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Quark confinement in 2 + 1 dimensional pure Yang–Mills theory

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Abstract. We report some progress on the quark confinement problem in 2 + 1 dim. pure Yang–Mills theory, using Euclidean instanton methods. The instantons are regularized Wu–Yang 'monopoles', whose long range Coulomb field is screened by collective effects. Such configurations are stable to small perturbations unlike the case of singular, undressed monopoles. Using exact non-perturbative results for the 3-dim. Coulomb gas, where Debye screening holds for arbitrarily low temperatures, we show in a self-consistent way that a mass gap is dynamically generated in the gauge theory. The mass gap also determines the size of the monopoles. We also identify the disorder operator of the model in terms of the Sine–Gordon field of the Coulomb gas and hence obtain a dual representation whose symmetry is the centre of $SU(2)$.

Keywords. Quark; confinement; instanton; disorder.

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1. Introduction

In this talk we will discuss some progress towards an understanding of the dynamics of pure Yang–Mills theory in 2 + 1 dims. [1]. Our discussion uses the Euclidean instanton approach which has been successful in elucidating the dynamics of 2 + 1 dim. Yang–Mills theory in the presence of adjoint Higgs [2, 3]. However the gauge theory without the Higgs fields is a much more difficult problem for several reasons. Firstly, perturbation theory, though ultraviolet finite, is hopelessly infrared divergent. Secondly, because this theory has only one length scale set by the gauge coupling g, unlike the theory with the Higgs fields, one does not have the luxury of having another length scale m_W to enable a controlled semi-classical approximation. The classical monopole configurations are *singular* configurations in the continuum limit and hence the renormalization of the monopole gas is very nontrivial.

It is possible to regulate the singularity by modifying the fields inside a 'core' of some size λ. The classical action now depends on λ. We do not vary the size λ of the monopoles. Rather we treat it as a parameter of the theory to be self consistently determined by the mass gap.

In the absence of Higgs fields a simple monopole is unstable to small fluctuations of gauge fields around it. The instability can be traced to the long range Coulomb tail of the monopole in the small oscillations operator. However a Coulomb gas of magnetic monopoles in 3 dims. always screens the long range magnetic field of a monopole. Therefore the small oscillations of the gauge field around a screening plasma of monopoles need not be unstable. In this way we construct a self-consistent picture of instanton gas. The Debye length then self-consistently determines the size of the monopoles. The resulting long distance effective theory we obtain is a sine-Gordon theory in $2+1$ dim. We also show that the Sine–Gordon theory is actually a dual representation of $2+1$ YM theory in the sense that the disorder (magnetic) operators of this theory have a local formula in terms of the Sine–Gordon field χ: $\phi_D = e^{(i/2)\chi(x)}$.

2. Topology of the gauge condition

We begin with a review the topological properties of the vacuum of $SU(2)$ Yang–Mills theory in $2+1$ dimensions following [4]. The physical degrees of freedom are most transparent in a unitary type of gauge. For pure gauge theories this is defined as follows. Consider some local operator $X(x)$ which transforms according to the adjoint representation of $SU(2)$. The unitary gauge is now defined by

$$[X(x), \tau_3] = 0, \tag{1}$$

(τ^i, $i = 1,2,3$ stands for the Pauli matrices). This gauge condition retains the $U(1)$ generated by τ^3 as an unbroken symmetry. If the model contains adjoint Higgs fields in addition to the gluons, X may be the Higgs field itself and (1) is the conventional unitary gauge in such a Georgi–Glashow model. In pure Yang–Mills theory X has to be constructed out of the gauge fields alone.

We may write the matrix operator X in the form

$$X = \lambda I + \sum_{a=1}^{3} \epsilon_a(x)\tau^a, \tag{2}$$

where I is the identity matrix. Then the points x_0 where

$$\epsilon^a(x_0) = 0 \tag{3}$$

are singularities of the gauge condition (1). As shown in [4] these singularities are nothing but magnetic monopoles with respect to the unbroken $U(1)$. In $3+1$ dimensions these monopoles are 'particles' which represent dynamical degrees of freedom (in this gauge) other than the conventional fields. In $2+1$ dimensions these monopoles are instanton configurations which will play a crucial role in determining the properties of the vacuum.

