.

Here we will summarize the result. The solution contains, besides D_1 and D_5 brane charges, D_3 brane charges that are induced by the $B_{\rm NS}$. For simplicity we consider only non-zero values for B_{79} and B_{68}. The asymptotic values are given by $B_{79}^{(\infty)} = b_{79}$ and $B_{68}^{(\infty)} = b_{68}$. It is important that at least 2 components of the $B_{\rm NS}$ are non-zero, in order to be able to discuss the self-dual and anti-self-dual components.

Below we present the full solution which can be derived by a solution generating technique. Details can be found in [16].

$$\begin{aligned} ds^2 &= (f_1 f_5)^{-1/2}(-dt^2 + (dx^5)^2) + (f_1 f_5)^{1/2}(dr^2 + r^2 d\Omega_3^2) \\ &\quad + (f_1 f_5)^{1/2}\left\{ Z_\varphi^{-1}((dx^6)^2 + (dx^8)^2) + Z_\psi^{-1}((dx^7)^2 + (dx^9)^2)\right\}, \end{aligned} \tag{35}$$

$$e^{2\phi} = f_1 f_5 / Z_\varphi Z_\psi, \tag{36}$$

$$\begin{aligned} B_{\rm NS}^{(2)} &= (Z_\varphi^{-1}\sin\varphi\cos\varphi(f_1 - f_5) + b_{68}) dx^6 \wedge dx^8 \\ &\quad + (Z_\psi^{-1}\sin\psi\cos\psi(f_1 - f_5) + b_{79}) dx^7 \wedge dx^9, \end{aligned} \tag{37}$$

$$F^{(3)} = \cos\varphi\cos\psi \tilde{K}^{(3)} + \sin\varphi\sin\psi K^{(3)}, \tag{38}$$

$$\begin{aligned} F^{(5)} &= Z_\varphi^{-1}(-f_5\cos\varphi\sin\psi K^{(3)} + f_1\cos\psi\sin\varphi\tilde{K}^{(3)}) \wedge dx^6 \wedge dx^8 \\ &\quad + Z_\psi^{-1}(-f_5\cos\psi\sin\varphi K^{(3)} + f_1\cos\varphi\sin\psi\tilde{K}^{(3)}) \wedge dx^7 \wedge dx^9, \end{aligned} \tag{39}$$

$$\begin{aligned} Z_{\varphi,\psi} &= 1 + \frac{\mu_{\varphi,\psi}}{2}\left(\frac{\alpha'}{r^2}\right), \quad \mu_\varphi = \mu_1 \sin^2\varphi + \mu_5\cos^2\varphi, \\ &\quad \mu_\psi = \mu_1\sin^2\psi + \mu_5\cos^2\psi. \end{aligned} \tag{40}$$

Here b_{68} and b_{79} are arbitrary constants which we have added at the end by a T-duality transformation that shifts the NS B-field by a constant. Note that for $\varphi = \psi = 0$ and $b_{68} = b_{79} = 0$, the above solution reduces to the known solution for D_1–D_5 system without B-field.

The above solution depends upon 4 parameters μ_1, μ_5, and the angles ϕ and ψ, and in general represents a system of D_1, D_5 and D_3 branes. Since we are seeking a solution that has no source D_3 branes we require that the D_3 brane charges are only induced by the presence of the non-zero $B_{\rm NS}$. This leads to certain conditions on the solutions which we do not derive here, but whose physical implication we analyze. We discuss both the asymptotically flat and near horizon geometry.

Asymptotically flat geometry

In this case the induced D_3 brane charges along the (5,7,9) and (5,6,8) directions are

$$Q_3 = B^{(\infty)}_{79} Q_5, \quad Q'_3 = B^{(\infty)}_{68} Q_5, \tag{41}$$

where

$$B^{(\infty)}_{79} = b_{79}, \quad B^{(\infty)}_{68} = b_{68}. \tag{42}$$

There is a induced contribution to the D_1 brane charge. The charge $Q_{1\rm s}$ of the source D_1 branes is

$$Q_{1\rm s} = Q_1 - b_{68}\ b_{79}\ Q_5, \tag{43}$$

while the D_5 brane charge remains unaffected by the moduli.

Mass

Let us now study the mass formula as a function of the charges and the moduli. The mass corresponding to the $\frac{1}{4}$ BPS solution [15], which coincides with the ADM mass, is given in terms of the appropriate charges by

$$M^2 = (Q_1 + Q_5)^2 + (Q_3 - Q'_3)^2. \tag{44}$$

This can in turn be expressed in terms of $Q_{1\rm s}$, Q_5 and b_{68}, b_{79}

$$M^2 = (Q_{1\rm s} + b_{68} b_{79} Q_5 + Q_5)^2 + Q_5^2 (b_{68} - b_{79})^2 \tag{45}$$

We must consider the mass as a function of the moduli, holding $Q_{1\rm s}$ and Q_5 fixed. We see that for non-zero moduli we have a true bound state that turns marginal when the moduli are set to zero. To locate the values of the moduli which minimize the mass, we extremize the mass w.r.t the moduli. The extremal values of the moduli are

$$b_{68} = -b_{79} = \pm\sqrt{Q_{1\rm s}/Q_5 - 1}, \tag{46}$$

This says that the $B_{\rm NS}$ moduli are self-dual, in the asymptotically flat metric. The mass at the critical point of the true bound state is then given by

$$M^2 = 4 Q_{1\rm s} Q_5. \tag{47}$$

Near horizon geometry

In this case, absence of D_3-brane sources is ensured if we set

$$Q_3^{(h)} = B_{79}^{(h)} Q_5, \quad Q_3^{(h)'} = B_{68}^{(h)} Q_5, \tag{48}$$

where

$$B_{68}^{(h)} = \frac{\mu_1 - \mu_5}{\mu_\varphi} \sin\varphi \cos\varphi + b_{68}, \tag{49}$$

$$B_{79}^{(h)} = \frac{\mu_1 - \mu_5}{\mu_\psi} \sin\psi \cos\psi + b_{79}, \tag{50}$$

are the horizon values of the two nonzero components of the B-field. Moreover, we see that in this case

$$\frac{B_{68}^{(h)}}{\mu_\psi} = -\frac{B_{79}^{(h)}}{\mu_\varphi}, \tag{51}$$

which is the self-duality condition on the B-field in the near horizon geometry. We also note that the volume of T^4 at the horizon is given by

$$V_{T^4}^{(h)} = \frac{\mu_1 \mu_5}{\mu_\varphi \mu_\psi} = \frac{Q_{1\mathrm{s}}^{(h)}}{Q_5}. \tag{52}$$

The D_1-brane charge that arises from source D_1-branes in this case is given by

$$Q_{1\mathrm{s}}^{(h)} = Q_1^{(h)} - B_{68}^{(h)} B_{79}^{(h)} Q_5. \tag{53}$$

One can show that

$$Q_{1\mathrm{s}}^{(h)} = Q_{1\mathrm{s}}, \tag{54}$$

where $Q_{1\mathrm{s}}$ is given by (53). Thus we see that not only do the parameters b_{68} and b_{79} have the same values here as in the asymptotically flat case, even the source D_1-branes are identical, despite the total D_1-brane charges being very different in the two cases.

Mass

The $\frac{1}{4}$ BPS mass formula in terms of the various charge densities in this case is

$$\left(\frac{M^{(h)}}{V_{T^4}^{(h)}}\right)^2 = \left(\frac{Q_1^{(h)}}{V_{T^4}^{(h)}} + Q_5\right)^2 + \left(\frac{Q_3^{(h)}}{\sqrt{g_{77} g_{99}}} - \frac{Q_3^{(h)'}}{\sqrt{g_{66} g_{88}}}\right)^2. \tag{55}$$

Using (48)–(54) it can be easily seen that

$$\left(M^{(h)}\right)^2 = V_{T^4}^{(h)} \left(4 Q_{1\mathrm{s}} Q_5\right). \tag{56}$$

Apart from the extra factor of the T^4 volume in the near horizon geometry, this is exactly the same as (47). The extra volume factor correctly takes into account the difference in the 6-dimensional Newton's constant between the asymptotically flat and near horizon geometries because of the difference in the T^4 volume in the two cases. We have already seen that the B-field is automatically self-dual in the near horizon geometry and that the volume of T^4 satisfies the condition given by (52) and (53). We now see that the mass of the bound state is already at the fixed point value. Thus the solution we have here provides an explicit demonstration of the attractor mechanism [19].

The significance of this solution is that it is the description of a stable bound state in the near horizon geometry. As we shall discuss later this situation corresponds to a non-singular dual CFT.

6.5 *Semi-classical absorption cross-section and semi-classical Hawking radiation formula*

Now that we have discussed the various classical solutions we want to summarize the basic steps in the calculation of the semi-classical absorption cross-section and its relation to the emission rate of Hawking radiation from a black hole [20–22]. We do the calculation for minimal scalars in the s-wave. These fields satisfy a linear equation in which only the Einstein metric is present, leading to a great simplification in the calculation.

$$D_\mu \partial^\mu \varphi = 0. \tag{57}$$

For the 5-dim. black hole discussed earlier the s-wave radial equation becomes

$$\left[\frac{h}{r^3}\frac{d}{dr}\left(hr^3\frac{d}{dr}\right) + fw^2\right] R_w(r) = 0 \tag{58}$$

where $f = f_1 f_5$ and

$$\varphi = R_w(r)\exp[-iwt]. \tag{59}$$

Introducing $\psi = r^{3/2}R$ and $r_* = r + \frac{r_0}{2}\ln\left|\frac{r-r_0}{r+r_0}\right|$ we have the Schrödinger type equation

$$\left[-\frac{d^2}{dr_*^2} + V_w(r_*)\right]\psi = 0 \tag{60}$$

where

$$V_w(r_*) = -w^2 f + \frac{3}{4r^2}(1 + 2r_0^2/r^2 - 3r_0^4/r^4). \tag{61}$$

The basic idea is to solve the equation in 2 regions with appropriate boundary conditions and then match the solution in the overlapping region. In order to do so we need to choose the parameters characterizing the solution to be in the following range,

$$\begin{aligned} r_0, r_n &\ll r_1, r_5, \\ wr_5 &\ll 1, \\ r_1 &\sim r_5,\ r_0 \sim r_n. \end{aligned} \tag{62}$$

Also the wave length of the incident radiation $1/w$ is comparable to the thermal wavelength specified by the $1/T_H$. The far and near solutions are matched at a point r_m such that

$$r_0, r_n \ll r_m \ll r_1, r_5, \qquad wr_1 \ll r_m/r_1 \tag{63}$$

Far Zone ($r \geq r_m$):

Here the potential V_w becomes (in terms of $\rho = wr$)

$$V_w(\rho) = -w^2 \left(1 - \frac{3}{4\rho^2}\right). \tag{64}$$

This is Bessel's equation, so that

$$\begin{aligned} \psi &= \alpha F(\rho) + \beta G(\rho) \\ F(\rho) &= \sqrt{\pi\rho/2} J_1(\rho), \quad G(\rho) = \sqrt{\pi\rho/2} N_1(\rho). \end{aligned} \tag{65}$$

For $\rho \to \infty$ one can easily see the coefficients of the incoming wave e^{-iwr} and the outgoing wave e^{iwr}.

Near zone ($r \leq r_m$):

This is the region near the pit of the potential, or the throat region. Here we get a hypergeometric equation,

$$\frac{h}{r^3}\frac{d}{dr}\left(hr^3 \frac{d}{dr} R\right) + \left[\frac{(wr_n r_1 r_5)^2}{r^6} + \frac{w^2 r_1^2 r_5^2}{r^4}\right] R_w(r) = 0 \tag{66}$$

which is solved by

$$\begin{aligned} R &= AR_{\mathrm{in}} + BR_{\mathrm{out}}, \\ R_{\mathrm{in}} &= z^{-i(a+b)/2} F(-ia, -ib, 1 - ia - ib, z), \\ R_{\mathrm{out}} &= z^{i(a+b)/2} F(-ia, -ib, 1 - ia - ib, z), \\ z &= (1 - r_0^2/r^2), \\ a &= w/(4\pi T_R), b = w/(4\pi T_L). \end{aligned} \tag{67}$$

The temperatures $T_{R,L}$ are given by

$$T_{L,R} = \frac{r_0}{2\pi r_1 r_5} \mathrm{e}^{\pm\sigma}. \tag{68}$$

The important boundary condition that we impose is $B = 0$. This says that at the black hole horizon there is no outgoing wave.

