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R-charged AdS_5 black holes and large N unitary matrix models

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ABSTRACT: Using the AdS/CFT, we establish a correspondence between the intricate thermal phases of R-charged AdS_5 blackholes and the R-charge sector of the N=4 gauge theory, in the large N limit. Integrating out all fields in the gauge theory except the thermal Polyakov line, leads to an effective unitary matrix model. In the canonical ensemble, a logarithmic term is generated in the non-zero charge sector of the matrix model. This term is important to discuss various supergravity properties like *i*) the non-existence of thermal AdS as a solution, *ii*) the existence of a point of cusp catastrophe in the phase diagram and *iii*) the matching of saddle points and the critical exponents of supergravity and those of the effective matrix model.

Contents

1. Introduction

The AdS/CFT correspondence[1] implies that the phases of string theory can be
studied by studying those of the dual gauge theory([2, 3, 4]). In the case of
type IIB string theory in $AdS^5 \times S^5$ the phases can be studied using the large

N limit of a unitary matrix model([5, 6, 7, 8])[1]. The unitary matrix is the finite temperature Polyakov loop which does not depend on the points of S^3. This fortunate circumstance is due to the fact that in the Hamiltonian formulation, N=4 SYM theory at a given time slice, is defined on the compact space S^3 and the $SO(6)$ scalars are massive because of their coupling to the curvature of S^3. These facts imply that, in principal, one can integrate out almost all the fields and obtain an effective theory of the zero mode of the gauge potential A_0.

Using this method a detailed correspondence of the critical points of the gauge theory effective lagrangian and the critical points of supergravity (discussed by Hawking and Page[11]) can be constructed at the leading order of the $1/N$ expansion. These are AdS_5 and the small and big black holes (as [12] we refer to these as SSB and BBH.) It turns out that in the gauge theory these critical points are in the gaped phase, where the density of eigenvalues vanishes in a finite arc of the circle around which the eigenvalues are distributed. The closing of the gap, corresponds to the Gross-Witten(GW) phase transition([13, 14, 15]). In a window around this transition, the supergravity description of string theory is likely to smoothly cross over into a description in terms of heavy string modes. [2]

In this paper we extend the discussion of the correspondence between R-charged AdS_5 blackholes([19, 20, 17, 18, 21]), and the effective unitary matrix model([13, 14, 15, 16]). R-charged black holes are known to have a rich phase structure in the canonical and grand canonical ensemble. In the canonical ensemble the fixed charge constraint, contributes an additional logarithmic term $\log(TrUTrU^\dagger)$ involving the order parameter, to the gauge theory effective action. This term is crucial for matching with supergravity. We analyze the implications of this term in the large N limit and compare with the various supergravity properties like the existence of only blackhole solutions in the canonical ensemble and also the existence of a point of cusp-catastrophe in the phase diagram.

The plan of this paper is as follows. In section 2 we give a brief review of charged AdS_5 blackholes. In section 3 and section 4 we discuss the effective action of the gauge theory at zero and small coupling, in the fixed charged sector. At zero coupling there is exactly one saddle point and the value of $TrUTrU^{-1}$ at the saddle point is always non-zero. For a small positive coupling there are two stable and one unstable saddle points, all with a non-zero value of $TrUTrU^{-1}$. They merge at the GW point. In section (5) we discuss the model effective action at strong coupling. Here too, there are three saddle points,two stable(I,III) and one unstable(II). In the region $\rho > \frac{1}{2}$, I and III can be identified with a stable small blackhole and stable big blackhole respectively. Saddle point II is identified with the small unstable black

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[1]Phases of large N gauge theory is also discussed in [9, 10]

[2]In [12], besides the vicinity of the Gross-Witten transition, authors also studied $1/N$ corrections and presented formulas for the partition function in the vicinity of blackhole nucleation and the Hagedorn transition.

hole. The merging of saddle points leads to critical phenomenon whose exponents can be calculated and shown to agree with supergravity. This is discussed in section 6. We have also calculated the $o(1)$ part of the partition function near the critical point.

2. R-charged blackholes in AdS_5 and critical phenomena

The R-charged AdS_5 black hole and relevant phase structure were discussed by A. Chamblin, R. Emparan, C. V. Johnson and R. C. Myers ([18],[17]). Here we review their result. The Einstein–Maxwell–anti–deSitter (EMadS$_{n+1}$) action may be written as

$$I = -\frac{1}{16\pi G} \int_M d^{n+1}x \sqrt{-g} \left[\widetilde{R} - F^2 + \frac{n(n-1)}{R^2} \right] \ , \tag{2.1}$$

where $\widetilde{R}$ is the Ricci scalar and R is the characteristic length scale of AdS. The metric of the Reissner–Nordström–anti–deSitter (RNadS) solution is given in static coordinates by

$$ds^2 = -(1 - \frac{m}{r^{n-2}} + \frac{q^2}{r^{2n-4}} + \frac{r^2}{R^2})dt^2 + \frac{dr^2}{1 - \frac{m}{r^{n-2}} + \frac{q^2}{r^{2n-4}} + \frac{r^2}{R^2}} + r^2 d\Omega_{n-1}^2 \ , \tag{2.2}$$

The parameter q is proportional to the charge

$$Q = \sqrt{2(n-1)(n-2)} \left(\frac{\omega_{n-1}}{8\pi G} \right) q \tag{2.3}$$

and m is proportional to the ADM mass M of the blackhole.

$$M = \frac{(n-1)\omega_{n-1}}{16\pi G} m \tag{2.4}$$

ω_{n-1} is the volume of the unit $(n-1)$–sphere, and the gauge potential is given by

$$A_0 = \left(-\frac{1}{\sqrt{\frac{2(n-2)}{n-1}}} \frac{q}{r^{n-2}} + \Phi \right) \tag{2.5}$$

where Φ is the electrostatic potential difference between the black hole horizon and infinity.

For n=4 the solution (2.2) can be considered as a rotating black hole in $AdS_5 \times S^5$ with angular momentum in the internal space S^5.[3] The symmetry group of S^5 is

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[3]The EMadS system described here may be thought as the dimensional reduction of $AdS_5 \times S^5$ to AdS_5. Generally introducing angular momentum in S^5 will distort S^5. This distortion is not taken into in the EMadS reduction [22, 23]. We thank S.Trivedi for pointing this out to us. However these details do not change our main consideration.

$SO(6)$ and the black hole we are discussing has equal $U(1)$ charges for all the three commuting $U(1)$ subgroups of $SO(6)$, the R-symmetry group of the $N = 4$ SYM. Hence we are dealing with a system which has the same chemical potential μ for all three $U(1)$ charges in the grand canonical ensemble or equivalently three fixed equal U(1) charges in the canonical ensemble.