In the Georgi–Glashow model the monopoles are given by the well-known 't Hooft–Polyakov solution, while for pure gauge theory they are described by the regularized Wu–Yang solution.

3. Confinement in pure Yang–Mills theory

In pure Yang–Mills theory as we have mentioned there is no Higgs field. This has two consequences. First, in the absence of the second scale (i.e. m_W), the perturbation expansion is infrared divergent and the theory cannot be defined perturbatively in the infinite volume limit. Secondly the classical monopole solutions are Wu–Yang monopoles [5] which have zero size and infinite action. We will regulate these monopoles by assigning a size λ, which is explained in the next subsection. We will then construct an expansion around a plasma of such monopoles and, as explained in the introduction, argue that these monopoles are stable against fluctuations.

3.1 *The monopole solution*

We will consider monopoles which have some size λ. The field due to single monopole at $\mathbf{x} = 0$ is given in the unitary gauge by

$$\tilde{A}_\mu(x) = \frac{1}{2}\begin{bmatrix} q\tilde{A}^3_\mu & \tilde{A}^1_\mu - iq\tilde{A}^2_\mu \\ \tilde{A}^1_\mu + iq\tilde{A}^2_\mu & -q\tilde{A}^3_\mu \end{bmatrix},$$
$$\tilde{A}^1_\mu(x) = -\frac{K(r/\lambda)}{r}\,[\hat{\phi}\cos\phi + \hat{\theta}\sin\phi]_\mu,$$
$$\tilde{A}^2_\mu(x) = \frac{K(r/\lambda)}{r}\,[-\hat{\phi}\sin\phi + \hat{\theta}\cos\phi]_\mu,$$
$$\tilde{A}^3_\mu(x) = -\frac{1}{r}\tan\frac{\theta}{2}\,[\hat{\phi}]_\mu = D_\mu \tag{4}$$

in spherical coordinates. Here $K(r/\lambda)$ is a structure function regulating the fields at $r = 0$. The function $K(r/\lambda)$ goes to 1 as $r \to 0$ as follows

$$K(r/\lambda) \sim 1 - \frac{r^2}{\lambda^2} \quad \text{for } r \to 0 \tag{5}$$

while at $r = \lambda$, $K(r/\lambda) = K'(r/\lambda) = 0$ and remains zero for $r > \lambda$. Furthermore $K'(r/\lambda)$ is continuous at $r = \lambda$. As shown by Banks, Myerson and Kogut [6], one can choose a $K(r/\lambda)$ so that the configuration (4) is a classical solution. The action of single monopole is

$$s = \frac{4\pi}{g^2}\int_0^\infty \mathrm{d}r\left[\left(\frac{\mathrm{d}K}{\mathrm{d}r}\right)^2 + \left(\frac{K^2-1}{2r^2}\right)\right]. \tag{6}$$

Note that the monopole field is abelian outside the monopole core. Consider a gas of such monopoles such that if the positions of the monopoles are x_a, x_b etc., one always has $|x_a - x_b| > 2\lambda$. The field configuration in the regions outside the core of the monopoles is approximated by

$$\mathcal{A}^{\mathrm{cl}}_\mu(x) = \sum_{a=1}^{N}\tilde{A}_\mu(x - x_a), \tag{7}$$

where x_a denotes a monopole position and N is the number of monopoles. In our self-consistent approach, we assume that (7) represents the dominant field configuration in the Euclidean path integral. The total action of this gas is

$$S_{\mathrm{cl}} = Ns + \frac{2\pi}{g^2}\sum_{a\neq b}\frac{q_a q_b}{|x_a - x_b|}. \tag{8}$$