R and $\frac{d}{dr} R$ can now be matched in the overlapping region (below the potential barrier) at some point r_m. The matching conditions imply

$$\begin{aligned} \sqrt{\pi/2} w^{3/2} \alpha/2 &= Ae_1, \\ e_1 &\equiv \frac{\Gamma(1 - ia - ib)}{\Gamma(1 - ib)\Gamma(1 - ia)}, \\ \beta/\alpha &\ll 1. \end{aligned} \tag{69}$$

Now that we have constructed the solution we can calculate the flux from the Schrödinger equation

$$\mathcal{F}(r) = \frac{1}{2i}[R^* h r^3 dR/dr - \text{c.c.}] \tag{70}$$

This flux is 'conserved' $\frac{d}{dr}\mathcal{F} = 0$.

The fraction of the flux that gets absorbed at the horizon is given by the ratio of the flux calculated from the solution at the horizon (where we used the horizon boundary condition) and the flux due to the incoming spherical wave from infinity,

$$R_1 = \mathcal{F}(r_0)/\mathcal{F}^{\text{in}}(\infty) = r_0^2 \frac{a+b}{w|e_1|^2} w^3 \pi/2. \tag{71}$$

Absorption cross-section: To calculate the absorption cross-section of an incident plane wave as opposed to the spherical wave that we did the above calculation with, we have to introduce a conversion factor. This is easily done by the expansion of a plane wave in terms of spherical waves,

$$\mathrm{e}^{-iwz} = (4\pi/w^3)\mathrm{e}^{-iwr} Z_{000} + \text{other partial waves.} \tag{72}$$

Taking this into account we get [22]

$$\begin{aligned}\sigma_{\text{abs}} &= (4\pi/w^3)R_1 \\ &= 2\pi^2 r_1^2 r_5^2 \frac{\pi w}{2} \frac{\exp(w/T_H) - 1}{(\exp(w/2T_R) - 1)(\exp(w/2T_L) - 1)}.\end{aligned} \tag{73}$$

In the $w \to 0$ limit, one gets [20]

$$\sigma_{\text{abs}} = A_{\text{h}} \tag{74}$$

where A_{h} denotes the area of the event horizon.

The decay rate is given by the well known formula of Hawking,

$$\Gamma = \text{Prob}_{\text{decay}} \frac{V_4}{\tilde{R}T} \frac{d^4k}{(2\pi)^4}, \;\; \tilde{R} = Q_1 Q_1 R \tag{75}$$

giving

$$\Gamma_H = \sigma_{\text{abs}} (\mathrm{e}^{w/T_H} - 1)^{-1} \frac{d^4k}{(2\pi)^4}. \tag{76}$$

With this we conclude our discussion of supergravity aspects and now turn to explaining some of the important thermodynamical formulas from the viewpoint of string theory.

7. Microscopic modeling of the black hole in terms of the D_1–D_5 system

Our aim here is to study the low energy collective excitations of the D_1–D_5 system. There are two ways to proceed and we shall discuss both of them. The first method is a description in terms of a 2-dim. gauge theory and the second method involves identifying D_1 branes with instantons of a 4 dim. gauge theory. The latter description is more accurate and is valid for instantons of all sizes. The 2-dim. gauge theory description is valid for small instanton size but it is more physical and gives a feeling for the dynamics. We will discuss this more approximate description first.

8. The D_1–D_5 system and the $\mathcal{N}=4$, $U(Q_1)\times U(Q_5)$ gauge theory in 2-dimensions

Consider type IIB string theory with five coordinates, say $x^5, \cdots, x^9$, compactified on $S^1 \times T^4$. The microscopic model consists Q_1 D_1-branes and Q_5 D_5-branes [7,12]. The D_1-branes are parallel to the x^5 coordinate compactified to a circle S^1 of radius R, while the D_5-branes are parallel to x^5 and $x^6, \cdots, x^9$ compactified on a torus T^4 of volume V_4. The charge N is related to the momenta of the excitations of this system along S^1. We take the T^4 radii to be of the order of α' and smaller than R which, in turn, is much smaller than the black hole radius.

We shall see that the low-energy dynamics of this D-brane system is described by a $U(Q_1) \times U(Q_5)$ gauge theory in two dimensions with $N = 4$ supersymmetry [13,23]. The gauge theory will be assumed to be in the Higgs phase because we are interested in the bound state where the branes are not seperated from each other in the transverse direction. In order to really achieve this and prevent branes from splitting off we will turn on the Fayet–Illiopoulos parameters. In supergravity these correspond to the vev of the Neveu–Schwarz $B_{\rm NS}$. In principle we can also turn on the θ term in the gauge theory. This corresponds to a vev of a certain linear combination of the RR 0-form and 4-form.

The elementary excitations of the D-brane system correspond to open strings with two ends attached to the branes and there are three classes of such strings: the (1,1), (5,5) and (1,5) strings. The associated fields fall into vector multiplets and hypermultiplets, using the terminology of $N = 2$, $D = 4$ supersymmetry.

(1,1) strings

The part of the spectrum coming from (1,1) strings is simply the dimensional reduction, to $1 + 1$ dimensions (the (t, x^5)-space), of the $N = 1$, $U(Q_1)$ gauge theory in $9 + 1$ dimensions [27].

The bosonic fields of this theory can be organized into the vector multiplet and the hypermultiplet of $N = 2$ theory in four-dimensions as

$$\begin{aligned}&\text{Vector multiplet: } A_0^{(1)}, A_5^{(1)}, Y_m^{(1)}, m = 1,2,3,4\\ &\text{Hypermultiplet: } Y_i^{(1)}, i = 6,7,8,9.\end{aligned} \tag{77}$$

The $A_0^{(1)}, A_5^{(1)}$ are the $U(Q_1)$ gauge fields in the non-compact directions. The $Y_m^{(1)}$'s and $Y_i^{(1)}$'s are gauge fields in the compact directions of the $N = 1$ super Yang-Mills in ten-dimensions. They are hermitian $Q_1 \times Q_1$ matrices transforming as adjoints of $U(Q_1)$. The hypermultiplets of $N = 2$ supersymmetry are doublets of the $SU(2)_R$ symmetry of the theory. The adjoint matrices $Y_i^{(1)}$'s can be arranged as doublets under $SU(2)_R$ as

$$N^{(1)} = \begin{pmatrix} N_1^{(1)} \\ N_2^{(1)\dagger} \end{pmatrix} = \begin{pmatrix} Y_9^{(1)} + iY_8^{(1)} \\ Y_7^{(1)} - iY_6^{(1)} \end{pmatrix}. \tag{78}$$

(5,5) strings

The field content of these massless open strings is similar to the the $(1,1)$ strings except for the fact that the gauge group is $U(Q_5)$ instead of $U(Q_1)$. Normally one would have expected the gauge theory of the $(5,5)$ strings to be a dimensional reduction of $N=1$ $U(Q_5)$ super Yang–Mills to $5+1$ dimensions. Since we are ignoring the Kaluza–Klein modes on T^4 this is effectively a theory in $1+1$ dimensions. The vector multiplets and the hypermultiplets are given by

$$\begin{aligned} &\text{Vector multiplet: } A_0^{(5)}, A_5^{(5)}, Y_m^{(5)}\; m=1,2,3,4 \\ &\text{Hypermultiplet: } Y_i^{(5)}\; i=6,7,8,9. \end{aligned} \tag{79}$$

The $A_0^{(5)}, A_5^{(5)}$ are the $U(Q_5)$ gauge fields in the non-compact directions. The $Y_m^{(5)}$'s and $Y_i^{(5)}$'s are gauge fields in the compact directions of the $N=1$ super Yang–Mills in ten-dimensions. They are hermitian $Q_5 \times Q_5$ matrices transforming as adjoints of $U(Q_5)$. The hypermultiplets $Y_i^{(5)}$'s can be arranged as doublets under $SU(2)_R$ as

$$N^{(5)} = \begin{pmatrix} N_1^{(5)} \\ N_2^{(5)\dagger} \end{pmatrix} = \begin{pmatrix} Y_9^{(5)} + iY_8^{(5)} \\ Y_7^{(5)} - iY_6^{(5)} \end{pmatrix}. \tag{80}$$

Since x^m are compact, the (1,1) strings can also have winding modes around the T^4. These are, however, massive states in the $(1+1)$-dimensional theory and can be ignored. This is because their masses are proportional to $R \gg \sqrt{\alpha'}$. Similarly, the part of the spectrum coming from (5,5) strings is the dimensional reduction, to $5{+}1$ dimensions, of the $N=1, U(Q_5)$ gauge theory in $9+1$ dimensions. In this case, the gauge field components $A_m^{(5)}$ ($m=6,7,8,9$) also have a dependence on x^m. Momentum modes corresponding to this dependence are neglected because the size of the 4-torus is of the order of the string scale $\sqrt{\alpha'}$. The neglect of the winding modes of the $(1,1)$ strings and the KK modes of the $(5,5)$ strings is consistent with T-duality. A set of four T-duality transformations along x^m interchanges D_1- and D_5-branes and also converts the momentum modes of the (5,5) strings along T^4 into winding modes of (1,1) strings around the dual torus [28]. Since these winding modes have been ignored, a T-duality covariant formulation requires that we should also ignore the associated momentum modes.

(1,5) and (5,1) strings

The field content obtained so far is that of $N=2$, $U(Q_1) \times U(Q_5)$ gauge theory, in $1+5$ dimensions, reduced to $1+1$ dimensions on T^4.

The $SO(4) \sim SU(2)_L \times SU(2)_R$ rotations on the tangent space of the torus act on the components of the adjoint hypermultiplets $X_m^{(1,5)}$ as an R-symmetry. To this set of fields we have to add the fields from the (1,5) sector that are constrained to live in $1+1$ dimensions by the ND boundary conditions. These strings have their ends fixed on different types of D-branes and, therefore, the corresponding fields transform in the fundamental representation of both $U(Q_1)$ and $U(Q_5)$. The ND boundary conditions have the important consequence that the (1,5) sector fields form a hypermultiplet which is chiral w.r.t. $SO(4)_I$. The chirality projection is due to the GSO projection. Hence the R-symmetry group is $SU(2)_R$.

$$\chi_{a\bar{b}} = \begin{pmatrix} A_{a\bar{b}} \\ B^{\dagger}_{a\bar{b}} \end{pmatrix}. \tag{81}$$

A few comments are in order:

1. The inclusion of these fields breaks the supersymmetry by half, to the equivalent of $N = 1$ in $D = 6$, and the final theory only has $SU(2)_R$ R-symmetry.
2. The fermionic superpartners of these hypermultiplets which arise from the Ramond sector of the massless excitations of $(1,5)$ and $(5,1)$ strings carry spinorial indices under $SO(4)_E$ and they are singlets under $SO(4)_I$.
3. The $U(1) \times \overline{U(1)}$ subgroup is important. One combination leaves the hypermultiplet invariant. The other combination is active and $(A_{a'a}, B_{a'a})$ have $U(1)$ charges $(+1, -1)$.
4. χ is a chiral spinor of $SO(4)_I$ with convention $\Gamma_{6789}\, \chi = -\chi$.
5. Since we are describing the Higgs phase in which all the branes sit on top of each other we have $Y_i^{(1,5)} = 0$.
6. In the above discussion, the fields Y_i and X_i along the torus directions are assumed to be compact. However it is not obvious how to compactify the range of χ so that the integration over this field in the path integral is finite.

In summary, the gauge theory of the D_1–D_5 system is a $1+1$ dimensional $(4,4)$ supersymmetric gauge theory with gauge group $U(Q_1) \times U(Q_5)$. The matter content of this theory consists of hypermultiplets $Y^{(1)}$'s, $Y^{(5)}$'s transforming as adjoints of $U(Q_1)$ and $U(Q_5)$ respectively. It also has the hypermultiplets χ's which transform as bi-fundamentals of $U(Q_1) \times \overline{U(Q_5)}$.