2.1 Equation of state

In order to discuss the thermodynamics, we consider the Euclidean continuation $(t \to i\tau)$ of the solution, and identify the imaginary time period β with the inverse temperature. Using the formula for the period, $\beta = 4\pi/V'(r_+)$ (for a review see [24]), we get

$$\beta = \frac{4\pi l^2 r_+^{2n-3}}{nr_+^{2n-2} + (n-2)l^2 r_+^{2n-4} - (n-2)q^2 l^2} \ . \tag{2.6}$$

This may be rewritten in terms of the potential as:

$$\beta = \frac{4\pi l^2 r_+}{(n-2)l^2(1 - c^2\Phi^2) + nr_+^2} \ . \tag{2.7}$$

The condition for euclidean regularity used to derive (2.6) is equivalent to the condition that the black hole is in thermodynamical equilibrium. The equation (2.6) may therefore be written as an equation of state $T=T(\Phi, Q)$. From this equation of state we see that for fixed Φ we get two branches, one for each sign, when the discriminant under the square root is positive[18]. For fixed Q, $T(\Phi)$ has three branches for $Q<Q_{crit}$ (Let us call them I, II, III) and one for $Q>Q_{crit}$. The critical charge is determined at the "point of inflection" by $(\partial Q/\partial \Phi)_T = (\partial^2 Q/\partial \Phi^2)_T = 0$.

The qualitative features of $\beta(r_+)$ for varying q are shown in Fig 1. There is a
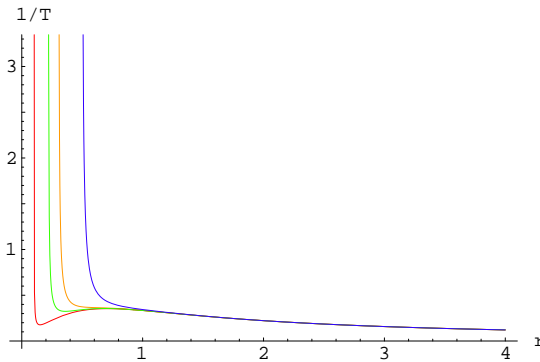
Figure 1: Plot of $\beta(r_+)$ for q increasing from the left. The third graph from the left is for q_{crit}

critical charge, q_{crit}, below which there are three solutions for r_+ for a range of values

of T, corresponding to small (I), unstable(II), and large (III) black holes. For fixed $q < q_{crit}$ only branch I is available at low temperatures. At $T = T_{02}(q)$, there is a nucleation of two new solutions, II the unstable small black hole solution, and III the stable big black hole solution. As the temperature is increased further, the black hole II approaches black hole I and at $T = T_{02}(q)$ the two solutions merge.

As q is increased further, T_{02} increases, whereas T_{01} decreases. At $q = q_{crit}$ we have $T_{crit} = T_{01} = T_{02}$. At q_{crit} and T_{crit} all three solutions merge. In the language of catastrophe theory this is a cusp catastrophe. As we increase q beyond q_{crit} there will be just one solution for all temperatures T.

We wish to take note of some properties of the phase diagram.

1. Thermal AdS is not a solution and all three branches of the solution represent black holes.

2. There exists a critical point where the three solutions of the system merge. It is a point of fold catastrophe.

2.2 Critical Phenomena

The critical point (Q_{cit}, T_{crit}) may be approached from various directions in the parameter space. If we set $T = T_{crit}$, then the equation determining $(r - r_{crit})$ takes the form (r_{crit} is the radius of the critical black hole)

$$(r - r_{crit})^3 = \mathcal{C}(Q - Q_{crit}) \tag{2.8}$$

$\mathcal{C}$ is a numerical constant. The critical exponent here is $\frac{1}{3}$, since

$$(r - r_{crit}) \propto (Q - Q_{crit})^{\frac{1}{3}} \tag{2.9}$$

As discussed in [18] (Fig16), the critical point may also be approached through the coexistence line in the parameter space. The coexistence line is the line with the property
$$S_I = S_{III} \tag{2.10}$$
for the parametric range $q < q_{crit}$. It is the line where the Hawking-Page (first order) transition from the small black hole to the big black hole takes place. As we approach the critical point through this line, we have the relation

$$(r_I - r_{II}) \propto (T - T_{crit})^{\frac{1}{2}} \tag{2.11}$$

In the following we will present an understanding of these properties. Before we do the matching with supergravity we would like to present a discussion of the gauge theory in the limit when $\lambda = g^2 N = 0$ and also when $\lambda << 1$. In these cases we of course can not compare with supergravity which requires $\lambda >> 1$.

3. Free YM theory

3.1 Effective action with chemical potential

In this section we briefly review (see [8],[7]) the effective action for a free $SU(N)$ Yang-Mills theory (with adjoint matter) on a compact manifold $\Im$ in the large N limit. The basic idea is to integrate out all fields in the theory except for the zero mode of the Polyakov line. The partition function is then reduced to a single unitary matrix integral.

Expanding all fields in the gauge theory in terms of harmonics on $\Im$, the theory reduces to a zero dimensional problem of free $N \times N$ Hermitian matrices

$$\mathcal{L} = \frac{1}{2} \sum_a Tr \left[(D_t M_n)^2 - \omega_n^2 M_n^2 \right] \tag{3.1}$$

The sum in (3.1))is over all field types and their Kaluza-Klein modes on $\Im$. ω_n is the frequency of each mode. The covariant derivative in (3.1) is

$$D_t M_n = \partial_t M_n - i[A_0, M_n] \tag{3.2}$$

A_0 comes from the zero mode (i.e. the mode independent of coordinates on $\Im$) of the time component of the gauge field and is not dynamical. The partition function of the theory at finite temperature can be written as a unitary matrix integral by integrating out all fields in (3.1) except for A_0. Hence we have

$$Z = \int DU \prod_n \left(\det{}_{adj} \left(1 - \epsilon_n e^{-\beta \omega_n} U \right) \right)^{-\epsilon_n} \tag{3.3}$$

where $U = \exp(i\beta A_0)$ is a $U(N)$. $\det_{adj}$ denotes the determinant in the adjoint representation and $\epsilon_n = 1$ (-1) for bosonic (fermionic) M_n. The above equation can be expressed as

$$Z = \int DU \exp(\sum_{n=1}^{\infty} \frac{z_n(\beta)}{n}) Tr U^n Tr U^{-n}) \tag{3.4}$$

where

$$z_n(\beta) = z_B(n\beta) + (-1)^{n+1} z_F(n\beta). \tag{3.5}$$

Here $z_B(\beta), z_F(\beta)$ are the single particle partition functions of the bosonic and fermionic sectors respectively (see [7] for the explicit formulas for $z(\beta)$ in various gauge theories).