Finally we record the single monopole field configuration in the radial gauge

$$A_\mu^a = -\frac{\epsilon_{\mu aj} x^j}{r^2}(1 - K(r/\lambda)). \tag{9}$$

3.2 *The path integral*

The monopole configuration is used to evaluate the path integral by the saddle point method. One expands the field around $\mathcal{A}_\mu^{\rm cl}$ which is a classical solution outside the core. The form of the solution inside the core is unimportant for our purposes:

$$\mathcal{A}_\mu = \mathcal{A}_\mu^{\rm cl} + g a_\mu. \tag{10}$$

The path integral may be formally written as

$$Z = \int \prod_x \mathrm{d}\mathcal{A}_\mu(x)$$
$$\exp(-S_{\rm cl}) \exp\left(-\int a D^2 a \mathrm{d}^3 x\right), \tag{11}$$

where $S_{\rm cl}$ is given by (8). Here D^2 is the stability operator for small fluctuations:

$$\int a D^2 a\, \mathrm{d}^3 x \equiv \int \mathcal{L}''(\mathcal{A}_{\rm cl}) a^2 \mathrm{d}^3 x, \tag{12}$$

where $\mathcal{L}''(\mathcal{A}_{\rm cl})$ is the second functional derivative of the Lagrangian density evaluated at $\mathcal{A}_{\rm cl}$.

Equation (11) as it stands is meaningless since D^2 has a zero eigenvalue for each symmetry of the original Lagrangian. The local gauge symmetry is fixed by requiring the fluctuations to satisfy the background gauge condition

$$\nabla_\mu(\mathcal{A}^{\rm cl}) a_\mu(x) = 0, \tag{13}$$

where $\nabla_\mu(\mathcal{A}^{\rm cl})$ is the covariant derivative evaluated at the configuration $\mathcal{A}^{\rm cl}$:

$$\nabla_\mu(\mathcal{A}_{\rm cl}) \equiv \partial_\mu + i[\mathcal{A}_\mu^{\rm cl}, \ ]. \tag{14}$$

This gives rise to the usual Fadeev-Popov determinant (det $\Delta_{\rm FP}$). The zero modes arising from the breaking of the global translation invariance and the $U(1)$ transformations obeying the background gauge condition but nonvanishing at infinity are replaced by integration over corresponding collective coordinates. Finally we have to sum over all N with the standard division by $N!$. One finally has the formal expressions

$$Z = \sum_{N=0}^{\infty} \frac{1}{N!} Q_N$$
$$Q_N = \sum_{\{q_a\}} \left(\frac{8}{\sqrt{\pi}} s^{3/2}\right)^N \int \prod_{a=1}^{N} \mathrm{d}x_a \mathrm{e}^{-S_{\rm cl}} \mathcal{J} \tag{15}$$
$$\mathcal{J} = \left(\frac{\det D^2}{\det -\partial^2}\right)^{-1/2} \left(\frac{\det \Delta_{\rm FP}}{\det -\partial^2}\right). \tag{16}$$

3.3 *Instability of the single undressed monopole*

In the usual semi-classical method the dilute instanton gas [7] is noninteracting and one writes

$$\det D^2 = (\det d^2)^N, \tag{17}$$

where d^2 is the stability operator for the single monopole configuration.

We will see, however, that the d^2 has negative eigenvalues signifying the instability of a single monopole. In the background gauge (13) the operator d^2 has the form:

$$d^2 = \delta_{\mu\nu}\nabla_\alpha(\tilde{A}_{\rm cl})\nabla^\alpha(\tilde{A}_{\rm cl}) + i\,[\tilde{F}^{\rm cl}_{\mu\nu},\ ], \tag{18}$$

where $\tilde{A}_{\rm cl}$ is defined in (4) and $\tilde{F}^{\rm cl}_{\mu\nu}$ is the corresponding field. We refer to the literature for details of the stability analysis [9,1]. Except we mention that the source of the instability of d^2 is the long range Coulomb tail of the single regularized monopole solution.

It should also be noted that in the presence of Higgs fields the $1/r^2$ Coulomb tail is cancelled and we have a screened potential $e^{-m_{\rm H}r}/r^2$, which removes the potential instability. $m_{\rm H}$ is the Higgs mass.