8.1 *The potential terms*

The lagrangian of the above gauge theory can be worked out from the dimensional reduction of $d = 4$, $\mathcal{N} = 2$ gauge theory.The potential energy density of the vector and hyper multiplets is a sum of 4 positive terms. In this section for convenience of notation we have defined $Y_i^{(1)} = Y_i$, $Y_i^{(5)} = X_i$, $Y_m^{(1)} = Y_m$, $Y_m^{(5)} = X_m$

$$V = V_1 + V_2 + V_3 + V_4, \tag{82}$$

$$V_1 = -\frac{1}{4g_1^2} \sum_{m,n} \mathrm{tr}_{U(Q_1)} [Y_m, Y_n]^2 - \frac{1}{4g_5^2} \sum_{m,n} \mathrm{tr}_{U(Q_5)} [X_m, X_n]^2, \tag{83}$$

$$V_2 = -\frac{1}{2g_1^2} \sum_{i,m} \mathrm{tr}_{U(Q_1)} [Y_i, Y_m]^2 - \frac{1}{2g_5^2} \sum_{i,m} [X_i, X_m]^2, \tag{84}$$

$$V_3 = \frac{1}{4}\sum_m \mathrm{tr}_{U(Q_1)}(\chi X_m - Y_m\chi)(X_m\chi^\dagger - \chi^\dagger Y_m)^2, \tag{85}$$

$$V_4 = \frac{1}{4}\mathrm{tr}_{U(Q_1)}\left(\chi i\Gamma^T_{ij}\chi^+ + i[Y_i, Y_j]^+ - \zeta^+_{ij}\frac{1}{Q_1}\right)^2$$

$$+\frac{1}{4}\mathrm{tr}_{U(Q_5)}\left(\chi^+ i\Gamma_{ij}\chi + i[X_i, X_j]^+ - \zeta^+_{ij}\frac{1}{Q_5}\right)^2. \tag{86}$$

The potential energy V_4 comes from a combination of F and D terms of the higher dim. gauge theory. $\Gamma_{ij} = \frac{i}{2}[\Gamma_i, \Gamma_j]$ are spinor rotation matrices. The notation a^+_{ij} denotes the self-dual part of the anti-symmetric tensor a_{ij}.

In V_4 we have included the Fayet–Iliopoulos (FI) terms ζ^+_{ij}, which form a triplet under $SU(2)_R$. Their inclusion is consistent with $N = 4$ SUSY. The FI terms can be identified with the self dual part of B_{ij}, the anti-symmetry tensor of the NS sector of the closed string theory [29]. This identification at this stage rests on the fact that (i) ζ^+_{ij} and B^+_{ij} have identical transformation properties under $SU(4)_I$ and (ii) at the origin of the Higgs branch where $\chi = X = Y = 0$, $V_4 \sim \zeta^+_{ij}\zeta^+_{ij}$. This signals a tachyonic mode from the view point of string perturbation theory. The tachyon mass is easily computed and this implies the relation $\zeta^+_{ij}\zeta^+_{ij} \sim B^+_{ij}B^+_{ij}$.

8.2 *D-flatness equations and the moduli space*

The supersymmetric ground state (semi-classical) is characterized by the 2-sets of D-flatness equations which are obtained by setting $V_4 = 0$. They are best written in terms of the $SU(2)_R$ doublet fields $N^{(1)}_{a'b'}$ and $N^{(5)}_{ab}$:

$$N^{(1)} = \begin{pmatrix} N^{(1)}_1 \\ N^{(1)}_2 \end{pmatrix} = \begin{pmatrix} Y_9 + iY_8 \\ Y_7 + iY_6 \end{pmatrix},$$

$$N^{(5)} = \begin{pmatrix} N^{(5)}_1 \\ N^{(5)}_2 \end{pmatrix} = \begin{pmatrix} X_9 + iX_8 \\ X_7 + iX_9 \end{pmatrix}. \tag{87}$$

We also define $\zeta = \zeta^+_{69}$ and $\zeta_c = \zeta^+_{67} + i\zeta^+_{68}$. With these definitions the 2 sets of D-flatness conditions become:

$$(AA^+ - B^+B)_{a'b'} + [N^{(1)}_1, N^{(1)\dagger}_1]_{a'b'} - [N^{(1)}_2, N^{(1)\dagger}_2]_{a'b'} = \frac{\zeta}{Q_1}\delta_{a'b'}, \tag{88}$$

$$(AB)_{a'b'} + [N^{(1)}_1, N^{(1)\dagger}_2]_{a'b'} = \frac{\zeta_c}{Q_1}\delta_{a'b'}, \tag{89}$$

$$(A^+A - BB^+)_{ab} + [N^{(5)}_1, N^{(5)\dagger}_1]_{ab} - [N^{(5)}_2, N^{(5)\dagger}_2]_{ab} = \frac{\zeta}{Q_5}\delta_{ab}, \tag{90}$$

$$(A^{+}B^{+})_{ab} + [N_1^{(5)}, N_2^{(5)\dagger}]_{ab} = \frac{\zeta_c}{Q_5}\delta_{ab}. \tag{91}$$

The hypermultiplet moduli space is a solution of the above equations modulo the gauge group $U(Q_1) \times U(Q_5)$. A detailed discussion of the procedure was given in [23,16]. Here we summarize.

If we take the trace parts of (88) we get the *same* set of 3 equations as the D-flatness equations for a $U(1)$ theory with $Q_1 Q_5$ hypermultiplets, with $U(1)$ charge assignment $(+1,-1)$ for $(A_{a'b}, B^T_{a'b})$. Thus,

$$\sum_{a'b}(A_{a'b}A^*_{a'b} - B^T_{a'b}B^{T*}_{a'b}) = \zeta, \tag{92}$$

$$\sum_{a'b} A_{a'b}B^T_{a'b} = \zeta_c. \tag{93}$$

For a given point on the surface defined by (92), (93) the traceless parts of (88) lead to $3Q_1^2 + 3Q_5^2 - 6$ constraints among $4Q_1^2 + 4Q_5^2 - 8$ degrees of freedom corresponding to the traceless parts of the adjoint hypermultiplets $N^{(1)}$ and $N^{(5)}$. Using $Q_1^2 + Q_5^2 - 2$ gauge conditions corresponding to $SU(Q_1) \times SU(Q_5)$ we have $(3Q_1^2 + 3Q_5^2 - 6) + (Q_1^2 + Q_5^2 - 2) = 4Q_1^2 + 4Q_5^2 - 8$ conditions for the $(4Q_1^2 + 4Q_5^2 - 8)$ degrees of freedom in the traceless parts of $N^{(1)}$ and $N^{(5)}$. The 8 degrees of freedom corresponding to $\mathrm{tr}X_i$ and $\mathrm{tr}Y_i$, $i = 6,7,8,9$ correspond to the centre-of-mass of the D_5 and D_1 branes respectively.

8.3 *The bound state in the Higgs phase*

Having discussed the moduli space that characterizes the SUSY ground state we can discuss the fluctuations of the transverse vector multiplet scalars X_m and Y_m, $m = 1,2,3,4$. In the Higgs phase since $\langle X_m\rangle = \langle Y_m\rangle = 0$ and $\chi = \overline{\chi}$ lies on the surface defined by (92), (93). The relevant action of fluctuations in the path integral is,

$$\begin{aligned} S = \sum_m \int dt dx_5 (\mathrm{tr}_{U(Q_5)}\partial_\alpha X_m \partial^\alpha X_m + \mathrm{tr}_{U(Q_1)}\partial_\alpha Y_m \partial^\alpha Y_m) \\ + \int dt dx_5 (V_2 + V_3). \end{aligned} \tag{94}$$

We restrict the discussion to the case when $Q_5 = 1$ and Q_1 is arbitrary. In this case the matrix X_m is a real number which we denote by x_m.

χ is a complex column vector with components $(A_{a'}, B_{a'})$, $a' = 1, ..., Q_1$. Since we are looking at the fluctuations of the Y_m only to quadratic order in the path integral, the integrals over the different Y_m decouple from each other and we can treat each of them separately. Let us discuss the fluctuation Y_1 and set $(Y_1)_{a'b'} = \delta_{a'b'}y_{1a'}$. Then the potential V_3, (85) becomes

$$V_3 = \sum_{a'}(|A_{a'}|^2 + |B_{a'}|^2)(y_{1a'} - x_1)^2. \tag{95}$$

We will prove that $|A_{a'}|^2 + |B_{a'}|^2$ can never vanish if the FI terms are non-zero. In order to do this let us analyze the complex D-term equation (93)

$$A_{a'}B_{b'} + [N_1^{(1)}, N_2^{(1)\dagger}]_{a'b'} = \frac{\zeta_c}{Q_1}\delta_{a'b'}. \tag{96}$$

We can use the complex gauge group $GL(C, Q_1)$ to diagonalize the complex matrix $N_1^{(1)}$ [24]. Then, (96) becomes

$$A_{a'}B_{b'} + (n_{a'} - n_{b'})(N_2^{(1)\dagger})_{a'b'} = \frac{\zeta_c}{Q_1}\delta_{a'b'}. \tag{97}$$

For $a' \neq b'$, this determines the non-diagonal components of $N_2^{(1)}$

$$(N_2^{(1)\dagger})_{a'b'} = -\frac{A_{a'}B_{b'}}{n_{a'} - n_{b'}}. \tag{98}$$

For $a = b$, we get the equations

$$A_{a'}B_{a'} = \frac{\zeta_c}{Q_1}, a' = 1, ..., Q_1 \tag{99}$$

which imply that

$$|A_{a'}||B_{a'}| = \frac{|\zeta_c|}{Q_1} \tag{100}$$

with the consequence that $|A_{a'}|$ and $|B_{a'}|$ are non-zero for all $a' = 1, .., Q_1$. This implies that $(|A_{a'}|^2 + |B_{a'}|^2) > 0)$, and hence the fluctuation $(y_{1a'} - x_1)$ is massive. If we change variables $y_{1a'} \to y_{1a'} + x_1$, then x_1 is the only flat direction. This corresponds to the global translation of the 5-brane in the x_1 direction.

A similar analysis can be done for all the remaining directions $m = 2, 3, 4$ with identical conclusions. This shows that a non-zero FI term implies a true bound state of the $Q_5 = 1$, $Q_1 = N$ system. If FI $= 0$, then there is no such guarantee and the system can easily fragment, due to the presence of flat directions in $(Y_m)_{a'b'}$.

What the above result says is that when the FI parameters are non-zero the zero mode of the fields $(Y_m)_{a'b'}$ is massive. If we regard the zero mode as a collective coordinate then the Hamiltonian of the zero mode has a quadratic potential which agrees with the near horizon limit of the Liouville potential derived in [29,16].

The general case with an arbitrary number of Q_1 and Q_5 branes seems significantly harder to prove, but the result is very plausible on physical grounds. If the potential for a single test D_1 brane is attractive, it is hard to imagine any change in this fact if there are 2 test D_1 branes, because the D_1 branes by themselves can form a bound state.

8.4 *The conformally invariant limit of the gauge theory*

The sigma model that describes the low energy modes corresponding to the hyper-multiplet moduli defined by the equations (88) is given by the lagrangian (bosonic part),

$$S = \sum_m \int dtdx_5 (\mathrm{tr}_{U(Q_5)} \partial_\alpha X_i \partial^\alpha X_i + \mathrm{tr}_{U(Q_1)} \partial_\alpha Y_i \partial^\alpha Y_i)$$
$$+ \int dtdx_5 (\partial_\alpha \chi \partial^\alpha \chi^\dagger). \tag{101}$$

This is a very difficult non-linear system, with $N = 4$ SUSY. Since we are interested in the low energy dynamics we may ask whether there is a SCFT fixed point. Such a SCFT must have (4,4) supersymmetry (16 real supersymmetries) with a central charge $c = 6(Q_1 Q_5 + 1)$. Now note that the equations (92), (93) describe a hyper-Kahler manifold and hence the sigma model defined on it is a SCFT with (4,4) SUSY. We can then consider the part of the action involving the X_i and Y_i which are solved in terms of the χ as giving a deformation of the SCFT. Now this deformation clearly reduces the SUSY to $N = 4$, but seems to preserve the original degrees of freedom. For this reason the deformation can be identified with a set of marginal or irrelevant operators. Inspite of the simplification at the fixed point this theory is difficult to work with.

The sigma model action at the conformally invariant point is

$$\int dtdx_5 \sum_{a'b} (\partial_\alpha A_{a'b} \partial_\alpha A^*_{a'b} - \partial_\alpha B^T_{a'b} \partial_\alpha B^{T*}_{a'b}). \tag{102}$$

The sigma model fields are constrained to be on the surface defined by (92), (93). Further after appropriate gauge fixing the residual gauge invariance inherited from the gauge theory is the Weyl group $S(Q_1) \times S(Q_5)$ [23]. The Weyl invariance can be used to construct gauge invariant strings of various lengths. If Q_1 and Q_5 are relatively prime it is indeed possible to prove the existence of a single winding string with minimum unit of momentum given by $\frac{1}{Q_1 Q_5}$. This is associated with the longest cyclic subgroup of $S(Q_1) \times S(Q_5)$. Cyclic subgroups of shorter length cycles lead to strings with minimum momentum $\frac{1}{l_1 l_5}$, where l_1 and l_5 are the lengths of the cycles. In a different way of describing these degrees of freedom we shall see in the next sections that strings of various lengths are associated with chiral primary operators of the conformal field theory on the moduli space of instantons on a 4-torus.