If we introduce a chemical potential $\mu = (\log m)$, the formula for the partition function changes to

$$Z[\beta, \mu] = \int DU e^{(\sum_{n=1}^{\infty} \frac{z_n(\beta,\mu)}{n} Tr U^n Tr U^{-n})} \tag{3.6}$$

where

$$z_n(\beta, \mu) = z_B(n\beta, n\mu) + (-1)^{n+1} z_F(n\beta, n\mu) \tag{3.7}$$

$z_B(\beta, \mu), z_F(\beta, \mu)$ are now the single particle partition functions with chemical potential μ. Hence

$$z_B(\beta, \mu) = \sum_{bosons} \exp(-E_i\beta + Q_i\mu) \tag{3.8}$$

$$z_F(\beta, \mu) = \sum_{fermions} \exp(-E_i\beta + Q_i\mu) \tag{3.9}$$

Q_i is the charge of the state whose energy is E_i (see [7]).

As we have already mentioned, we wish to describe a system which has the same chemical potential μ for all three $U(1)$ charges of the R-symmetry group $SO(6)$ in grand canonical ensemble. Equivalently, we can work with fixed and equal values of the $U(1)$ charges in the canonical ensemble.

Let us for simplicity confine ourselves to the bosonic sector of the $N = 4$ SYM theory. The gauge fields have no R-charge. The six scalars ϕ_i ($i = 1, ..., 6$) can be grouped in pairs of two. We define

$$\phi^+ = \phi^1 + i\phi^2, \quad \phi^- = \phi^1 - i\phi^2 \tag{3.10}$$

(We can similarly define complex fields for the other two pairs.) $\phi^\pm$ have charge ± 1 for each of the three commuting $U(1)s$ of $SO(6)$. Hence, if we consider the single particle partition function for these fields, it will be

$$z[x, \mu] = (exp(+\mu) + exp(-\mu))z[x, 0]/2 = \cosh(\mu)z[x, 0] \tag{3.11}$$

where $z[x, 0]$ is the single particle partition function without any chemical potential.

3.2 Canonical Ensemble

We will now discuss the free gauge theory partition function for a canonical ensemble with constant charge Q_0, by introducing a delta function $\delta(\widehat{Q} - Q_0)$ in the functional integration of the gauge theory. $\widehat{Q} = Q[\phi]$ is the corresponding functional for the charge which we want to keep fixed. In our case $\widehat{Q}$ is just the functional for R-charge in gauge theory.

The fixed charge partition function is defined by

$$Z(\beta, Q_0) = \int DX e^{\int_0^\beta S[X]} \delta(\widehat{Q} - Q_0) \tag{3.12}$$

$$= \int DX e^{\int_0^\beta S[X]} \int \exp(i\mu\widehat{Q}) \exp(-i\mu Q_0) d\mu \tag{3.13}$$

$$= \int d\mu \exp(-i\mu Q_0) (\int DX e^{\int_0^\beta S[X]} e^{i\mu\widehat{Q}}) \tag{3.14}$$

$$= \int d\mu \exp(-i\mu Q_0) \int DU \exp(\Sigma z_n[\beta, i\mu] TrU^n TrU^{-n}) \tag{3.15}$$

where $z_n[\beta, i\mu] = z_n^V[\beta, 0] + cos(\mu)z_n^S[\beta, 0] + cos(\frac{\mu}{2})z_n^F[\beta, 0]$.

We can now make the approximation [4] that $|z_n[x, i\mu]|$ for $n > 1$ is negligible in comparison to $|z_1[x, i\mu]|$. Neglecting the contribution from the $n > 1$ modes we arrive at a model which contains only $TrUTrU^{-1}$. Using the specific formula for $z[x, \mu]$, of the bosonic [5] sector ,

$$
\begin{aligned}
Z(\beta, Q_0) &= \int d\mu \exp(-i\mu Q_0) \int DU \exp((a + c\cos(\mu))TrUTrU^{-1}) \\
&= \int DU \exp(aTrUTrU^{-1}) \int d\mu \exp(-i\mu Q_0)\exp(c\cos(\mu)TrUTrU^{-1}) \\
&= \int DU \exp(aTrUTrU^{-1})I_{Q_0}(cTrUTrU^{-1})
\end{aligned}
\tag{3.16}
$$

Here $a(\beta) = z_n^V[\beta, 0]$, $c(\beta) = z_n^S[\beta, 0]$ and for convenience we did not show the explicit β dependence in the equations. $I_n(x)$ is the Bessel function.

Hence we end up with a matrix model with an effective potential

$$
S_{eff} = a(TrUTrU^{-1}) + \log[I_{Q_0}(cTrUTrU^{-1})]
\tag{3.17}
$$

where $a > 0, c > 0$. We define $\rho^2 = TrUTrU^{-1}/N^2$ to get

$$
S_{eff}(\rho) = N^2(a\rho^2 + (1/N^2)\log[I_{Q_0}(cN^2\rho^2)])
\tag{3.18}
$$

It may seem that the logarithmic term is suppressed by a factor of $1/N^2$ and hence negligible for large N. But this need not be the case because, in the semicalssical large N limit, we must deal with a system with a charge of order N^2. Hence we define $Q_0 = N^2q$ ($q \sim o(1)$). Using the asymptotic expansion of $I_n(nx)$ for large n, the effective action [6] becomes

$$
S_{eff}(\rho) = N^2(a\rho^2 + q(\sqrt{(1 + \frac{c^2}{q^2}\rho^4)} + \log(\frac{\frac{c}{q}\rho^2}{1 + \sqrt{1 + \frac{c^2}{q^2}\rho^4}}))) + O(1)
\tag{3.19}
$$

3.3 Phase Structure

To understand the phase diagram of this model at large N, we have to locate the saddle points of (3.19) after including the relevant contribution from the path integral measure depending on whether $\rho < \frac{1}{2}$ or $\rho > \frac{1}{2}$.[7]

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[4]This approximation can be thought of as a low temperature approximation. This is because at low temperatures $z_n^S[\beta, 0]$ approaches zero as $e^{-\beta n}$. Hence the higher z_n are suppressed relative to z_1. It is also true that for all temperatures, $z_n < z_1$ and for very high temperatures we have $z_n \sim \frac{z_1}{n}$. As an example, the total contribution for all other z_n, even near hagedorn transition in free N=4 SYM theory, is only about 7% of z_1[7].

Unitary matrix models involving $TrU^n, n > 1$ has been discussed in [25, 26].

[5]Effect of the fermions is discussed in appendix A.

[6]It should be noted that when Q=0 we should use the asymptotic expansion of $I_0(x)$ for large x. Then we get the expected answer $S_{eff} = (a + c)TrUTrU^{-1}$ which is same as a model without any constraint on charge.

[7]The term in the right hand side of equation (3.21) originates from the path integral measure over an unitary matrix.(see [27],Appendix of [12]).