3.4 *Debye screening in the monopole gas*

Our main point is that when the instantons are interacting, the fluctuation problem cannot in general be factored into N copies of the fluctuation problem for an isolated monopole. Rather one should consider the stability of the neutral plasma of monopoles as a whole. This statement also applies to the Yang–Mills–Higgs system in the previous section. The main reason behind this is that we have an integration over the positions of the monopole and the charges, or equivalently a functional integration over the charge density field. The effect of this averaging over the charge density is very nontrivial. The results of [8] show that the charge density field clusters so that the theory of the density field generates a mass gap dynamically.

In fact the results of [8] mean that in the neutral plasma the fluctuations of the magnetic field are bounded and the $1/r^2$ magnetic field of a single monopole is Debye screened to e^{-Mr}/r^2, where $M(g^2,\lambda,z)$ is the non-perturbative mass gap, which depends on the coupling g^2, the monopole size λ which is effectively the cut-off of the Coulomb gas and the fugacity z. Recall that the source of instability for a single isolated monopole is the long range tail of the Coulomb potential. One might, therefore expect that in the screened neutral plasma a 'dressed' monopole whose Coulomb tail has been screened can in fact be stable.

In the following we shall assume that the monopole gas is dilute. Using translation invariance we focus on one monopole at $x = x_\alpha$ and its neighbourhood. We are thus considering the problem in the presence of a single source at $x = x_\alpha$. Recall that the fields outside the monopole cores of size λ are abelian. Thus, in the unitary gauge it follows from (7) and (4) that outside the core of this monopole the field is

$$\tilde{A}^{\rm out}_\mu = \sum_{a=1}^{N} \frac{1}{2} q_a \begin{pmatrix} D_\mu(x-x_a) & 0 \\ 0 & -D_\mu(x-x_a) \end{pmatrix}, \tag{19}$$

where we have set $x_N = x_\alpha$. Inside the core the effect of the core field of the other monopoles can be ignored and we have

$$\tilde{A}^{\rm in}_\mu = \sum_{a=1}^{N-1} \frac{1}{2} q_a \begin{pmatrix} D_\mu(x-x_a) & 0 \\ 0 & -D_\mu(x-x_a) \end{pmatrix} + \frac{1}{2} \begin{pmatrix} qD_\mu(x-x_\alpha) & \tilde{W}^-_\mu(x-x_\alpha) \\ \tilde{W}^+_\mu(x-x_\alpha) & -qD_\mu(x-x_\alpha) \end{pmatrix}, \tag{20}$$

where $\tilde{W}^\pm_\mu \equiv \tilde{A}^1_\mu \pm iq\tilde{A}^2_\mu$ and $\tilde{A}^1_\mu, \tilde{A}^2_\mu$ are as in (4). Introducing the charge density

$$\rho(x) \equiv \sum_{a=1}^{N} q_a \delta(x - x_a) \tag{21}$$

and assuming that in the mean, for large N, $\rho_N \simeq \rho_{N-1}$ we can rewrite (20) as

$$A_\mu(x; x_\alpha, [\rho]) = \int \mathrm{d}^3y \frac{\tau^3}{2} \rho(y) D_\mu(x-y) + \theta(\lambda - |x - x_\alpha|) \begin{pmatrix} D_\mu(x-x_\alpha) & W^-_\mu(x-x_\alpha) \\ W^+_\mu(x-x_\alpha) & -D_\mu(x-x_\alpha) \end{pmatrix}. \tag{22}$$

In the above expression the sharp θ function may be replaced by a smoother version.

Debye screening means that in the presence of a source the density $\rho(y)$ has a mean value $\bar{\rho}(y, x_\alpha)$, fluctuations around which are small. $A_\mu(x; x_\alpha, [\bar{\rho}])$ then represents a 'dressed' monopole configuration. In our case this 'source' is provided by the particular monopole at $x = x_\alpha$ in the plasma and the statement pertains to the field in the neighbourhood of this particular monopole.