We conclude this section by showing that certain deductions about thermodynamic properties can be made just by the knowledge of the central charge and the level of the Virasoro algebra. This information is sufficient to calculate the number of micro-states. To find the microstates of the D_1–D_5 black hole we look for states with $L_0 = N_L$ and $\bar{L}_0 = N_R$. The assymptotic number of distinct states of this SCFT is given by Cardy's formula

$$\Omega = \exp\left(2\pi\left(\sqrt{Q_1 Q_5 N_L} + \sqrt{Q_1 Q_5 N_R}\right)\right). \tag{103}$$

From the Boltzmann formula one obtains

$$S = 2\pi\left(\sqrt{Q_1 Q_5 N_L} + \sqrt{Q_1 Q_5 N_R}\right). \tag{104}$$

This exactly reproduces the Bekenstein–Hawking entropy. For the extremal (BPS) case, $N_R = 0$, and for the near extremal case $N_L = N + n$ and $N_R = n$, where $n \ll N$. For the near extermal cases (104) also gives the correct Hawking temperature T_H

$$T_L^{-1} = \frac{1}{R}\frac{\partial S}{\partial N_L} = \frac{\pi}{R}\sqrt{\frac{Q_1 Q_5}{N_L}},$$
$$T_R^{-1} = \frac{1}{R}\frac{\partial S}{\partial N_R} = \frac{\pi}{R}\sqrt{\frac{Q_1 Q_5}{N_R}},$$
$$T_H^{-1} = \frac{1}{2}(T_L^{-1} + T_R^{-1}). \tag{105}$$

9. D_1 branes as instantons of the D_5 gauge theory

In this section we take a different approach to the description of D_1 branes [30]. We will see that we can find D_1 branes within D_5 branes! We begin with a theory of Q_5, D_5 branes along the compact coordinates $x_i, i = 5, 6, 7, 8, 9$. The low energy degrees of freedom of this system are described by a $N = 2$, $U(Q_5)$ gauge theory in 6 dimensions. This gauge theory has a dimensional coupling constant g_6, and hence it is not renormalizable. This means that it cannot capture degrees of freedom at the string scale and hence is valid for wavelengths much larger than the string scale which acts as a short distance cutoff.

In this gauge theory let us look for configurations which break the 16 supersymmetries to 8. The reason is that we know that the presence of D_1 branes would do exactly that. Further since the D_1 branes are strings moving in time along the x_5 direction and smeared all over the 4-torus ($x_i, i = 6, 7, 8, 9$) we look for gauge field configurations which, to begin with, depend only on the torus coordinates. Such configurations are well known and easy to find.

We consider the variation of the gaugino under a supersymmetry transformation and set it to zero

$$\delta_\epsilon \lambda = \Gamma_{ab} F^{ab} \epsilon = 0 \tag{106}$$

where a, b run over $6, 7, 8, 9$. It is easy to see that this is equivalent to

$$F_{ab} = \epsilon_{abcd} F^{cd},\ a, b, \ldots = 6, \ldots, 9 \tag{107}$$

where

$$\Gamma_{6789}\epsilon = \epsilon. \tag{108}$$

These are the instanton solutions of euclidean SYM_4. We assume that the instanton number is positive and equal to Q_1. These solutions are characterized by moduli whose variation does not change the action of SYM_6. Promoting these moduli to slowly varying functions of x_5, t we obtain stringy solitons of the 5-brane theory. In order to identify these solitons as D_1 branes we have to show that the instanton number density is a source of the Ramond 2-form $C^{(2)}$.

To do this consider the Chern–Simons terms of the world volume theory of the D_1 branes, [27].

$$\mu_5 \int d^6x [C^{(2)} \wedge F \wedge F] \tag{109}$$

which shows that non-zero values of F_{67}, F_{89} can act as a source term for $C_{05}^{(2)}$. The latter corresponds to a D_1-brane stretching in the 5 direction.

We can also verify the mass of the D_1 branes by simply evaluating the instanton action, and it turns out to be

$$M = \frac{1}{g_s^2}(a_1 g_s Q_1), \tag{110}$$

where

$$a_1 = \frac{R}{\alpha'}. \tag{111}$$

This is a beautiful realization of a brane as a soliton bound within another brane. The motions of these D_1 branes inside the 5-branes represent the low lying collective modes of the D_1–D_5 system. $R \gg \sqrt{\alpha'}$ once more implies that we neglect the winding modes of the soliton strings. The KK modes of the YM_6 are also neglected.

The technically difficult part here is that the moduli space of instantons on the 4-torus is not a well known mathematical object. For example the ADHM construction [32] is valid for R^4 and not T^4. There is a possibility that the ADHM construction for this case, in the limit of small instanton size involves the 2 sets of D-terms that we discussed in the last section. The advertised moduli space $\mathcal{M}$, of instantons on T^4, is the Hilbert Scheme of the symmetric product $(\tilde{T}^4)^{Q_1 Q_5}/S(Q_1 Q_5)$. ($\tilde{T}^4$ can be different from the compactification torus T^4.)

One can give physical arguments to support at least the topological aspect of the above result [33–35]. One uses the fact that the configuration of D_1–D_5, we are working with, is U-dual to a fundamental string with winding modes. The BPS states of this fundamental string (that is, states with either purely left moving or right moving oscillators) maps to the ground states of the D_1–D_5 system which is given by the dimension of the cohomology of $\mathcal{M}$.

Our attitude will be to consider the sigma model on $\mathcal{M}$, as a resolution of the sigma model on the orbifold $(\tilde{T}^4)^{Q_1 Q_5}/S(Q_1 Q_5)$. It so turns out that $\mathcal{M}$ is a hyper-Kahler manifold and hence one can define a $N = (4,4)$ SCFT. We will explicitly construct the $N = (4,4)$ orbifold SCFT and discuss its blowing up modes, which turn out to be 4 marginal operators of the SCFT.

Before we do that we would like to discuss the validity of our considerations in the strong coupling region where $g_s Q_1 Q_5 \gg 1$.

First we note that (107) is derived as a condition from supersymmetry and is independent of the coupling constant. These instantons also do not receive any stringy corrections [31]. After this we used the collective coordinate method to arrive at the long wavelength approximation. However in the standard procedure we have to assume that g_s is small and hence we can neglect the interactions of the collective coordinates with the other Yang-Mills quanta. However it can be shown that the moduli space and the corresponding sigma model does not receive any corrections in the string coupling. This is basically because the hypermultiplet moduli space does not get renormalized by the interactions [25,26]. This fact is crucial because it says that the SCFT that we found at weak coupling is valid at strong coupling and hence we can use it to make comparisons with supergravity calculations.

10. The $\mathcal{N} = 4$ super conformal algebra

We now discuss the super conformal algebra generated by the holomorphic stress tensor $T(z)$, a doublet of supersymmetry generators $G^a(z), G^{b\dagger}(z)$ and $J^i(z)$ the $SU(2)$ R local symmetry. There are also corresponding anti-holomorphic generators $\widetilde{J}(\bar{z}), \widetilde{G}(\bar{z})$ and $\widetilde{T}(\bar{z})$.

$$\begin{aligned}
T(z)T(w) &= \frac{\partial T(w)}{z-w} + \frac{2T(w)}{(z-w)^2} + \frac{c}{2(z-w)^4},\\
G^a(z)G^{b\dagger}(w) &= \frac{2T(w)\delta_{ab}}{z-w} + \frac{2\sigma^i_{ab}\partial J^i}{z-w} + \frac{4\sigma^i_{ab}J^i}{(z-w)^2} + \frac{2c\delta_{ab}}{3(z-w)^3},\\
J^i(z)J^j(w) &= \frac{i\epsilon^{ijk}J^k}{z-w} + \frac{c}{12(z-w)^2},\\
T(z)G^a(w) &= \frac{\partial G^a(w)}{z-w} + \frac{3G^a(z)}{2(z-w)^2},\\
T(z)G^{a\dagger}(w) &= \frac{\partial G^{a\dagger}(w)}{z-w} + \frac{3G^{a\dagger}(z)}{2(z-w)^2},\\
T(z)J^i(w) &= \frac{\partial J^i(w)}{z-w} + \frac{J^i}{(z-w)^2},\\
J^i(z)G^a(w) &= \frac{G^b(z)(\sigma^i)^{ba}}{2(z-w)},\\
J^i(z)G^{a\dagger}(w) &= -\frac{(\sigma^i)^{ab}G^{b\dagger}(w)}{2(z-w)}.
\end{aligned} \tag{112}$$

The global R-parity group, is given by the zero modes of the currents $J^i(z)$ and $\widetilde{J}(\bar{z})$. It is denoted by $SU(2)_R \times \widetilde{SU(2)}_R$, and it is an outer automorphism of the $N = (4,4)$ current algebra. The $N = (4,4)$ SCFT admits another global $SO(4)$ symmetry which we shall discuss subsequently.

10.1 *The supergroup* $SU(1,1|2)$

We now discuss the zero mode part of the current algebra of the previous section. This is the Lie super-algebra $SU(1,1|2)$ generated by the global charges: $L_{\pm,0}, J_R^{(1),(2),(3)}$ and $G^a_{1/2,-1/2}$. The global charges of the supersymmetry currents $G^a(z)$ are in the Neveu–Schwarz sector.

$$\begin{aligned}
&[L_0, L_\pm] = \mp L_\pm \qquad [L_+, L_-] = 2L_0,\\
&\{G^a_{1/2}, G^{b\dagger}_{-1/2}\} = 2\delta^{ab}L_0 + 2\sigma^i_{ab}J_R^{(i)},\\
&\{G^a_{-1/2}, G^{b\dagger}_{1/2}\} = 2\delta^{ab}L_0 - 2\sigma^i_{ab}J_R^{(i)},\\
&\left[J_R^{(i)}, J_R^{(j)}\right] = i\epsilon^{ijk}J_R^{(k)},
\end{aligned}$$

$$
\begin{aligned}
\left[L_0, G^a_{\pm 1/2}\right] &= \mp\frac{1}{2}G^a_{\pm 1/2} \qquad & \left[L_0, G^{a\dagger}_{\pm 1/2}\right] &= \mp\frac{1}{2}G^{a\dagger}_{\pm 1/2},\\
\left[L_+, G^a_{1/2}\right] &= 0 \qquad & \left[L_-, G^a_{-1/2}\right] &= 0,\\
\left[L_-, G^a_{1/2}\right] &= -G^a_{-1/2} \qquad & \left[L_+, G^a_{-1/2}\right] &= G^a_{1/2},\\
\left[L_+, G^{a\dagger}_{1/2}\right] &= 0 \qquad & \left[L_-, G^{a\dagger}_{1/2}\right] &= 0,\\
\left[L_-, G^{a\dagger}_{1/2}\right] &= -G^a_{-1/2} \qquad & \left[L_+, G^{a\dagger}_{-1/2}\right] &= G^a_{1/2},\\
\left[J^{(i)}_R, G^a_{\pm 1/2}\right] &= \frac{1}{2}G^b_{\pm 1/2}(\sigma^i)^{ba} \qquad & \left[J^{(i)}_R, G^{a\dagger}_{\pm 1/2}\right] &= -\frac{1}{2}(\sigma^i)^{ba}G^{b\dagger}_{\pm 1/2}.
\end{aligned} \tag{113}
$$

The anti-holomorphic sector leads to an identical algebra so that the global Lie super-algebra is $SU(1,1|2) \times SU(1,1|2)$.

One can clearly see that the global group $SU(1,1|2)$ has the conformal group and $SU(2)$ as a sub-algebras. It also has 8 real supercharges (with spin and conformal dimension $= 1/2$) and hence including both the holomorphic and anti-holomorphic sectors we have a total of 16 real supercharges. In contrast to this the supergravity background, with zero KK momentun along the circle x_5, has the isometry $SU(1,1) \times SO(4)$ and only 8 real Poincaré supersymmetries.

10.2 *Maldacena duality: geometry dual to SCFT*

The puzzle of the doubling of the SUSYS is resolved by the remarkable discovery of Maldacena [17] that the geometry dual to the SCFT is infact not the asymptotically flat space times the internal torus T^4 (which has the isometry $SO(1,1) \times SO(4)_E$ and 8 real Poincaré supersymmetries). Instead the dual geometry is Ad$S_3 \times S^3 \times \tilde{T}^4$, with isometry $SO(2,2) \times SO(4)$. The duality conjecture, as far as the symmetries are concerned, states that the $SU(1,1|2) \times SU(1,1|2)$ symmetry of the near horizon geometry is matched with the global part of the $\mathcal{N} = (4,4)$ SCFT on $\mathcal{M}$ together with the identification of the $SO(4)_I$ algebra of T^4 and $\tilde{T}^4$.

