A GENERAL CLASS OF QUASIFACTORIAL AND RELATED DESIGNS

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1. Introduction

A general class of quasifactorial designs was introduced by Nair and Rao (1942) and used in constructing balanced confounded designs for asymmetrical factorial experiments. The full details leading to the construction of asymmetrical designs are reported in another paper by Nair and Rao (1948). The object of this paper is to construct some useful quasifactorial designs for varietal trials. A number of designs closely resembling the quasifactorial system have also been given.

A quasifactorial design is defined as follows. There are $v = p_1 \times p_2 \times \ldots \times p_n$ varieties which can be identified by the multiple system

$$(x_1, x_2, \ldots, x_n) \qquad \ldots \quad (1.1)$$

$$x_i = 1, 2, \ldots, p_i; \; i = 1, \ldots, n$$

and b blocks each containing k different varieties such that

(i) every variety is used r times, and

(ii) the two varieties represented by $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ occur together in $\lambda_{c_1, \ldots, c_n}$ blocks where $c_i = 1$ or 0 according as $x_i = y_i$ or $x_i \neq y_i$. There are $(2^n - 1)$ parameters $\lambda_{c_1, \ldots, c_n}$ which need not be all different. The parameters satisfy the following relationships

$$vr = bk \qquad \ldots \quad (1.2)$$

$$r(k-1) = \Sigma p(c_1, \ldots, c_n) \, \lambda_{c_1, \ldots, c_n}$$

$$p(c_1, \ldots, c_n) = (p_{m1}-1)(p_{m2}-1) \ldots (p_{mt}-1) \qquad \ldots \quad (1.3)$$

where $c_{m1}, \ldots, c_{mt}$ are unity and the rest zero.

The quasifactorial design as defined above satisfies the parametric requirements of a partially balanced design (Bose and Nair, 1939; Nair and Rao, 1942). We shall consider only the two dimensional quasifactorial designs which are of special interest.

2. Two dimensional quasifactorial

In the case of two dimensional quasifactorial, the varieties, $v = p_1 \times p_2$, can be arranged in a rectangular lattice with p_1 rows and p_2 columns. Given any variety, the rest fall into three groups, (p_1-1) in the same column, (p_2-1) in the same row and $(p_1-1)(p_2-1)$ in the rest of the lattice. These are respectively the first, second and third associates with λ parameters equal to λ_{01}, λ_{10} and λ_{11}. This is a partially balanced design with the second system of parameters given by

$$p^{01}_{ij} = \begin{pmatrix} p_1-2 & 0 & 0 \\ 0 & 0 & p_2-1 \\ 0 & p_2-1 & (p_2-1)(p_1-2) \end{pmatrix}, p^{10}_{ij} = \begin{pmatrix} 0 & 0 & p_1-1 \\ 0 & p_2-2 & 0 \\ p_1-1 & 0 & (p_1-1)(p_2-2) \end{pmatrix}$$

$$p^{11}_{ij} = \begin{pmatrix} 0 & 1 & p_1-2 \\ 1 & 0 & p_2-2 \\ p_1-2 & p_2-2 & (p_1-2)(p_2-2) \end{pmatrix}.$$

When some of the λ_{ij} are equal, the design may reduce to a partially balanced design with only two associates though not necessarily. Some sufficient conditions for reduction to two associates are given below:

(i) $p_1 = p_2$, $\lambda_{10} = \lambda_{01} = \lambda_1$, $\lambda_{11} = \lambda_2 \neq \lambda_1$,

(ii) $\lambda_{10} = \lambda_{11} = \lambda_2$, $\lambda_{01} = \lambda_1 \neq \lambda_2$ for any p_1 and p_2,

(iii) $\lambda_{01} = \lambda_{11} = \lambda_2$. $\lambda_{10} = \lambda_1 \neq \lambda_2$ for any p_1 and p_2.

The second system of parameters for the case (i) is, using p for the common value

$$p^1_{ij} = \begin{pmatrix} p-2 & p-1 \\ p-1 & (p-1)(p-2) \end{pmatrix}, \quad p^2_{ij} = \begin{pmatrix} 2 & 2(p-2) \\ 2(p-2) & (p-2)^2 \end{pmatrix}$$

and for case (ii)

$$p^1_{ij} = \begin{pmatrix} p_1-2 & 0 \\ 0 & p_1(p_2-1) \end{pmatrix}, \quad p^2_{ij} = \begin{pmatrix} 0 & p_1-1 \\ p_1-1 & p_1(p_2-2) \end{pmatrix}$$

and for case (iii), p_{ij} are obtained by interchanging p_1 and p_2 in the expressions for case (ii). We will consider only designs with three associate classes since most of the partially balanced designs with two associate classes have been listed by Bose, Clatworthy and Shrikhande (1954).