Differentiating $S_{eff}(\rho)$ we get

$$
\begin{aligned}
\frac{\partial}{\partial \rho^2} S_{eff}(\rho) &= a + \frac{\partial}{\partial \rho^2}(\frac{1}{N^2} \log[I_Q(Q\frac{cN^2\rho^2}{Q})]) \\
&= a + b\frac{I'_Q(Q\frac{c\rho^2}{q})}{I_Q(Q\frac{c\rho^2}{q})} \\
&= a + \frac{q}{\rho^2}(1 + \frac{c^2}{q^2}\rho^4)^{\frac{1}{2}} + O(1/Q)
\end{aligned} \tag{3.20}
$$

Hence the equations to solve are

$$
a\rho + \frac{q}{\rho}(1 + \frac{c^2}{q^2}\rho^4)^{\frac{1}{2}} = \rho, \rho < \frac{1}{2}
$$
$$
a\rho + \frac{q}{\rho}(1 + \frac{c^2}{q^2}\rho^4)^{\frac{1}{2}} = \frac{1}{4(1 - \rho)}, \rho > \frac{1}{2} \tag{3.21}
$$

The left side in (3.21) can be written as

$$
a\rho + \frac{q}{\rho}(1 + \frac{c^2}{q^2}\rho^4)^{\frac{1}{2}} = a\rho + c\rho + \frac{q}{\rho}\frac{1}{(1 + \frac{c^2}{q^2}\rho^4)^{\frac{1}{2}} + \frac{c}{q}\rho^2} \tag{3.22}
$$

So fixing the charge gives rise to a term of type $\frac{q}{\rho}\frac{1}{(1+\frac{c^2}{q^2}\rho^4)^{\frac{1}{2}}+\frac{c}{q}\rho^2}$ which has some important properties.

1. For all values of $q > 0, c > 0$, this term is a decreasing positive function of ρ and it diverges as $\rho \to 0$.

2. For all values of $c > 0$ it is a monotonically increasing function of q.

We can now discuss the solution of this model at $N = \infty$. Let us assume that we are discussing the phase where $a(T) + c(T) < 1$. This condition is valid for low temperatures since $a(T), c(T) \to 0$ as $T \to 0$. It should also be recalled that without any charge fixing the hagedorn transition occurs when $a(T) + c(T) = 1$. Unlike the situation with no charge, here we have a function, on the left hand side of (3.21), which diverges as $\rho \to 0$. Hence $\rho = 0$ can not be a solution. We get only one solution at a finite value of ρ which we will describe in the next paragraph.

Equation (3.21) is solved in the region $\rho < \frac{1}{2}$ with solution

$$
\rho^4 = \frac{q^2}{(1 - a)^2 - c^2} = \frac{q^2}{(1 - a - c)(1 - a + c)} \tag{3.23}
$$

The self consistency condition for a solution in the region $\rho < \frac{1}{2}$ is

$$
\rho^4 = \frac{q^2}{(1 - a)^2 - c^2} < \frac{1}{16} \tag{3.24}
$$

At low temperatures the condition is naturally satisfied for a small enough value of q. If we gradually increase the temperature (i.e. the value of $a(T)$ and $c(T)$) while keeping the value of the q fixed, then the value of ρ at this saddle point will increase. At some temperature $T_3(q)$, ρ will become equal to $\frac{1}{2}$. Since the measure part (i.e. right hand side of (3.21)) has a third order discontinuity at $\rho = \frac{1}{2}$, we will get a third order phase transition at the temperature T_3. From (3.24) we have the following condition at T_3

$$\frac{q^2}{(1-a)^2 - c^2} = \frac{1}{16} \tag{3.25}$$

If the temperature is increased beyond T_3 then we have to solve (3.21) in the region $\rho > \frac{1}{2}$.

If we increase q, then the value of ρ at the saddle point for a fixed temperature will increase. At some q_3 we get a third order phase transition satisfying

$$16 q_3(T)^2 = (a-1)^2 - c^2 \tag{3.26}$$

Since the minimum value of $a(T)$ and $c(T)$ is zero, the maximum possible value of $q_3^2(T)$ is $q_{crit}^2 = \frac{1}{16}$. If we increase the q beyond q_{crit}, the saddle point will always be confined in the parameter range $\rho > \frac{1}{2}$. Consequently as we increase the temperature from zero we will not get a third order phase transition for $q > q_{crit} = \frac{1}{4}$.

This free model, at zero gauge coupling ($l_s >> R$ in bulk), has some similarities with AdS_5 black holes in a fixed charge ensemble. However unlike the three black hole branches in AdS_5, we get only one branch in the free theory. But most importantly the solution always has a nonzero value of ρ.

It should be recalled that before the Hagedorn transition, a free gauge theory with zero charge has the solution $\rho = 0$ [5, 7].

Some properties of the free theory will be important when we analyze the situation for the weakly coupled gauge theory. Just above the temperature $T_3(q)$, the difference of the two sides of (3.21) can be expanded in the region $\rho > \frac{1}{2}$. Defining $\rho = \frac{1}{2} + x, (x > 0)$ the difference is

$$-\epsilon(q)x - C_1 x^2 \tag{3.27}$$

Here $\epsilon(q) > 0$ and $\epsilon(q) \to 0$ as $q \to 0$. It is important to note that $C_1 > 2$ because the measure function (i.e. right hand side of (3.21) has a third order discontinuity at the point $\rho = \frac{1}{2}$. We will also discuss the significance of this in what follows.

4. Small coupling model

We will now discuss the problem with a small non-zero gauge coupling $\lambda = g_{YM}^2 N$. By AdS/CFT correspondence it corresponds to a finite string length in AdS. It has

been shown in [12] that by considering a phenomenological model of type

$$S[TrUTrU^{-1}] = a(\lambda, T)(TrUTrU^{-1}) + \frac{b(\lambda, T)}{N^2}(TrUTrU^{-1})^2, b > 0 \quad (4.1)$$

we can map out the possible phase diagram of type IIB string theory in AdS_5.[8] Even though the model in (4.1) can be derived from a weak coupling analysis of the gauge theory, it can be thought of as a phenomenological model describing supergravity in AdS_5.

We are motivated to discuss the fixed charge ensemble in the same spirit. Let us add a small interaction term to (3.18). The effective action is then given by

$$S_{eff}(\rho) = N^2(a\rho^2 + b\rho^4 + q(\sqrt{(1 + \frac{c^2}{q^2}\rho^4)} + \log(\frac{\frac{c}{q}\rho^2}{1 + \sqrt{1 + \frac{c^2}{q^2}\rho^4}}))) + O(1) \quad (4.2)$$

Here b is proportional to λ and is also a function of charge. Depending on the theory considered, the sign of b can be either positive or negative. It has been shown in [28] that b is positive for a pure YM theory. In the following discussion we will assume that this is the case in order to motivate a similarity with the supergravity picture. The equations determining the saddle points of (4.2) , including the contribution from the path integral measure, are

$$(a + c)\rho^2 + 2b\rho^4 + \frac{q}{(1 + \frac{c^2}{q^2}\rho^4)^{\frac{1}{2}} + \frac{c}{q}\rho^2} = \rho^2, \rho < \frac{1}{2}$$

$$(a + c)\rho^2 + 2b\rho^4 + \frac{q}{(1 + \frac{c^2}{q^2}\rho^4)^{\frac{1}{2}} + \frac{c}{q}\rho^2} = \frac{\rho}{4(1 - \rho)}, \rho > \frac{1}{2} \quad (4.3)$$

In what follows it will be useful to introduce a function

$$M(\rho) = \rho^2, \rho < \frac{1}{2}$$

$$= \frac{\rho}{4(1 - \rho)}, \rho > \frac{1}{2} \quad (4.4)$$

$M(\rho)$ is an increasing convex function and is the right hand side of (4.3). It is also useful to introduce $D(x) = S'_{eff}(x) - M(x)$. Eqn (4.3) is then equivalent to $D(\rho) = 0$.