Symmetries of the bulk	Symmetries of SCFT
(a) Isometries of AdS_3 $SO(2,2) \simeq SL(2,R) \times \widetilde{SL(2,R)}$	The global part of the Virasoro group $SL(2,R) \times \widetilde{SL(2,R)}$
(b) Isometries of S^3 $SO(4)_E \simeq SU(2) \times SU(2)$	R-symmetry of the SCFT $SU(2)_R \times \widetilde{SU(2)}_R$
(c) Sixteen near horizon symmetries	Global supercharges of $\mathcal{N} = (4,4)$ SCFT
(d) $SO(4)_I$ of T^4	$SO(4)_I$ of $\tilde{T}^4$

With this we conclude the general discussion of matching symmetries of the SCFT and the Ad$S_3 \times S^3 \times \tilde{T}^4$ geometry that is dual to it. In the following section we discuss a specific representation which has further symmetries that enable us to match operators and fields on both sides of the dual description.

11. $\mathcal{N} = (4,4)$ SCFT on the orbifold $(\tilde{T}^4)^{Q_1Q_5}/S(Q_1Q_5)$

The $\mathcal{N} = 4$ superconformal algebra with $c = 6Q_1Q_5$ can be constructed out of Q_1Q_5 copies of $c = 6$, $\mathcal{N} = 4$ superconformal algebra on $\tilde{T}^4$. The discussion in this and subsequent sections is mainly taken from [44,10].

The Lagrangian is given by

$$S = \frac{1}{2}\int d^2z\ \left[\partial x^i_A \bar{\partial} x_{i,A} + \psi^i_A(z)\bar{\partial}\psi^i_A(z) + \widetilde{\psi}^i_A(\bar{z})\partial\widetilde{\psi}^i_A(\bar{z})\right]. \tag{114}$$

Here i runs over the $\widetilde{T^4}$ coordinates 1, 2, 3, 4 (we have renamed the internal coordinates) and $A = 1, 2, \ldots, Q_1Q_5$ labels various copies of the four-torus. The symmetric group $S(Q_1Q_5)$ acts by permuting the copy indices.

Let us introduce some definitions. The complex bosons X and the fermions Ψ are defined as:

$$\begin{aligned}
X_A(z) &= (X^1_A(z), X^2_A(z)) = \sqrt{1/2}(x^1_A(z) + ix^2_A(z), x^3_A(z) + ix^4_A(z)),\\
\Psi_A(z) &= (\Psi^1_A(z), \Psi^2_A(z)) = \sqrt{1/2}(\psi^1_A(z) + i\psi^2_A(z), \psi^3_A(z) + i\psi^4_A(z)),\\
X^\dagger_A(z) &= \begin{pmatrix} X^{1\dagger}_A(z) \\ X^{2\dagger}_A(z) \end{pmatrix} = \sqrt{\frac{1}{2}}\begin{pmatrix} x^1_A(z) - ix^2_A(z) \\ x^2_A(z) - ix^2_A(z) \end{pmatrix},\\
\Psi^\dagger_A(z) &= \begin{pmatrix} \Psi^{1\dagger}_A(z) \\ \Psi^{2_A\dagger}(z) \end{pmatrix} = \sqrt{\frac{1}{2}}\begin{pmatrix} \psi^1_A(z) - i\psi^2_A(z) \\ \psi^3_A(z) - i\psi^4_A(z) \end{pmatrix}.
\end{aligned} \tag{115}$$

In terms of these we can write the generators of the SCFT,

$$\begin{aligned}
T(z) &= \partial X_A(z)\partial X^\dagger_A(z) + \frac{1}{2}\Psi_A(z)\partial\Psi^\dagger_A(z) - \frac{1}{2}\partial\Psi_A(z)\Psi^\dagger_A(z),\\
G^a(z) &= \begin{pmatrix} G^1(z) \\ G^2(z) \end{pmatrix} = \sqrt{2}\begin{pmatrix} \Psi^1_A(z) \\ \Psi^2_A(z) \end{pmatrix}\partial X^2_A(z)\\
&\quad + \sqrt{2}\begin{pmatrix} -\Psi^{2\dagger}_A(z) \\ \Psi^{1\dagger}_A(z) \end{pmatrix}\partial X^1_A(z),\\
J^i_R(z) &= \frac{1}{2}\Psi_A(z)\sigma^i\Psi^\dagger_A(z).
\end{aligned} \tag{116}$$

The charges corresponding to the R-parity current are

$$J^i_R = \frac{1}{2}\int \frac{dz}{2\pi i}\Psi_A(z)\sigma^i\Psi^\dagger_A(z). \tag{117}$$

In the above the summation over A which runs from 1 to Q_1Q_5 is implied.

11.1 *The $SO(4)$ global symmetry of the SCFT*

We now discuss global symmetries that are particular to the free field representation that we have discussed above. There are two global $SU(2)$ symmetries which correspond to the $SO(4)$ rotations of the 4 bosons x^i. The corresponding charges are given by

$$\begin{aligned}
I_1^i &= \frac{1}{4}\int \frac{dz}{2\pi i} X_A \sigma^i \partial X_A^\dagger - \frac{1}{4}\int \frac{dz}{2\pi i} \partial X_A \sigma^i X_A^\dagger \\
&\quad + \frac{1}{2}\int \frac{dz}{2\pi i} \Phi_A \sigma^i \Phi_A^\dagger, \\
I_2^i &= \frac{1}{4}\int \frac{dz}{2\pi i} \mathcal{X}_A \sigma^i \partial \mathcal{X}_A^\dagger - \frac{1}{4}\int \frac{dz}{2\pi i} \partial \mathcal{X}_A \sigma^i \mathcal{X}_A^\dagger.
\end{aligned} \tag{118}$$

Here

$$\begin{aligned}
\mathcal{X}_A &= (X_A^1, -X_A^{2\dagger}) \quad \mathcal{X}^\dagger = \begin{pmatrix} X_A^{1\dagger} \\ -X_A^2 \end{pmatrix} \\
\Phi_A &= (\Psi_A^1, \Psi_A^{2\dagger}) \quad \Phi_A^\dagger = \begin{pmatrix} \Psi_A^{1\dagger} \\ \Psi_A^2 \end{pmatrix}.
\end{aligned} \tag{119}$$

These charges generate the $SU(2) \times SU(2)$ algebra:

$$\begin{aligned}
\left[I_1^i, I_1^j\right] = i\epsilon^{ijk} I_1^k \qquad & \left[I_2^i, I_2^j\right] = i\epsilon^{ijk} I_2^k \\
\left[I_2^i, I_2^j\right] &= 0.
\end{aligned} \tag{120}$$

The new global charges have the following commutation relations with the local currents,

$$\begin{aligned}
\left[I_1^i, G^a(z)\right] &= 0 \quad \left[I_1^i, G^{a\dagger}(z)\right] = 0, \\
\left[I_1^i, T(z)\right] &= 0 \quad \left[I_1^i, J(z)\right] = 0, \\
\left[I_2^i, \mathcal{G}^a(z)\right] &= \frac{1}{2}\mathcal{G}^b(z)\sigma_{ba}^i \quad \left[I_2^i, \mathcal{G}^{a\dagger}(z)\right] = -\frac{1}{2}\sigma_{ab}^i \mathcal{G}^{b\dagger}(z), \\
\left[I_2^i, T(z)\right] &= 0 \quad \left[I_2^i, J(z)\right] = 0,
\end{aligned} \tag{121}$$

where

$$\mathcal{G} = (G^1, G^{2\dagger}) \quad \mathcal{G}^\dagger = \begin{pmatrix} G^{1\dagger} \\ G^2 \end{pmatrix}. \tag{122}$$

The charges I_1, I_2 constructed above generate $SO(4)$ transformations only on the *holomorphic* bosons $X_A(z)$. Similarly, we can construct charges $\widetilde{I}_1, \widetilde{I}_2$ which generate $SO(4)$ transformations only on the *antiholomorphic* bosons $\widetilde{X_A}(\bar{z})$. Normally one would expect these charges to give rise to a global $SO(4)_{\rm hol} \times SO(4)_{\rm antihol}$ symmetry. However a boson field is a sum of a holomorphic and anti-holomorphic part, and hence it has a well defined transformation property only under the charges

$$J_I = I_1 + \widetilde{I}_1, \quad \widetilde{J}_I = I_2 + \widetilde{I}_2. \tag{123}$$

These generate the $SO(4)_I = SU(2)_I \times \widetilde{SU(2)}_I$, and fall into representations of the $\mathcal{N} = (4,4)$ algebra (as can be proved by using the commutation relations (124) of the I's). The bosons $X(z, \bar{z})$ transform as $(\mathbf{2}, \mathbf{2})$ under $SU(2)_I \times \widetilde{SU(2)}_I$.

The following commutations relation show that the bosons transform as $(\mathbf{2}, \mathbf{2})$ under $SU(2)_{I_1} \times SU(2)_{I_2}$

$$\begin{aligned} \left[I_1^i, X_A^a\right] &= \frac{1}{2} X_A^b \sigma_{ba}^i, \quad \left[I_1^i, X_A^{a\dagger}\right] = -\frac{1}{2} \sigma_{ab}^i X_A^{b\dagger}, \\ \left[I_2^i, \mathcal{X}_A^a\right] &= \frac{1}{2} \mathcal{X}_A^b \sigma_{ba}^i, \quad \left[I_2^i, \mathcal{X}_A^{a\dagger}\right] = -\frac{1}{2} \sigma_{ab}^i \mathcal{X}_A^{b\dagger}. \end{aligned} \tag{124}$$

The fermions transform as $(\mathbf{2}, \mathbf{1})$ under $SU(2)_{I_1} \times SU(2)_{I_2}$ as can be seen from the commutations relations given below

$$\begin{aligned} \left[I_1^i, \Phi_A^a\right] &= \frac{1}{2} \Phi_A^b \sigma_{ba}^i, \quad \left[I_1^i, \Phi_A^{a\dagger}\right] = -\frac{1}{2} \sigma_{ab}^i \Phi_A^{b\dagger}, \\ \left[I_2^i, \Psi^a\right] &= 0, \quad \left[I_2^i, \bar{\Psi}^a\right] = 0. \end{aligned} \tag{125}$$

12. $SU(1,1|2)$ classification of states of the SCFT

The $SU(1,1|2)$ algebra has 2 sub-algebras: the global Virasoro algebra and the $SU(2)_R$ algebra. Their representations are labeled by the conformal weight h and the $SU(2)_R$ spin j. The highest weight states $|\mathrm{hw}\rangle = |h, \mathbf{j}_R, j_R^3 = j_R\rangle$ are defined by,

$$\begin{aligned} L_1|\mathrm{hw}\rangle &= 0 \quad L_0|\mathrm{hw}\rangle = h|\mathrm{hw}\rangle, \\ J_R^{(+)}|\mathrm{hw}\rangle &= 0 \quad J_R^{(3)}|\mathrm{hw}\rangle = j_R|\mathrm{hw}\rangle, \\ G_{1/2}^a|\mathrm{hw}\rangle &= 0 \quad G_{1/2}^{a\dagger}|\mathrm{hw}\rangle = 0, \end{aligned} \tag{126}$$

J_R^+ is the raising operator for spin j_R^3.

Amongst these highest weight states those for which $h = j$ satisfy additional conditions $G_{-1/2}^{2\dagger}|\mathrm{hw}\rangle = 0$, $G_{-1/2}^1|\mathrm{hw}\rangle = 0$. These states are called *chiral primaries*. From the chiral primaries one can generate multiplets by the action of the operators $G_{-1/2}^{1\dagger}$ and $G_{-1/2}^2$. These multiplets are called *short multiplets*. The $h = j$ short multiplet is given in the following table:

States	j	L_0	Degeneracy
$\lvert\mathrm{hw}\rangle_S$	h	h	$2h+1$
$G_{-1/2}^{1\dagger}\lvert\mathrm{hw}\rangle_S$, $G_{-1/2}^2\lvert\mathrm{hw}\rangle_S$	$h-1/2$	$h+1/2$	$2h+2h=4h$
$G_{-1/2}^{1\dagger}G_{-1/2}^2\lvert\mathrm{hw}\rangle_S$	$h-1$	$h+1$	$2h-1$

(127)

The short multiplets are usually denoted by the degeneracy of the $h = j$ state.

Since we have $SU(1,1|2)$ also coming from the anti-holomorphic sector the global super-algebra is $SU(1,1|2) \times SU(1,1|2)$ and its short multiplets will be denoted by $(\mathbf{2h+1}, \mathbf{2h'+1})_{\mathbf{S}}$. The top component of the short multiplet is the states belonging to the last row in (127). These states are annihilated by *all* the super-charges.