The crucial point is that since Debye screening holds, we can assume self-consistently that the gas of 'dressed' monopoles is weakly interacting, unlike the 'bare' monopoles. The field around a dressed monopole decay exponentially over a distance scale set by the Debye screening length $l_D = 1/M$. If the average distance between the monopoles is much larger than l_D then the interaction between such dressed monopoles vanishes and the operator D^2 has a N-fold degeneracy. In other words the potential appearing in the stability equation resembles N far separated potential wells. In this situation we have, using translation invariance,

$$\det D^2 \simeq (\det D^2[\bar{\rho}])^N, \tag{23}$$

where $D^2[\bar{\rho}]$ now denotes the stability operator for a *single* dressed monopole which is the same for any monopole in the plasma. For finite distances between monopoles the exact degeneracy is lifted and eigenvalues of D^2 organize themselves in bands. This would lead to corrections to the result (23) which may be expanded in powers of l_D/l_m where l_m denotes the average distance between the monopoles.

In this regard there is a difference between the pure Yang Mills system and the Yang–Mills–Higgs system. The presence of the Higgs field means that the potential appearing in the stability operator around a single monopole decays over length scales of the order of $1/m_W$. Since the Debye length is much larger than $1/m_W$, corrections to the extensivity of the small fluctuation determinant appear as powers of $1/l_m m_W$.

3.5 *The Sine–Gordon transform and dynamical generation of mass gap*

We can now rewrite the theory in terms of a Sine–Gordon model. The path integral is written as

$$Z = \sum_{N=0}^{\infty} \frac{1}{N!}\left(\frac{8g^6}{\sqrt{\pi}}\bar{s}^{3/2}\mathrm{e}^{-\bar{s}}\right)^N \sum_{\{q_a\}}\prod_{i=1}^{N}\int \mathrm{d}x_i$$

$$\exp\left(-\frac{2\pi}{g^2}\sum_{a\neq b}\frac{q_a q_b}{|x_a - x_b|}\right)(\Theta[\bar{\rho}])^N, \tag{24}$$

where we have defined $\Theta[\bar{\rho}]$ (functional of the charge densities) as

$$\Theta[\bar{\rho}] \equiv \left(\frac{\det D^2[\bar{\rho}]}{\det -\partial^2}\right)^{-1/2}\left(\frac{\det \Delta_{\mathrm{FP}}(\bar{\rho})}{\det -\partial^2}\right) \tag{25}$$

and $\bar{s}$ is the action for a single monopole.

It may be noted that we are dealing with a super renormalizable theory. Thus the expression (25) is ultraviolet finite.

We then have

$$Z = \sum_{N=0}^{\infty}\frac{1}{N!}\bar{J}^N\sum_{\{q_a\}}\prod_{i=1}^{N}\int \mathrm{d}x_i \exp\left(-\frac{2\pi}{g^2}\sum\frac{q_a q_b}{|x_a - x_b|}\right), \tag{26}$$

where the mean fugacity $\bar{J}$ is given by the formula

$$\bar{J} = \frac{8g^6}{\sqrt{\pi}}\bar{s}^{3/2}\mathrm{e}^{-\bar{s}}\Theta[\bar{\rho}]. \tag{27}$$

We can now perform the Sine–Gordon transform and we write (26) as

$$Z = \int \mathcal{D}\chi(x)\exp\left[-\frac{g^2}{32\pi^2}\int \mathrm{d}^3x\{(\nabla\chi)^2, \quad -2\bar{M}^2(1-\cos\chi)\}\right], \tag{28}$$

where

$$\bar{M}^2 = \frac{16\pi^2\bar{J}}{g^2}. \tag{29}$$

It is important to emphasize in accordance with the discussion of [8], that the quadratic term $\int \mathrm{d}^3x(\nabla\chi)^2$ in (28) must be understood in a regularized sense, so that the Coulomb potential between the monopoles is valid only upto distances greater than the core size λ. In this sense λ is the cut off (lattice spacing) of the Sine–Gordon theory.