It has been shown in [12] that for an interacting model with zero charge, we get nucleation of black holes along the curve given by $a = \frac{1-2w}{(1-w)^2(1+w)}$ and $b = \frac{2w}{(1-w)^2(1+w)^3}$, $1 > w > 0$. Here we want to analyze a similar type of phenomenon. Let us consider the different cases.

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[8]In fact in [12] an arbitrary convex function is considered, and shown to map out the phase diagram of IIB theory in AdS_5. The simplified model (4.1) leads to similar qualitative result.

T is varying and q is small

Let us start with a value of charge which is small (i.e. $q \ll 1$) and let us increase the temperature from zero. At low temperatures all the parameters a, c will be small (for small T these parameters have a dependence like $e^{-c\beta}$, c is a constant). Hence we get just one solution for $\rho < \frac{1}{2}$ which we call I. There is no solution for $\rho > \frac{1}{2}$ (see the topmost curve of Fig 2) because the left hand side of (4.3) is less than the right hand side.

The situation is quite similar to supergravity where for small charge and low enough temperatures we get a stable small black hole solution. [9]
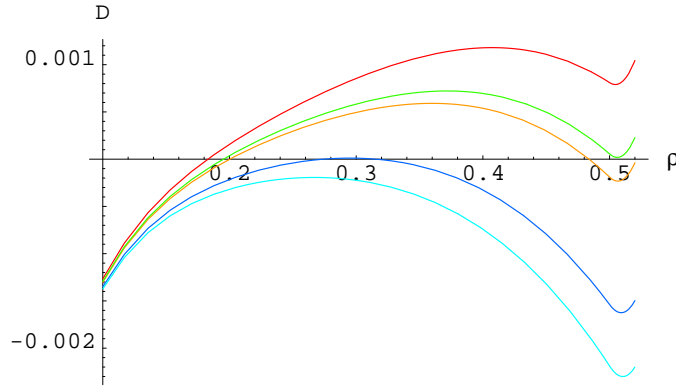
Figure 2: Plots of $D(\rho)$ with increasing $a = c$ from the above and with fixed b and $q < q_{crit}$

The function $M(\rho)$ (i.e. right hand side of (4.3)) is a convex increasing function. Hence we will generate new solutions of (4.3) in the region $\rho > \frac{1}{2}$ as we increase temperature (i.e. $a(T), c(T)$ as discussed in appendix B) keeping q fixed (Fig 2, Fig 3). The new solutions will always come in pairs (Fig 2). Let us call the solution nucleation temperature as $T_{01}(q)$. At $T = T_{01}$ we have

$$D(\rho) = S'_{eff}(\rho) - M(\rho) = 0$$
$$D'(\rho) = 0 \qquad (4.5)$$

As we increase the temperature further (i.e. $a(T), c(T)$) the function on the left hand side of (4.3) will also increase. Hence the solutions will start to separate. Let us call them II and III. Here $\rho_{III} > \rho_{II}$. Also, III is a stable saddle point whereas II is an unstable one. They are similar to stable big and unstable small black holes in supergravity. As the temperature is increased beyond T_{01} the value ρ_{II} decreases whereas ρ_{III} increases. At some temperature $T_H(q)$, we will have $S(III) < S(I)$ and consequently we expect a first order phase transition. At a temperature T_{HP},

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[9]We should keep in mind that $\rho < \frac{1}{2}$ is not the supergravity regime. The solutions of gauge theory effective action there should be represented as excited string states.
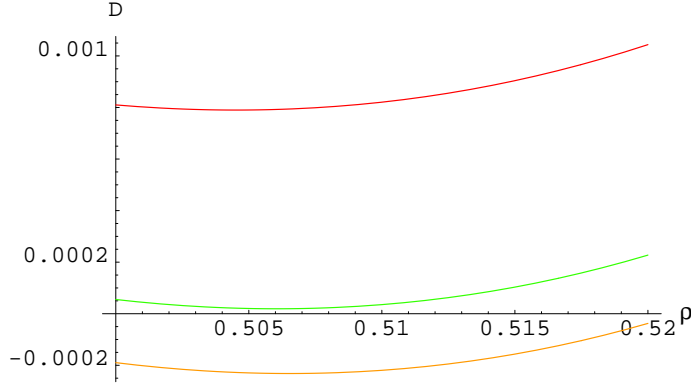
Figure 3: Top three graphs of Fig 2 showing emergence of two saddle points in the region $\rho > \frac{1}{2}$

the dominant saddle point of the system changes from I to III. As the temperature increases the saddle point II goes through a third order transition when $\rho(II)$ crosses the $\rho = \frac{1}{2}$ point. Call this temperature T_3 which is determined by the following relation between the parameters

$$(a + \frac{b}{2}) + 4q(1 + \frac{c^2}{16q^2})^{\frac{1}{2}} = 1 \tag{4.6}$$

Increasing the temperature further, the saddle point II approaches the saddle point I and they merge at a temperature T_{02} ($4th$ graph from above in Fig 2). In the language of catastrophe theory this is a fold catastrophe. For $T > T_{02}$, the only saddle point is III. This then is the thermal history as we increase the temperature for a small q.

In summary at low temperatures, we have only one saddle point I and then two new saddle point II, III are created at T_{01}. As the temperature increases the saddle points I, II merge at a temperature T_{02}. Beyond that we have only one saddle point III. In the next paragraph we will discuss what happens when we increase the value of the q.

Varying q:

Let us discuss how the various temperatures discussed above change as we increase the value of q.

1. The first is $T_{01}(q)$, the nucleation temperature for saddle points II and III. T_{01} will decrease as we increase the value of q. This is so because all three terms in the right hand side of (4.3) are positive and increasing functions of ρ and the left hand side is a positive convex function. Consequently the saddle point value of ρ at T_{01} will also decrease.

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2. The temperature T_{02} at which the stable saddle point I and unstable saddle point II merge and the value of ρ at T_{02}, will increase as we increase q. The reason is that all the coefficients in (4.3) are positive and hence increasing the value of q increases the function on the left hand side.