12.1 *The chiral primaries and marginal operators of the untwisted sector*

The short multiplet $(\mathbf{2},\mathbf{2})_{\mathbf{S}}$ is special, it terminates at the middle row of (127). For this case, the top component is the middle row. These states have $h = \bar{h} = 1$ and transform as $(\mathbf{1},\mathbf{1})$ of $SU(2)_R \times \widetilde{SU(2)}_R$. There are 4 such states for each $(\mathbf{2},\mathbf{2})_{\mathbf{S}}$. Hence these give rise to 16 marginal deformations of the SCFT. We shall see that there are 4 more marginal operators which come from the Z_2 twisted sector of the SCFT.

The 16 marginal operators of the untwisted sector arise as short multiplets belonging to the 4 chiral primaries $(\mathbf{2},\mathbf{2})_{\mathbf{S}}$,

$$\begin{aligned} &\Psi^1_A(z)\widetilde{\Psi}^1_A(\bar{z}), \quad \Psi^1_A(z)\widetilde{\Psi}^{2\dagger}_A(\bar{z}), \\ &\Psi^{2\dagger}_A(z)\widetilde{\Psi}^1_A(\bar{z}), \quad \Psi^{2\dagger}_A(z)\widetilde{\Psi}^{2\dagger}_A(\bar{z}), \end{aligned} \tag{128}$$

where summation over A is implied. These four operators have conformal dimension $(h,\bar{h}) = (1/2, 1/2)$ and $(j^3_R, \tilde{j}^3_R) = (1/2, 1/2)$ under the R-symmetry $SU(2)_R \times \widetilde{SU(2)}_R$. These are the relevant operators of the SCFT.

The short multiplets corresponding to each of the above chiral primaries can be constructed following the table in the previous section. Each such multiplet leads to 4 marginal operators with conformal weights $(1,1)$ and transform as $(\mathbf{1},\mathbf{1})$ under $SU(2)_R \times \widetilde{SU(2)}_R$. These operators can be derived from the pole terms of the operator product expansion of the chiral primaries with the SUSY currents. The result agrees with the expectation that the 16 top components of the $4(\mathbf{2},\mathbf{2})_{\mathbf{S}}$ short multiplets are $\partial x^i_A \bar{\partial} x^j_A$. These top components can be added to the SCFT as perturbations without violating the $\mathcal{N} = (4,4)$ supersymmetry.

It is clear that these operators can be classified using the global $SO(4)_I$ symmetry of the SCFT. The four torus $\tilde{T}^4$ actually breaks this symmetry but we assume that the target space is R^4 for the classification of states. We have the following table of various quantum numbers,

Operator	$SU(2)_I \times \widetilde{SU(2)}_I$	$SU(2)_R \times \widetilde{SU(2)}_R$	$(h,\bar{h})$
$\partial x^{\{i}_A(z)\bar{\partial} x^{j\}}_A(\bar{z}) - \frac{1}{4}\delta^{ij}\partial x^k_A(z)\bar{\partial} x^k_A(\bar{z})$	$(\mathbf{3},\mathbf{3})$	$(\mathbf{1},\mathbf{1})$	$(1,1)$
$\partial x^{[i}_A(z)\bar{\partial} x^{j]}_A(\bar{z})$	$(\mathbf{3},\mathbf{1}) + (\mathbf{1},\mathbf{3})$	$(\mathbf{1},\mathbf{1})$	$(1,1)$
$\partial x^i_A(z)\bar{\partial} x^i_A(\bar{z})$	$(\mathbf{1},\mathbf{1})$	$(\mathbf{1},\mathbf{1})$	$(1,1)$

(129)

12.2 *Chiral primaries and marginal operators of the* Z_2 *twist sector*

The orbifold SCFT has twisted sectors corresponding to conjugacy classes of the symmetric group $S(Q_1Q_5)$. These classes are labeled by cyclic groups of various lengths. If n is the length of the cycle and N_n is it's multiplicity then we have the basic equation,

$$\sum nN_n = Q_1Q_5. \tag{130}$$

The simplest conjugacy class consists of 1 cycle of length 2, and $Q_1Q_5 - 2$ cycles of length 1. An example of an element of this class is

$$(X_1 \to X_2,\ X_2 \to X_1),\ X_A \to X_A, A = 1, \ldots, Q_1Q_5 - 2. \tag{131}$$

Clearly the group action has a fixed point at $X_1 = X_2$. The linear combination that carries a representation of Z_2 is

$$X_{cm} = X_1 + X_2 \ \text{ and } \ \phi = X_1 - X_2. \tag{132}$$

Under the group action (131) X_{cm} is invariant and $\phi \to -\phi$. Thus the singularity is *locally* of the type R^4/Z_2 or equivalently C^2/Z_2. The bosonic twist operators for this orbifold singularity are given by following OPE's [36]

$$\begin{aligned}
&\partial\phi^1(z)\sigma^1(w,\bar{w}) = \frac{\tau^1(w,\bar{w})}{(z-w)^{1/2}}, \quad \partial\phi^{1\dagger}(z)\sigma^1(w,\bar{w}) = \frac{\tau'^1(w,\bar{w})}{(z-w)^{1/2}},\\
&\partial\phi^2(z)\sigma^2(w,\bar{w}) = \frac{\tau^2(w,\bar{w})}{(z-w)^{1/2}}, \quad \partial\phi^{2\dagger}(z)\sigma^2(w,\bar{w}) = \frac{\tau'^2(w,\bar{w})}{(z-w)^{1/2}},\\
&\bar{\partial}\widetilde{\phi}^1(\bar{z})\sigma^1(w,\bar{w}) = \frac{\widetilde{\tau}'^1(w,\bar{w})}{(\bar{z}-\bar{w})^{1/2}}, \quad \bar{\partial}\widetilde{\phi}^{1\dagger}(\bar{z})\sigma^1(w,\bar{w}) = \frac{\widetilde{\tau}^1(w\bar{w})}{(\bar{z}-\bar{w})^{1/2}},\\
&\bar{\partial}\widetilde{\phi}^2(\bar{z})\sigma^2(w,\bar{w}) = \frac{\widetilde{\tau}'^2(w,\bar{w})}{(\bar{z}-\bar{w})^{1/2}}, \quad \bar{\partial}\widetilde{\phi}^{2\dagger}(\bar{z})\sigma^2(w,\bar{w}) = \frac{\widetilde{\tau}^2(w,\bar{w})}{(\bar{z}-\bar{w})^{1/2}}.
\end{aligned} \tag{133}$$

The τ's are excited twist operators. The fermionic twists are constructed from bosonized currents defined by

$$\begin{aligned}
&\chi^1(z) = \mathrm{e}^{iH^1(z)}, \quad \chi^{1\dagger}(z) = \mathrm{e}^{-iH^1(z)},\\
&\chi^2(z) = \mathrm{e}^{iH^2(z)}, \quad \chi^{2\dagger}(z) = \mathrm{e}^{-iH^2(z)},
\end{aligned} \tag{134}$$

where the χ's, defined as $\Psi_1 - \Psi_2$, are the superpartners of the bosons ϕ.

From the above the supersymmetric twist fields which act both on fermions and bosons are:

$$\begin{aligned}
\Sigma_{(12)}^{(\frac{1}{2},\frac{1}{2})} &= \sigma^1(z,\bar{z})\sigma^2(z,\bar{z})\mathrm{e}^{iH^1(z)/2}\mathrm{e}^{-iH^2(z)/2}\mathrm{e}^{i\widetilde{H}^1(\bar{z})/2}\mathrm{e}^{-i\widetilde{H}^2(\bar{z})/2}\\
\Sigma_{(12)}^{(\frac{1}{2},-\frac{1}{2})} &= \sigma^1(z,\bar{z})\sigma^2(z,\bar{z})\mathrm{e}^{iH^1(z)/2}\mathrm{e}^{-iH^2(z)/2}\mathrm{e}^{-i\widetilde{H}^1(\bar{z})/2}\mathrm{e}^{i\widetilde{H}^2(\bar{z})/2}\\
\Sigma_{(12)}^{(-\frac{1}{2},\frac{1}{2})} &= \sigma^1(z,\bar{z})\sigma^2(z,\bar{z})\mathrm{e}^{-iH^1(z)/2}\mathrm{e}^{+iH^2(z)/2}\mathrm{e}^{i\widetilde{H}^1(\bar{z})/2}\mathrm{e}^{-i\widetilde{H}^2(\bar{z})/2}\\
\Sigma_{(12)}^{(-\frac{1}{2},-\frac{1}{2})} &= \sigma^1(z,\bar{z})\sigma^2(z,\bar{z})\mathrm{e}^{-iH^1(z)/2}\mathrm{e}^{+iH^2(z)/2}\mathrm{e}^{-i\widetilde{H}^1(\bar{z})/2}\mathrm{e}^{+i\widetilde{H}^2(\bar{z})/2}.
\end{aligned} \tag{135}$$

The subscript (12) refers to the fact that these twist operators were constructed for the representative group element (131) which exchanges the 1 and 2 labels of the coordinates of $\widetilde{T}^4$. The superscript stands for the $(j_R^3, \tilde{j}_R^3)$ quantum numbers. The full twist operators for the Z_2 conjugacy class are obtained by summing over Z_2 twist operators for all representative elements of this class. For $(j_R^3, \tilde{j}_R^3) = (1/2, 1/2)$, we have

$$\Sigma^{(\frac{1}{2},\frac{1}{2})} = \sum_{i=1}^{Q_1 Q_5} \sum_{j=1, j\neq i}^{Q_1 Q_5} \Sigma_{(ij)}^{(\frac{1}{2},\frac{1}{2})}. \tag{136}$$

A similar construction holds for three other operators. The conformal dimensions of these operators is $(1/2, 1/2)$. They transform as $(\mathbf{2}, \mathbf{2})$ under the $SU(2)_R \times \widetilde{SU(2)}_R$ symmetry of the SCFT. They belong to the bottom component of the short multiplet $(\mathbf{2}, \mathbf{2})_{\mathbf{S}}$. The operator $\Sigma^{(\frac{1}{2},\frac{1}{2})}$ is a chiral primary. As before the 4 top components of this short multiplet, which we denote by

$$\begin{aligned} &T^{(\frac{1}{2},\frac{1}{2})}, \;\; T^{(\frac{1}{2},-\frac{1}{2})} \\ &T^{(-\frac{1}{2},\frac{1}{2})}, \;\; T^{(-\frac{1}{2},-\frac{1}{2})}, \end{aligned} \tag{137}$$

are given by the leading pole in the following OPE's respectively

$$\begin{aligned} &G^2(z)\widetilde{G}^2(\bar{z})\Sigma^{(\frac{1}{2},\frac{1}{2})}(w,\bar{w}), \;\; G^2(z)\widetilde{G}^{1\dagger}(\bar{z})\Sigma^{(\frac{1}{2},\frac{1}{2})}(w,\bar{w}), \\ &G^{1\dagger}(z)\widetilde{G}^2(\bar{z})\Sigma^{(\frac{1}{2},\frac{1}{2})}(w,\bar{w}), \;\; G^{1\dagger}(z)\widetilde{G}^{1\dagger}(\bar{z})\Sigma^{(\frac{1}{2},\frac{1}{2})}(w,\bar{w}). \end{aligned} \tag{138}$$

The table of the quantum numbers of these operators is given below.

Operator	$(j^3, \tilde{j}^3)_I$	$SU(2)_R \times \widetilde{SU(2)}_R$	$(h, \bar{h})$
$\mathcal{T}^1_{(1)} = T^{(\frac{1}{2},\frac{1}{2})}$	$(0,1)$	$(\mathbf{1},\mathbf{1})$	$(1,1)$
$\mathcal{T}^1_{(0)} = T^{(\frac{1}{2},-\frac{1}{2})} + T^{(-\frac{1}{2},\frac{1}{2})}$	$(0,0)$	$(\mathbf{1},\mathbf{1})$	$(1,1)$
$\mathcal{T}^1_{(-1)} = T^{(-\frac{1}{2},-\frac{1}{2})}$	$(0,-1)$	$(\mathbf{1},\mathbf{1})$	$(1,1)$
$\mathcal{T}^0 = T^{(-\frac{1}{2},-\frac{1}{2})} - T^{(-\frac{1}{2},-\frac{1}{2})}$	$(0,0)$	$(\mathbf{1},\mathbf{1})$	$(1,1)$.

(139)

The first three operators of the above table can be organized as a $(\mathbf{1}, \mathbf{3})$ under $SU(2)_I \times \widetilde{SU(2)}_I$. We will denote these 3 operators as $\mathcal{T}^1$. The last operator transforms as a scalar $(\mathbf{1}, \mathbf{1})$ under $SU(2)_I \times \widetilde{SU(2)}_I$ and is denoted by $\mathcal{T}^0$.

These marginal operators are the 4 blow up modes of the R^4/Z_2 singularity [37], [37a]. Since these are top components of the short multiplet $(\mathbf{2}, \mathbf{2})_{\mathbf{S}}$ they can be added to the free SCFT as perturbations without violating the $\mathcal{N} = (4, 4)$ supersymmetry of the SCFT.