3.6 *Stability of the dressed monopole*

We now discuss, in some more detail than previously, whether the function $\Theta[\bar{\rho}]$ which is used in the definition of the fugacity $\bar{J}$ in (27) is well defined. This issue is important because we have already indicated that as we average over $\rho(x)$ the operator $D^2[\rho]$ has negative eigenvalues when ρ corresponds to a single isolated monopole.

We will now argue that $D^2[\bar{\rho}]$ is a positive operator. Recall the form of $D^2[\bar{\rho}]$ in the unitary gauge, outside the core of the dressed monopole which we can choose to be at $x = 0$

$$D^2[\bar{\rho}] = -\delta_{\mu\nu}\nabla_\alpha(A(x,[\bar{\rho}]))\nabla^\alpha(A(x,[\bar{\rho}])) + i\,[\bar{F}^{(3)}_{\mu\nu}(x,[\bar{\rho}]),\,], \tag{30}$$

where $\bar{F}^{(3)}_{\mu\nu}(x,[\bar{\rho}])$ is related to the magnetic field by $\bar{B}^{(3)}_\lambda = \epsilon_{\mu\nu\lambda}\bar{F}^{(3)}_{\mu\nu}$ and $\bar{B}^{(3)}_\lambda$ is given by

$$\bar{B}^{(3)}_\lambda = \frac{\partial}{\partial x_\lambda}\int \mathrm{d}^3y\,\frac{\bar{\rho}(y)}{|x-y|}. \tag{31}$$

Debye screening means that

$$\bar{B}^{(3)}_\lambda = \frac{x^\lambda}{r^3}\mathrm{e}^{-Mr}f(Mr), \tag{32}$$

where the function $f(x)$ has the property that for $x \gg 1$, $f(x) \sim 1$, and M is the mass gap related to the Debye length l_D by $M = 1/l_D$.

For a nonzero mass gap, the field outside the monopole core cannot be transformed to the radial form of (9). However close to the core and distances much smaller than the Debye length l_D the field is close to the single monopole field outside the core and may be cast in the radial gauge. Furthermore at distances much larger than the Debye length, the field is close to zero and once again one may cast the gauge potential in the radial gauge trivially (i.e. with $K(r) = 1$). As mentioned in § 4 the source of the even parity S-wave instability is the long range Coulomb field of the monopole. Since screening cuts off this Coulomb field and replaces it approximately by an exponential, one expects stability.

The situation is in fact similar to that of the Yang–Mills–Higgs system in some respects. Recall that the S-wave stability of the 't Hooft–Polyakov monopole is ensured by the fact the Higgs field rises *exponentially* to 1 beyond the core and cancels the negative $1/r^2$ tail of the gauge field, preventing the potential in the Schroedinger problem from being negative at large distances. In our problem the gauge field itself falls off exponentially to zero and thus the potential for the S-wave is positive at large distances. In this sense we have a dynamical Higgs effect.

3.7 *Self consistent dynamical generation of monopole size*

As mentioned before the fugacity of the monopole gas depends on $\bar{\rho}$ and hence it has a dependence on the mass gap M and the cutoff λ of the Sine–Gordon theory. This means that the cutoff λ is not independent, but determined in terms of M and g^2, i.e.

$$M = M(\lambda, g^2, z(\lambda, M)) \tag{33}$$

implies that

$$\lambda = \frac{1}{g^2}F\left(\frac{M}{g^2}\right), \tag{34}$$

where F is a function obtained by inverting (33). This equation now fixes the monopole core size λ as a function of the mass gap M in a self-consistent manner. The above considerations must be considered as qualitative because the calculation of the function F in (34) is beyond our present technology.