As we increase the value of q further, T_{01} and T_{02} will become equal for some value of $q = q_{crit}$. In the language of catastrophe theory this is a cusp catastrophe. Corresponding to q_{crit} there will be a $T_{crit} = T_{01} = T_{02}$ and a value of ρ_{crit} (saddle point ρ at q_{crit}, T_{crit}). As we increase the q beyond q_{crit}, we do not get any new saddle point and consequently there is only one saddle point for all temperatures. We will discuss the physics near this phase transition in detail in what follows.

Value of ρ_{crit} :

Let us determine the value of ρ_{crit}. As already discussed at the end of the previous chapter, in (3.27) C is always a finite quantity. Hence, (4.3) cannot have three solutions in the region $\rho < \frac{1}{2}$ for small b. Therefore the saddle point I is always in the region $\rho < \frac{1}{2}$. Whereas the saddle point III will be in the region $\rho > \frac{1}{2}$. Hence the only place where these three saddle points can meet is $\rho_{crit} = \frac{1}{2}$ which is also the point of the third order phase transition.

Physical Interpretation

Before proceeding further we will briefly discuss the bulk interpretation of saddle points of weakly coupled gauge theory. Weak coupling in gauge theory means $l_s \gg R_{AdS_5}$. Hence the supergravity picture is not valid in the bulk. However at large N, the string coupling (i.e. $\frac{1}{N}$) will be small and we may conclude that the saddle points discussed above can be described by exact (in all orders in l_s) conformal field theories. These CFTs are characterized by the values of q and ρ at the saddle points.

We would like to end this section by emphasizing that the coincidence of the three saddle points at $\rho = \frac{1}{2}$, is a property of the weak coupling ($\lambda << 1$) limit of the gauge theory. In what follows we will see that this fact is not necessarily true at strong coupling. We will show that there the coincidence happens in the gaped phase where $\rho > \frac{1}{2}$.

5. Effective action and phase diagram at strong coupling

In this section we will discuss the effective action and the phase diagram in the strongly coupled gauge theory which is dual to the supergravity (discussed in section 2) regime of IIB string theory.

5.1 Finite temperature effective actions in the gauge theory

Let us first summarize the situation in the zero charge sector. The propagator of adjoint fields in the free gauge theory, on a compact manifold S^3, coupled to a space independent A_0 is given by (see [29])

$$G_{U\ kl}^{ij}(x,t,y,0) = \sum_{n=-\infty}^{n=\infty} (U^n)_j^i G_0(x,t+n\beta,y,0)(U^{-n})_j^k \tag{5.1}$$

where $G_0(x,t,y,0)$ is the zero temperature Green's function and U is the constant Polykov line. We know that at any temperature [10] $G_0(x,t+n\beta,y,0) > G_0(x,t,y,0)$ and also at low temperatures

$$G_0(y,t+n\beta,x,0) \sim e^{-n\beta} \tag{5.2}$$

Using the above Green's function one can develop the large N diagrammatic to arrive at an effective action involving Z_N invariant terms built out of products of trU^n. In fact one can imagine integrating out all the modes trU^n for $n > 1$ in favor of $trUtrU^{-1}$. In this way one gets an effective action of the form

$$S_{eff} = \sum_{n=1}^{n=\infty} a_n(\beta,\lambda)\left(\frac{TrUTrU}{N^2}\right)^n \tag{5.3}$$

As one increases the coupling constant we would expect that the form of the effective action would remain the same except that the dependence of the parameters on temperature and the 'thooft coupling would change.

5.2 Non-zero charge sector

If we include the fixed charge constraint, as in (3.15), then we get the following expression for the fixed charge path integral

$$Z_q = \int d\mu e^{i\mu Q} \int DU e^{N^2(\sum_{n=0}^{n=\infty} S_n(\rho,\lambda,\beta)\cos(n\mu))} \tag{5.4}$$

For large N we can do the μ integral by the saddle point method. The saddle point of μ is on the imaginary axis. Hence we set $\mu = im$, to get the saddle point equation

$$q = \sum_{n=1}^{n=\infty} nS_n(\rho,\lambda,\beta)\sinh(nm)), q = Q/N^2 \tag{5.5}$$

At small values of ρ, $S_n(\rho)$ goes as ρ^n [11] and hence in the $\rho \to 0$ limit we can approximate the equation (5.5) as

$$q \approx \mathcal{C}\rho\sinh(m) \tag{5.8}$$

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[10]Temperature here is measured in units of $\frac{1}{R_{S^3}}$.

[11]Introduction of a chemical potential changes the formula (5.1) as

$$G_{U\ kl}^{\mu\ ij}(x,t,y,0) = e^{\frac{i\mu t}{\beta}} \sum_{n=-\infty}^{n=\infty} (U^n)_j^i G_0^\mu(x,t+n\beta,y,0)(U^{-n})_j^k \tag{5.6}$$

where $\mathcal{C}$ is a constant independent of ρ. The solution is

$$m \approx q \log(\rho) + \mathcal{C}, \rho \to 0 \tag{5.9}$$

Substituting m in (5.4) we get a logarithmic term for ρ. We conclude that the logarithmic term is a general feature and it implies among other things that $TrU = 0$ is never a solution in the non-zero charge sector.

5.3 Model effective action at strong coupling

Following our previous discussion, we will include the generic logarithmic term in the effective potential for a fixed non-zero charge, and proceed to analyze the saddle point structure following [12].

Our proposal for the gauge theory effective action is

$$S_q = S(a(T), b(T), ..., \rho) + q \log(\rho) \tag{5.10}$$

and the saddle point equations are

$$\rho F(a(T), b(T), c(T), ...\rho^2) + q = \rho^2, \rho < \frac{1}{2}$$
$$\rho F(a(T), b(T), c(T), ...\rho^2) + q = \frac{\rho}{4(1-\rho)}, \rho > \frac{1}{2} \tag{5.11}$$

Where $F(\rho) = S'_{eff}(\rho)$. We assume that

1. $F(x, T)$ is a monotonically increasing function of x and F(0,T)=0

2. Value of $F(x, T)$ increases for fixed x as we increase the temperature and F(x,0)=0.

These global properties of $F(x)$ reproduce the phase diagram of supergravity.

Analysis of solution structure

Let us consider the function $D(T, \rho) = \rho F(\rho, T) - M(\rho)$ (Where $M(\rho)$ is the contribution from measure appearing at the right hand side of (5.11)). At $T = 0$, $F(\rho, T)$ is zero. Hence $D(T, \rho)$ is a monotonically decreasing function of ρ at $T = 0$.