In conclusion we have accounted for the 20 marginal operators of the SCFT: 16 from the untwisted sector and 4 from the twisted sector. Those from the twisted sector have a special significance because a non-singular SCFT corresponds to turning on non-zero values for the corresponding moduli.

It is also possible to show [40] using $\mathcal{N} = (4,4)$ supersymmetry that the moduli space of the these 20 marginal operators is the coset space

$$\frac{SO(4,5)}{SO(4)\times SO(5)}. \tag{140}$$

Further, the number of marginal operators is $4h_{11}$ where h_{11} is the Hodge number and corresponds to the number of chiral primaries of weight $(h,\bar{h}) = (1/2, 1/2)$.

Later we shall see that turning on these moduli also corresponds to a true rather than a marginal bound state of the brane system.

12.3 *Chiral primaries of higher twisted sectors*

Cyclic groups of length k lead to twisted sectors characterized by the discrete group Z_k. In the vicinity of the fixed points the orbifold has the structure

$$R^4 \times R^4/\omega \times R^4/\omega^2 \times \ldots \times R^4/\omega^{k-1}. \tag{141}$$

The coordinate ϕ_m is twisted by the phase ω^m (m runs from $1\ldots k$). The dimension of the supersymmetric twist operator which twists the coordinates by a phase $\mathrm{e}^{2\pi in/N}$ in 2 complex dimensions is $h(n,N) = n/N$ [36]. Hence the dimension of the twist operator corresponding to the cyclic group of length k is given by

$$h = \sum_{i=1}^{k-1} h(i,k) = (k-1)/2. \tag{142}$$

A similar formula holds in the anti-holomorphic sector. There is a twist operator corresponding to every element in the conjugacy class and by summing over all the elements we can construct a chiral primary operator $\Sigma^{(k-1)/2}$. It has conformal dimension $(h,\bar{h}) = ((k-1)/2, (k-1)/2)$ and $(j_R^3, \tilde{j}_R^3) = ((k-1)/2, (k-1)/2)$. It belongs to the bottom component of the short multiplet $(\mathbf{k},\mathbf{k})_\mathbf{S}$. The other components of the short-multiplet $(\mathbf{k},\mathbf{k})_\mathbf{S}$ corresponding to the k-cycle twists can be generated by the action of supersymmetry currents and the R-symmetry currents of the $\mathcal{N} = (4,4)$ theory.

As the largest cycle is of length Q_1Q_5, the maximal dimension and angular momentum of the k-cycle twist operator is $((Q_1Q_5-1)/2, (Q_1Q_5-1)/2)$. This implies the important conclusion that the maximal value of the angular momentum of the corresponding state is $(Q_1Q_5-1)/2$. This statement is called the *stringy exclusion principle* [41].

The chiral primaries with conformal weight $(h,\bar{h})$ of a $\mathcal{N} = (4,4)$ superconformal field theory on a manifold K correspond to the elements of the cohomology $\mathcal{H}_{2h\,2\bar{h}}(K)$ [42]. The chiral primaries are formed by the product of the chiral primaries corresponding to the cohomology of the diagonal $\tilde{T}^4$ denoted by B^4 (the sum of all copies of $\tilde{T}^4$) and the various k-cycle chiral primaries.

The cohomology of B^4 is constructed in terms of the complex fermions defined in (115). The elements of $\mathcal{H}_{11}(B^4)$ were already given in (128). $\mathcal{H}_{22}(B^4)$ consists of the top form of B^4

$$\Psi_A^1(z)\Psi_A^{2\dagger}(z)\widetilde{\Psi}_A^1(\bar{z})\widetilde{\Psi}_A^{2\dagger}(\bar{z}), \tag{143}$$

where summation over all indices of A is implied.

The chiral primaries of B^4 which correspond to the elements of the cohomology $\mathcal{H}_{10}(B^4)$ are given by

$$\sum_{A=1}^{Q_1 Q_5} \Psi_A^1(z) \quad \text{and} \quad \sum_{A=1}^{Q_1 Q_5} \Psi_A^{2\dagger}(z) \tag{144}$$

and those that correspond to the elements of the cohomology $\mathcal{H}_{21}(B^4)$ are

$$\Psi_A^1(z)\Psi_A^{2\dagger}(z)\widetilde{\Psi}_A^1(\bar{z}) \;\text{ and }\; \Psi_A^1(z)\Psi_A^{2\dagger}(z)\widetilde{\Psi}_A^{2\dagger}(\bar{z}). \tag{145}$$

$\mathcal{H}_{20}(B^4)$ has only one element which is given by

$$\Psi_A^1(z)\Psi_A^{2\dagger}(z) \tag{146}$$

where summation over A is implied.

Once we know cohomology of B^4 the cohomology of $\mathcal{M}$ can be easily constructed by combining the chiral the chiral primaries $\Sigma^{k/2}(z,\bar{z})$ of the various twisted sectors. For details see [10]. Below we present the answer for the set of chiral primaries with $h = \bar{h}$.

$$\begin{aligned} 5(\mathbf{2},\mathbf{2})_S &+ 6 \oplus_{\mathbf{m}\geq \mathbf{3}} (\mathbf{m},\mathbf{m})_S \\ &+ 5((Q_1Q_5+)1, (Q_1Q_1+1))_S \\ &+ ((Q_1Q_5+2), (Q_1Q_1+2))_S. \end{aligned} \tag{147}$$

In the above the maximaum value of $\mathbf{m} = Q_1Q_5$.

12.4 *Matching chiral primaries with states in supergravity*

We have already discussed the Maldacena duality conjecture in section 10.2. The global symmetries of the SCFT exactly match the symmetries of Ad$S_3 \times S^3$. Further the radius of S^3 is $\sqrt{\alpha'}(g_6^2 Q_1 Q_5)^{1/4}$. Since this is a very large number in the supergravity limit $g_s Q_1 >> 1, g_s Q_5 >> 1$, the masses of the Kaluza-Klien modes on S^3 are very small and we expect these to match the states of the short multiplets of the SCFT in the limit when Q_1 and Q_5 are very large. This indeed happens (except for short multiplets that correspond to non-propagating degrees of freedom). We refer the reader to the literature for details [43,45,46].

13. The supergravity moduli and correspondence with SCFT

In this section we will match the SUGRA moduli in the near horizon geometry and marginal operators of the SCFT.

Let us first discuss the moduli of supergravity. It is known that the moduli space of type IIB sugra compactified on a 4-torus consists of 25 scalars which parametrize the coset $SO(5,5)/(SO(5)\times SO(5))$. These correspond to 10 scalars h_{ij} which arise from compactification of the metric. $i,j,k\ldots$ stands for the directions of T^4, 6 scalars b_{ij}

which arise from the Neveu–Schwarz B-field, 6 scalars b'_{ij} from the Ramond–Ramond B'-field, 3 scalars are the ten-dimensional dilaton ϕ_{10}, the Ramond–Ramond scalar χ and the Ramond–Ramond 4-form C_{6789}. In the near horizon geometry 5 of the above scalars become massive. These correspond to h_{ii}, which is proportional to the volume of the 4-torus, b^-_{ij}, the anti-self dual part of the Neveu–Schwarz B field and a certain linear combination of the RR 4-form and scalar. This moduli space corresponds to the coset $SO(5,4)/(SO(5)\times SO(4))$.

Now since the above fields are massless we can use the isometries of AdS_3 to calculate the conformal dimensions $(h,\bar{h})$. The mass formula is given by [47,41]

$$h+\bar{h}=1+\sqrt{1+m^2}. \tag{148}$$

Using this we note that the massless fields have $(h,\bar{h})$= $(1,1)$ and hence they belong to the top component of the short multiplet $5(\mathbf{2},\mathbf{2})_{\mathbf{S}}$.

The quantum numbers of these massless states are summarized in the table below:

Field	$SU(2)_I\times \widetilde{SU(2)}_I$	$SU(2)_E\times \widetilde{SU(2)}_E$	Mass
$h_{ij}-\frac{1}{4}\delta_{ij}h_{kk}$	$(\mathbf{3},\mathbf{3})$	$(\mathbf{1},\mathbf{1})$	0
b'_{ij}	$(\mathbf{3},\mathbf{1})+(\mathbf{1},\mathbf{3})$	$(\mathbf{1},\mathbf{1})$	0
ϕ_6	$(\mathbf{1},\mathbf{1})$	$(\mathbf{1},\mathbf{1})$	0
$a_1\chi+a_2C_{6789}$	$(\mathbf{1},\mathbf{1})$	$(\mathbf{1},\mathbf{1})$	0
b^+_{ij}	$(\mathbf{1},\mathbf{3})$	$(\mathbf{1},\mathbf{1})$	0

(149)

In sections 2.1 and 2.2 we have presented a table of the quantum numbers of the SCFT marginal operators. We would now like to match those marginal operators with the sugra fields we have obtained above.

We give the answer below and then justify it:

Operator	Field	$SU(2)_I\times \widetilde{SU(2)}_I$
$\partial x_A^{\{i}(z)\bar{\partial}x_A^{j\}}(\bar{z})-1/4\delta^{ij}\partial x_A^k\bar{\partial}x_A^k$	$h_{ij}-1/4\delta_{ij}h_{kk}$	$(\mathbf{3},\mathbf{3})$
$\partial x_A^{[i}(z)\bar{\partial}x_A^{j]}(\bar{z})$	b'_{ij}	$(\mathbf{3},\mathbf{1})+(\mathbf{1},\mathbf{3})$
$\partial x_A^i(z)\bar{\partial}x_A^i(\bar{z})$	ϕ	$(\mathbf{1},\mathbf{1})$
$\mathcal{T}^1$	b^+_{ij}	$(\mathbf{1},\mathbf{3})$
$\mathcal{T}^0$	$a_1\chi+a_2C_{6789}$	$(\mathbf{1},\mathbf{1})$

(150)

The representations $(\mathbf{1},\mathbf{3})$ and $(\mathbf{1},\mathbf{1})$ occur twice in the above table and hence there may be an ambiguity in the identification of these sugra fields with the SCFT operators.

We resolve the ambiguity with the help of the following argument. We have noted that the operators $\mathcal{T}^1$ and $\mathcal{T}^0$ correspond to blowing up modes of the orbifold CFT. Turning these off would lead us to a singular SCFT. This singularity has been related to the fact that in a marginal bound state of the brane system one can have fragmentation into sub-systems. In the supergravity as we have explicitly seen turning on the self-dual b^+_{ij} Neveau–Schwarz field leads to a stable bound state of the D_1–D_5 system. This is also true of the modulus corresponding to $a_1\chi + a_2 C_{6789}$. Solutions with these moduli turned off correspond to marginal bound states. Hence we expect that the blowing up modes of the SCFT correspond to the stabilizing moduli.

13.1 *The maximally twisted sector and black hole states*

The black hole is represented by a density matrix

$$\rho = \frac{1}{\Omega} \sum_{\{i\}} |i\rangle\langle i|. \tag{151}$$

The states $|i\rangle$ belongs to the various twisted sectors of the orbifold theory. The total value of L_0 and $\bar{L}_0$ satisfy the constraint

$$L_0 = N_L \quad \bar{L}_0 = N_R. \tag{152}$$

This corresponds to the fact that the general non-extremal black hole will have Kaluza-Klein excitations along both the directions on the S^1.

As we have seen before this information is sufficient to give the entropy of the degenerate state which satify the above constraint. We will see that infact this contribution for large values of the charges comes from the maximally twisted sector of the SCFT. The maximally twisted sector of the orbifold CFT coresponds to the longest cycle of the symmetric group $S(Q_1Q_5)$. It has a corresponding chiral primary $\sum^{(Q_1Q_5-1)/2}$ and its associated short multiplet. The presence of the twist field is equivalent to the following boundary condition on the bosonic fields

$$X_A(\mathrm{e}^{2\pi i}z, \mathrm{e}^{-2\pi i}\bar{z}) = X_{A+1}(z,\bar{z}). \tag{153}$$

This implies that the momentum n_L, n_R in the twisted sector is quantized in units of $1/(Q_1Q_5)$, and hence the momentum quantum number can go up to an integer multiple of (Q_1Q_5). Hence the contribution to the entropy from the twisted sector is

$$S(\text{maximally twisted}) = 2\pi\sqrt{n_L} + 2\pi\sqrt{n_R}. \tag{154}$$

This equals the total entropy with the choice $n_L = Q_1Q_5N_L$ and $n_R = Q_1Q_5N_R$.

Hence we can identify the black hole micro-states with the states in the maximally twisted sector.

14. Hawking radiation

In this section we discuss the derivation of the Hawking process from the viewpoint of string theory. First let us collect all the ingredients we have to formulate the process.