4. Dual representation and the disorder operator

We now relate the Euclidean formalism in terms of disorder operators introduced by 'tHooft [10] . In the hamiltonian formalism the Schroedinger picture disorder operator $\Phi_D(\mathrm{x}_0)$ is defined as an operator which creates a Z_2 magnetic vortex at the point x_0 in two dimensional space. More specifically it implements a singular gauge transformation $\Omega^{[x_0]}$ which has the property that if we consider a closed spatial loop C parameterized by an angle θ one has

$$\Omega^{[x_0]}(\theta + 2\pi) = -\Omega^{[x_0]}(\theta) \tag{35}$$

when the loop C encloses the point x_0. If C does not enclose x_0 the gauge transformation is single valued. Consider now the two point function of the Heisenberg picture disorder operators

$$\langle \Phi_D^\dagger(x)\Phi_D(y)\rangle. \tag{36}$$

Here x and y stand for three dimensional coordinates (including the Euclidean time). This two point function is then a sum over all configurations of the gauge fields which have a Dirac string singularity along a line joining x and y with a monopole of charge $1/2$ at the point y and an anti-monopole of charge $-1/2$ at the point x. It is crucial that the magnetic charges of these monopoles is half that of the monopoles which populate the vacuum. They have magnetic charges so that the Dirac string is *visible* by the lowest electrically charged quarks which couple to the gauge field.

Repeating the steps which led to the Sine–Gordon representation with the difference that we have two external magnetic sources with charges $\pm 1/2$ we easily get

$$\langle \Phi_D^\dagger(x)\Phi_D(y)\rangle = \langle \mathrm{e}^{i/2(\chi(y)-\chi(x))}\rangle, \tag{37}$$

the average on the right hand side in (37) being performed in the Sine–Gordon theory. A similar identification holds for all higher point correlation functions of the disorder operators. Hence we can identify the disorder operator with

$$\Phi_D(x) = \mathrm{e}^{i/2\chi(x)} \tag{38}$$

In fact the Sine–Gordon action may be now written in terms of Φ_D as

$$S = \frac{g^2}{32\pi^2}\int \mathrm{d}^3x[\partial_\mu\Phi_D^\dagger\partial_\mu\Phi_D + M^2((\Phi_D)^2 + (\Phi_D^\dagger)^2)] \tag{39}$$

up to an irrelevant constant. The field Φ_D is not a conventional scalar field, since $\Phi_D^\dagger\Phi_D = 1$. This non-linear Z_2 sigma model can be generalized to a linear sigma model by the addition of the term $\lambda\int \mathrm{d}^3x(\Phi_D^\dagger\Phi_D - 1)^2$ to the action (39). The action then exactly has the form conjectured in [10].

The Sine–Gordon theory thus is itself a dual representation of the original Yang–Mills theory. The action (39) has the global Z_2 symmetry $\Phi_D \to \Phi_D^\dagger$ which is spontaneously broken leading to magnetic disorder and confinement. This is simply the symmetry $\chi \to \chi + 2\pi$ of the Sine–Gordon model. It is clear from the action (39) that the dimensionless coupling constant is $\sqrt{M}/g$. This is inversely related to the gauge coupling g as expected in a dual formulation. The dual theory is weakly coupled when M/g^2 is small. In this limit the minima of the potential $(\cos \chi)$ break the Z_2 symmetry spontaneously.

Finally we note that the above construction of the disorder operators can be easily extended to $SU(N)$ gauge theories following the treatment in [3].

5. Conclusions

We have argued that in $2+1$ dimensional pure Yang–Mills theory Debye screening in a gas of regularized and dressed Wu–Yang monopoles provides a consistent picture of quark confinement. We have used the results of [8] that in a three dimensional Coulomb gas the charge density field always clusters, leading to Debye screening even for arbitrarily low temperatures. Our line of argument has been self-consistent in nature, because Debye screening in turn implies a screened magnetic field and hence the stability operator around a single dressed monopole is expected to have no negative eigenvalues. The mass gap thus obtained is non-perturbative and determines the monopole size self-consistently. A related issue is that the mean configuration $\bar{\rho}(x)$ in the presence of a single monopole source is in general non-classical and hence the associated scalar potential $\bar{\chi}(x)$ does not satisfy a classical equation. Hence the explicit evaluation of the Wilson loops is not as easily done. However on general grounds the existence of a mass gap leads to qualitative conclusions that are similar to the case of the Yang–Mills–Higgs system. Finally we have obtained a representation of the disorder operators of the theory in terms of the Sine–Gordon field which leads to a dual representation of the gauge theory.

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