We know that at $T = T_{01}$ a pair of two new saddle points appear at $\rho_{01} > \frac{1}{2}$. Hence at $T = T_{01}$ we have $D(T_{01}, \rho_{01}) = 0$ and $D'(T_{01}, \rho_{01}) = 0$. $D(T_{01}, \rho)$ has a zero for $\rho = 0$ and it is a decreasing function in the neighborhood of $\rho = 0$. It again increases and become zero at ρ_{01} and then the function again decreases as $\rho \to 1$. This implies that the function has a local maximum and local minimum.
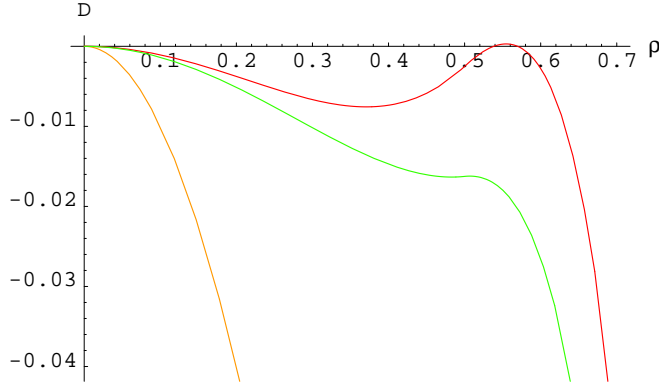
In summary:

where

$$G_0^\mu(x, t + n\beta, y, 0) = e^{in\mu} G_0(x, t + n\beta, y, 0) \tag{5.7}$$

Hence in each order in perturbation theory the terms containing $\cos(n\mu)$ also get multiplied by $(TrUTrU^{-1})^m, m > n$ or the higher operators like TrU^n which can be integrated out to give again a term like $(TrUTrU^{-1})^m$.

Figure 4: Plots of $D(T, \rho)$ for $T = 0$, $T = T_{crit}$, $T = T_{01}$ from below.

1. $D(0, \rho)$ is a monotonically decreasing function of ρ.

2. $D(T_{01}, \rho)$ has a maximum and minimum.

There is a temperature T_{crit} at which the local maximum and local minimum appear (Fig 5.3). Let us call this temperature T_{crit}. At T_{crit} the curve $D(T_{crit}, \rho)$ will have a point of inflection at $\rho = \rho_{crit}$, say. Let the value of $D(T_{crit}, \rho_{crit}) = q_{crit}$.

Increasing the value of q from zero we need to solve the equation $D(T, \rho) = q$. We will get a solution for a non-zero value of ρ. Denote this solution as I. As the temperature increases, two new solutions appear at $T = T_{01}$. Call the stable solution as III, and the unstable solution II. As the temperature is further increased to T_{02}, the unstable solution II and the stable solution I merge. For $T > T_{02}$, the only solution is III.

As q approaches q_{crit} from below the two temperatures T_{01} and T_{02} approach each other. At $q = q_{crit}$, we have $T_{01} = T_{02} = T_{crit}$. If we increase q beyond q_{crit} only one solution appears for all temperature. These facts are consistent with supergravity solutions(section 2).

With a sufficiently sharp rising function $F(T, \rho)$ in (5.11) we can obtain this critical point in the region $\rho > \frac{1}{2}$.[12] As the function $D(\rho, T)$ is smooth in the region $\rho > \frac{1}{2}$ the second derivative of the function $D(T, \rho)$ will vanish at the inflection point, and we will get a third order phase transition. We can calculate the partition function in suitable double scaling limit near the critical point. This is discussed in section 6.

A specific example

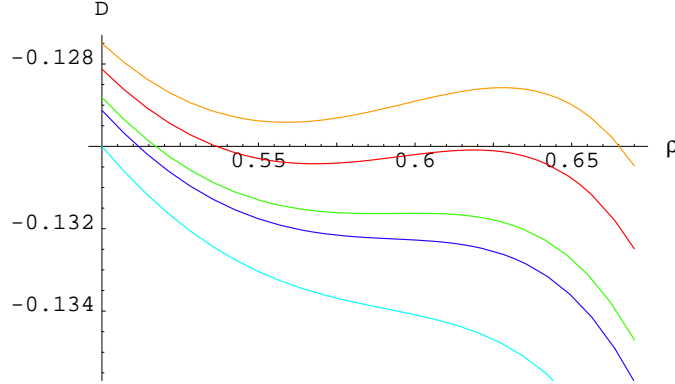We will now illustrate the above phenomenon in a simple model defined by

$$F(\rho) = a\rho + b\rho^3 \tag{5.12}$$

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[12]Which describes supergravity in the bulk[12].

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where $a, b > 0$.

We will determine the parameter ranges of a, b for which all the three saddle points of (5.12) are in the range $\rho > \frac{1}{2}$.

Figure 5: Plots of $D(T, \rho)$ with with fixed a and increasing b from the top, showing a critical transition in the region $\rho > \frac{1}{2}$ (from the top, the $3rd$ graph has a point of inflection)

At $\rho = \frac{1}{2}$ we have the constraints

$$\partial_\rho(\rho F(\rho) - \frac{\rho}{4(1-\rho)}) < 0$$
$$\partial_\rho^2(\rho F(\rho) - \frac{\rho}{4(1-\rho)} > 0 \tag{5.13}$$

Putting the value of $\rho = \frac{1}{2}$ in the above inequality we get the following constraints on the parameters $a + b < 1$ and $a + 3b > 2$. Simplifying we have $b > \frac{1}{2}$ and $a < \frac{1}{2}$.

As the coupling become stronger, we expect that b is not necessarily small and will be of $o(1)$ or greater. All the saddle points of (5.11) are then naturally shifted to the region $\rho > \frac{1}{2}$. Here, as was discussed in [12], we can expect to match the solutions of the gauge theory with those of supergravity. The stable saddle point I corresponds to the stable black hole branch I of supergravity. And unstable saddle point II is matched with the unstable black hole branch II. The stable saddle point III is matched with the big stable black hole in supergravity. With this identification the thermal history and critical behavior of the gauge theory, discussed earlier in this chapter, match with the thermal history and critical behavior of supergravity (discussed in section 2 and [18]).

6. Universal neighborhood of critical point and the critical exponents

Let us consider the effective action $S_{tot}(\rho, T, q)$ which includes the contribution from the path integral measure over an unitary matrix. The derivative of S_{tot}

with respect to ρ, say $G(\rho, T, q)$, gives the saddle point equations (5.11). We have already discussed that by a suitable choice of parameters the critical point appears in the region $\rho > \frac{1}{2}$. This critical point is a third order critical point because three saddle points of the system merges here. Hence the first and second derivatives of $G_{tot}(\rho, T, q)$ with respect to ρ vanish at $\rho = \rho_{crit}, q = q_{crit}, T = T_{crit}$. Expanding $G(\rho, T, q)$ around the critical point , we get

$$G(\rho, T, q) = (\delta\rho)^3 \frac{\partial_\rho^3 G}{3!} + (\delta T)\partial_T G + (\delta q)\partial_q G + (\delta\rho)(\delta T)\partial_\rho\partial_T G + (\delta q)(\delta T)\partial_\rho\partial_q G \quad (6.1)$$

Let us fix $T = T_{crit}$ or $\delta T = 0$. Then the equation (6.1) has one solution. In order to know how the saddle point value of ρ approaches ρ_{crit} $(\delta\rho \to 0)$ as $\delta q \to 0$, we equate the leading part of (6.1) to zero.

$$(\delta\rho)^3 \frac{\partial_\rho^3 G}{3!} + (\delta q)\partial_q G = 0 \qquad (6.2)$$

Hence $\delta\rho \propto \delta q^{\frac{1}{3}}$ and we get the same universal exponent $\frac{1}{3}$ as in supergravity [18].