1. We have a complete description of the low lying excitations of the D_1–D_5 system representated as a SCFT.
2. We have a description of the black hole microstates in terms of fractionally moded oscillators that correspond to the chiral primary $\sum^{(Q_1 Q_5 - 1)/2}$.
3. Both the above statements about the microscopic theory are valid at large values of the effective coupling $g_s\sqrt{Q_1 Q_5}$ where the supergravity description is also valid.

   If we model the black hole by a density matrix

$$\rho = \frac{1}{\Omega} \sum_i |i\rangle\langle i|, \tag{155}$$

   where $|i\rangle$ represents the microstates then this sector of the full Hilbert space accounts for the black hole entropy
4. We have a correspondence between the massless supergravity modes and the marginal operators of the SCFT.

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The Hawking process corresponds to a transition from one black hole state $|i\rangle$ to another $|f\rangle$ with the emission of a particle. The emitted particle in principle corresponds to any state allowed by the symmetries. However we shall restrict ourselves to the massless emissions which are a predominant decay mode.

The description of the above transition requires an interaction hamiltonian. In the absence of a Born–Infeld action we appeal to the symmetries of the problem. From the discussion of the classification and matching of the marginal operators of the SCFT and the massless fields of supergravity in the near horizon limit, we can formulate a simple interaction hamiltonian to first order which is consistent with the symmetries.

$$S_{\rm int} = \int d^2 z \varphi_n|_B O_n(z, \bar{z}). \tag{156}$$

In the above $\varphi_n|_B$ stands for the boundary value of the closed string (supergravity) mode, and $O_n(z, \bar{z})$ stands for the corresponding operator. We will assume that the normalization of the supergravity mode is such that in the bulk theory it leads to a standard kinetic energy term. Using the above interaction one can calculate the S-matrix element relevant to the transition from one micro-state to another with the emission of a particle that couples to the micro-state.

$$S_{if} = \langle f | H_{\rm int} | i \rangle. \tag{157}$$

Note that this S-matrix describes a transition from one pure state to another. If we use the density matrix description then we have to average over all the initial states to obtain the probability of absorption,

$${\rm Prob}_{\rm abs} = \frac{1}{\Omega} \sum_i \sum_f |S_{if}|^2 \tag{158}$$

where Ω is the number of initial microstates.

Let us now use the above principles to calculate the absorption cross-section and the Hawking rate corresponding to one of the 20 minimal scalars namely, h_{ij} whose corresponding operator was found to be $\partial x^i_A \bar{\partial} x^j_A$. Both transform $(\mathbf{3},\mathbf{3})$ under $SO(4)_I$. The invariant interaction is given by

$$S_{\text{int}} = \frac{\mu}{2} \int d^2 z \ \left[h_{ij} \partial_z x^i_A \partial_{\bar{z}} x^j_A \right]. \tag{159}$$

The undetermined constant μ can be absorbed in the normalization of the SCFT operator which in turn cannot be fixed within the frame of the conformal field theory. We fix this normalization by matching the 2-point function of $\partial x^i_A \bar{\partial} x^j_A$ in the SCFT and the 2-point function of the internal graviton h_{ij}, in accordance with the Ads/CFT conjecture. This matching implies (in our conventions) $\mu = 1$ [45].

The presence of the twisted boundary conditions on the bosonic field makes it neccessary to redefine variables so that a convenient mode expansion is possible.

$$\tilde{x}^i(\sigma + 2\pi(A-1), t) \equiv x^i_A(\sigma, t), \sigma \in [0, 2\pi), \tag{160}$$

$\tilde{x}^i$ has period $2\pi Q_1 Q_5 R$. It is easy to write the normal mode expansion

$$\begin{aligned} \tilde{x}^i(\sigma, t) = (4\pi)^{-1/2} \sum_{n>0} & \left[\left(\frac{a^i_n}{\sqrt{n}} \mathrm{e}^{in(-t+\sigma)/Q_1 Q_5} \right.\right. \\ & + \left.\left. \frac{\tilde{a}^i_n}{\sqrt{n}} \mathrm{e}^{in(-t-\sigma)/Q_1 Q_5} \right) + \text{h.c.} \right] \end{aligned} \tag{161}$$

The effect of the twisting on these oscillators is given by

$$\begin{aligned} & g : a^i_n \to a^i_n \mathrm{e}^{2\pi i n/Q_1 Q_5} \\ & g : \tilde{a}^i_n \to \tilde{a}^i_n \mathrm{e}^{-2\pi i n/Q_1 Q_5}. \end{aligned} \tag{162}$$

The black hole state can now be explicitly constructed using the above oscillators,

$$|i\rangle = \prod_{n=1}^{\infty} \prod_i C(n,i) (a^{i\dagger}_n)^{N^i_{L,n}} (\tilde{a}^{i\dagger}_n)^{N^i_{R,n}} |0\rangle, \tag{163}$$

where $C(n,i)$ are normalization constants ensuring unit norm of the state. $|0\rangle$ represents the NS ground state.

The present discussion is also valid in the Ramond sector, in which case the ground state will have an additional spinor index but that will not affect the S-matrix. This comment is important because the black hole states are in the Ramond sector of the SCFT. This can be inferred from the boundary conditions on the Killing spinors in the black hole background. For AdS_3 the dual boundary states are in the Neveu–Schwarz sector of the SCFT [49].

The creation operators create KK (Kaluza–Klein) momentum along S^1 (parameterized by x^5). The total left (right) moving KK momentum of (163) (in units of $1/\tilde{R}, \tilde{R} \equiv Q_1 Q_5 R_5$, R_5 being the radius of the S^1) is N_L (N_R), where

$$N_L = \sum_{n,i} n N^i_{L,n}, \quad N_R = \sum_{n,i} n N^i_{R,n}. \tag{164}$$

The total KK momentum carried by the state $|i\rangle$ is an integer given by

$$p_5 = (N_L - N_R)/\tilde{R} \tag{165}$$

where $\tilde{R} = Q_1 Q_5 R_5$. This fact also implies that $|i\rangle$ is invariant under the twisting action.

Now we can calculate the S-matrix element for the process:

$$h_{89}(w,0) \to x^8_L(w/2,-w/2) + x^9_R((w/2,w/2), \tag{166}$$

(the numbers in parenthesis denote (k_0, k_5))

$$S_{if} = \frac{\sqrt{2}\kappa_5 w_1 w_2 \tilde{R}\delta_{n_1,n_2} 2\pi\delta(w - w_1 - w_2)}{\sqrt{w_1 \tilde{R} w_2 \tilde{R} w V_4}} \sqrt{N^8_{L,n_1}} \sqrt{\tilde{N}^9_{R,n_2}} \tag{167}$$

V_4 = volume of the noncompact space. $N^i_{L,n}$ and $N^i_{R,n}$ denotes number of oscillators with left- and right-moving momentum n respectively (see (163)). The factors $\sqrt{N}$ appear from the normalization of the states

$$|N\rangle = (a^\dagger)^N/\sqrt{N!}|0\rangle, [a,a^\dagger] = 1 \tag{168}$$

and the fact that

$$\langle N-1|a|N\rangle = \sqrt{N}. \tag{169}$$

From the above S-matrix element we can evaluate the absorption probability for a quantum of frequency w

$$\begin{aligned} \mathrm{Prob_{abs}} &= \frac{1}{\Omega_i}\sum_{i,f}|S_{if}|^2, \\ &= \frac{\tilde{R}T}{V_4}\kappa_5^2 w \langle N_{iL}(w/2)\rangle\ \langle N_{iR}(w/2)\rangle. \end{aligned} \tag{170}$$

Here T is the total time of the process.

The decay probability is obtained by the formula

$$\begin{aligned} \mathrm{Prob_{decay}} &= \frac{1}{\Omega_f}\sum_{i,f}|S_{if}|^2, \\ &= \frac{\tilde{R}T}{V_4}\kappa_5^2 w \langle N_{fL}(w/2)\rangle\ \langle N_{fR}(w/2)\rangle. \end{aligned} \tag{171}$$

We are interested in the process

$$N_{iL,iR}(n_1) = N_{fL,fR}(n_1) + 1, \tag{172}$$

where $n_1/RQ_1Q_5 = w/2$. To compare the string calculation with the semi-classical absorption calculation, where the black hole does not emit, we have to subtract the $\mathrm{Prob_{decay}}$ from $\mathrm{Prob_{abs}}$.

A straightforward calculation then leads to

$$\sigma_{\rm abs} = 2\pi^2 r_1^2 r_5^2 \frac{\pi w}{2} \frac{\exp(w/T_H) - 1}{(\exp(w/2T_R) - 1)(\exp(w/2T_L) - 1)} \tag{173}$$

which exactly agrees with the semi-classical calculation (73).

Finally the decay rate is given by,

$$\Gamma = {\rm Prob}_{\rm decay} \frac{{\rm V}_4}{\tilde{R}{\rm T}} \frac{d^4k}{(2\pi)^4} \tag{174}$$

and hence,

$$\Gamma_H = \sigma_{\rm abs}({\rm e}^{w/T_H} - 1)^{-1} \frac{d^4k}{(2\pi)^4} \tag{175}$$

which also exactly reproduces the semiclassical result (76).

We now make a few comments about these results:

- We have done a lot of work to be able to calculate the absorption crossection and the Hawking rates which agree with the semi-classical supergravity calculations. The string theory calculations were originally done in [20,21] and were based on a model that was physically motivated by string dualities [50,51]. In particular the calculation in [21] based on the DBI action reproduced even the exact coefficient that matched with the semi-classical answer for the absorption cross section of the minimal scalars. However this method did not work when applied to the fixed scalars [52–54]. This fact was very discouraging because it meant the absence of a consistent starting point for string theory calculations. The discovery of Maldacena [17] finally enabled the string theory calculations [45,44] because it was able to make a precise connection of the near horizon geometry with the infra-red fixed point theory of brane dynamics.

- The Hawking radiation calculation that we have presented is physically motivated. However this method or twisted oscillators cannot be used to calculate the rates corresponding to the particles whose vertex operators come from the twisted sector. However these can also be done using a formulation that relates the Hawking rates to the thermal 2-point function of the corresponding operators. Such a formulation follows directly from the basic equations (10), (11). This approach has been discussed in [10] and it is originally due to Callan and Gubser [48].

- We recall that the semi-classical calculation was done in an asymptotically flat geometry and yet the absorption cross section matched with the SCFT calculation which is dual to the near horizon geometry. This emphasizes the fact that in the semi-classical calculation the absorption occurred entirely from near the horizon of the black hole.

- It is important to point out that in the Maldacena limit the closed string modes like the graviton decouple from the brane system. This means that as $\alpha' \to 0$ the interaction hamiltonian of the graviton and the SCFT would be vanishingly small. Hence one should not work in the strict $\alpha' = 0$ limit. However one can still obtain a sizable absorption cross section. To see this it is sufficient to note (74),

$$\sigma_{\rm abs}(w \to 0) = A_h = 4G_5\sqrt{Q_1 Q_5 N}. \tag{176}$$

From here we see that

$$A_h \sim \frac{g_6^2}{R_5}\alpha'^2\sqrt{Q_1Q_5N}. \tag{177}$$

The quantity under the square root sign grows tends to infinity in the supergravity limit and hence compensates the fact that $\alpha' \to 0$.

- As we have discussed before in §9 non-renormalization theorems guarantee the validity of the SCFT in the strong coupling region. This also means that the 2-point functions of operators belonging to the short multiplets, which determine the Hawking rates, do not get renormalized. This is in particular true for all the 20 minimal scalars. Hence the Hawking rates of these particles are indeed calculated and matched in the supergravity regime. Also note that the rates of all the 20 particles can be matched by fixing the normalization of any one of them, using the AdS/SCFT correspondence.
- We have explained earlier the importance of studying the D_1–D_5 system in the presence of the vevs of B_{NS}. This corresponds to stable rather than marginal bound states and non-singular SCFT. This raises the question, whether the Hawking rates depend upon these vevs. They do not. This was shown in [44].

15. Concluding remarks

Let us conclude by stating some of the outstanding problems.

1. How does one formulate the effective long wave length theory of the non-supersymmetric black holes?
2. How does one derive space-time from brane theory? In particular is there a way of deducing Ad$S_3 \times S^3$ (the infinitely stretched horizon) as a consequence of brane dynamics? The method of coadjoint orbits is a promising approach to this question. And what about the black hole horizon itself. These questions are intimately tied to explaining the geometric Bekenstein–Hawking formula or in other words understand the holographic principle [55].
3. The D_1–D_5 system has relevant perturbations. It would be interesting to study the holographic renormalization group in this situation. What is the end point of the RG flow?

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