6.1 Partition function near the critical point

Near the critical point we can write the S_{tot} as

$$S_{tot} = S_{tot}(\rho_{crit}, T_{crit}, q_{crit}) + (\delta\rho)^4 \frac{\partial_\rho^4 S}{4!} + (\delta q)\partial_q S + (\delta q)(\delta\rho)\partial_\rho\partial_q S + O(\delta\rho^5) \quad (6.3)$$

If we define a double scaling limit $N^{\frac{1}{2}}\rho = x, N^{\frac{3}{2}}q = z$, we can write the $o(1)$ part of the partition function, after suitable rescaling of the variables, as

$$Z_2 \propto \int dx e^{-(x^4 - zx)} \qquad (6.4)$$

This can be calculated in a power series

$$Z_2 \propto \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!}\Gamma(\frac{n}{2} + \frac{1}{4}) \qquad (6.5)$$

6.2 Approaching the critical point through a line of first order transitions

Another type of double scaling limit is possible in this problem. We can set[13]

$$(\delta T)\partial_T G + (\delta q)\partial_q G = 0 \qquad (6.6)$$

by choosing a suitable relation between δT and δq. Using (6.6) in (6.1) we get

$$(\delta\rho)^3 \frac{\partial_\rho^3 G}{3!} + (\delta\rho)((\delta T)\partial_\rho\partial_T G + (\delta q)\partial_\rho\partial_q G) = 0 \qquad (6.7)$$

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[13]It is same as following the HP(first order) transition line in parameter space.

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with the solutions

$$\delta\rho = 0$$
$$\delta\rho \propto \pm(\delta T)^{\frac{1}{2}} \tag{6.8}$$

We can expand S_{eff} as

$$S_{eff} \approx S_{crit} + (\delta\rho)^4 \frac{\partial_\rho^4 S}{4!} + C_1(\delta T)(\delta\rho)^2 + OT \tag{6.9}$$

where OT are terms independent of $\delta\rho$. Defining a suitable double scaling limit. $N^{\frac{3}{2}}\delta T = z, N^{\frac{1}{2}}\delta\rho = x$ and a suitable rescaling of the parameters we can evaluate the $o(1)$ factors in the partition function as

$$Z_2 \propto \int dx e^{-(x^4 + 2zx^2)} \propto \sum_{n=0}^{\infty} \frac{(2z)^n}{n!}\Gamma(\frac{n}{2} + \frac{1}{4}) \propto \sqrt{z}e^{\frac{z^2}{2}}K_{\frac{1}{4}}(\frac{z^2}{2}) \tag{6.10}$$

where $z < 0$.

7. Conclusion

In this paper we have studied the logarithmic matrix model generated by fixing the R-charge in the gauge theory partition function. In the free gauge theory it has been shown that there is no solution with $\rho = 0$ (AdS type solution). We then studied the effect of adding an interaction term in our model and discussed the generic nature of the logarithmic term even at arbitrary value of the coupling. We identified the supergravity saddle points and their critical behavior which was discussed in ([18]).

Our main aim was to give another example of the utility of unitary matrix methods in providing a non-perturbative dual description of blakholes in AdS and to understand the relation between matrix models and string theory in general. It would be interesting to consider an effective unitary matrix model to describe phases of Kerr-Ads black holes.

8. Acknowledgment

A. Appendix: Inclusion of Fermions

Including the contributions from the fermions of $N = 4\ SYM$ theory change (3.16) to

$$Z(\beta, Q_0) = \int DU \int d\mu \exp(N^2(a + c\cos(\mu) + d\cos(\frac{\mu}{2}))\rho^2 - i\mu Q_0) \qquad (A.1)$$

Where $d(\beta)$ is the single particle partition function for the fermions.

At large N , the integral in (A.1) could be evaluated by the saddle point method. The equations determining the saddle points of $\mu = im$ and ρ are

$$(c\sinh(m) + \frac{d}{2}\sinh(\frac{m}{2})) = \frac{q}{\rho^2} \qquad (A.2)$$

and

$$\rho(a + \cosh(m) + d\cosh(\frac{m}{2})) = \rho \qquad (A.3)$$

We would like to see weather there is a solution with $\rho = 0$. As the right hand side of (A.2) becomes large in the limit $\rho \to 0$, we can self consistently approximate $\cosh(m)$ and $\sinh(m)$ as e^m and we get

$$m \approx \log \frac{q}{c\rho^2} \qquad (A.4)$$

Hence a logarithmic potential for ρ is once again generated. One can also confirm this by putting (A.4) in (A.3).

B. Appendix: Positivity of the coefficient of the quadratic term in the effective action

Let us consider the partition function of YM theory on a compact manifold written as an integral over the effective action of $\rho = TrUTrU^{-1}$.

$$Z(\beta) = \int DU e^{N^2(S_{eff}(\rho))} \qquad (B.1)$$

$$= \int d\rho e^{N^2(S_{eff}(\rho) - S_M(\rho))} \qquad (B.2)$$

Where $S_M(\rho)$ is the contribution from the measure part[14] of path integral and

$$S_{eff}(\rho) = a(\beta)\rho^2 + \sum_{n=4} a_n(\beta)\rho^n \qquad (B.3)$$

i.e. a polynomial in ρ. As $\beta \to \infty$ we have $S_{eff}(\rho) \to 0$. Contribution from the measure part $S_M(0)$ has only one minimum at $\rho = 0$. Hence at low temperature the

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[14]See discussions before (3.20).

system will have a saddle point at $\rho = 0$. Expanding ρ around this saddle point as $\rho = 0 + \frac{\delta\rho}{N}$ we get

$$Z(\beta) = \int_{-\infty}^{\infty} d(\delta\rho)e^{-(1-a(\beta))(\delta\rho)^2 + o(\frac{1}{N^2})} \tag{B.4}$$

Like any thermal partition function, (B.2) or (B.4) is a decreasing function of the β. Hence $a(\beta)$ should also be a decreasing function of β. Since $a(\infty) = 0$, for any finite β, $a(\beta)$ is a positive decreasing function. Hagedorn transition happens when $a(\beta_H) = 1$, but whether $a(\beta)$ will reach 1 or not depends on the model.

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