3. Construction of two dimensional quasifactorial designs

3.1. *Series derivable from orthogonal Latin squares.*

$$v = pq, b = q(q-1), k = p, r = (q-1) \quad \ldots \quad (3.1.1)$$
$$\lambda_{01} = \lambda_{10} = 0, \quad \lambda_{11} = 1.$$

It is known from the work of Bose (1938) and Stevens (1938) that when q is a prime or a prime power, it is possible to construct $(q-1)$ orthogonal Latin squares in such a way that they differ only in a cyclical interchange of the rows from the 2nd to the q-th. Such squares $(q-1)$ are taken and the rows of each are bordered with numbers $1, \ldots, q$. In each square there are q^2 cells which may be identified by a pair of integers one representing the row and another the number in the cell (corresponding to the Latin square).

If we represent the varieties by an ordered pair of integers and consider the $q(q-1)$ columns from all the orthogonal squares as blocks we obtain a design with $\lambda_{10} = \lambda_{01} = 0$ and $\lambda_{11} = 1$. This is because varieties represented by (ij) and (rs) occur in no column if $i = r$ or $j = s$ and occur in just one column when $i \neq r$ and $j \neq s$. This result can be easily proved by using the special property of orthogonal squares derived by the method of interchanging $(q-1)$ rows cyclically. As it stands this is a design for q^2 varieties. Omitting $(q-p)$ rows of the Latin squares designs for pq varieties with $\lambda_{10} = \lambda_{01} = 0$ and $\lambda_{11} = 1$ are obtained. As an illustration, let us consider the designs for 2×4, 3×4 and 4×4 obtained from 4×4 orthogonal Latin squares.

TABLE 1. DESIGNS FOR 2×4, 3×4 AND 4×4

row no.	orthogonal latin squares		
(1)	(2)	(3)	(4)
1	1 2 3 4	1 2 3 4	1 2 3 4
2	2 1 4 3	3 4 1 2	4 3 2 1
3	3 4 1 2	4 3 2 1	2 1 4 3
4	4 3 2 1	2 1 4 3	3 4 1 2

The design for 3×4 is obtained by omitting the last row and considering the twelve columns. The actual design is

(11, 22, 33), (12, 21, 34), (13, 24, 31), (14, 23, 32)

(11, 23, 34), (12, 24, 33), (13, 21, 32), (14, 22, 31)

(11, 24, 32), (12, 23, 31), (13, 22, 34), (14, 21, 33)

where the 12 varieties are represented by pairs of integers 11, 12, 13, 14,..., 34.

Another method of construction which is more general than the above is as follows. [illegible]

[illegible]

As an illustration [illegible]

[illegible]

[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]

The design for [illegible] with twelve blocks is given in table [illegible].

TABLE [illegible]

[illegible]	ordered sets representing blocks											
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]

The first block has the varieties [illegible] the second [illegible] and so on.

If all the [illegible] orthogonal squares have a [illegible] then the rows and columns can be bordered with elements in the same order as in the diagonal. Each cell now gives rise to an ordered set of [illegible] elements, two corresponding to rows and columns and [illegible] to the orthogonal squares. Omitting the diagonal elements and considering the ordered sets in the $q(q-1)$ remaining cells we can build up as before by attaching integers, $1, 2, \ldots, (k+1)$ for the order a design for $(k+1)\times q$ varieties in $q(q-1)$ blocks of $(k+1)$ plots with $\lambda_{11} = 1$, $\lambda_{10} = \lambda_{01} = 0$.

A design for $2\times q$ always exist because it just depends on the existence of a Latin square. Since a Latin square of any order can be written so that it has a directrix, it follows that a design for $3\times q$ exists for all q. Designs for $4\times q$ depend on the existence of 3 orthogonal Latin squares of order q or at least two with a common directrix and so on.

Table 3 below gives the list of useful designs in the series (3.1.1)

$$v = pq,\ b = q(q-1),\ k = p,\ r = (q-1)$$

$$\lambda_{01} = \lambda_{10} = 0,\quad \lambda_{11} = 1.$$

The method of construction adopted (see Table 1) gives the design in groups of blocks representing separate replications wherever such a resolution is possible. The non-resolvable designs which may arise by adopting the method of Table 2 are marked with an asterisk. Designs for $p = q$ are omitted as they are partially balanced with two classes of associates.

TABLE 3. DESIGNS OF SERIES (3.1.1)($p \neq q$)

sl. no.	v	b	k	r	sl. no.	v	b	k	r
(1)	(2)	(3)	(4)	(5)	(1)	(2)	(3)	(4)	(5)
1	12	12	3	3	11	36	72	4	8
2	15	20	3	4	12	35	42	5	6
3*	18	30	3	5	13	40	56	5	7
4	21	42	3	6	14	45	72	5	8
5	24	56	3	7	15	42	42	6	6
6	27	72	3	8	16	48	56	6	7
7*	30	90	3	9	17	54	72	6	8
8	20	20	4	4	18	56	56	7	7
9	28	42	4	6	19	63	72	7	8
10	32	56	4	7	20	72	72	8	8

We now consider a second series of designs derivable from orthogonal Latin squares with the following parameters

$$v = pq,\ b = q^2,\ k = p,\ r = q \qquad \ldots \quad (3.1.2)$$

$$\lambda_{11} = 1 = \lambda_{10},\ \lambda_{01} = 0.$$

In this case the partially balanced design has only two associate classes as mentioned in section 2. The first and second system of parameters are

$$n_1 = q(p-1),\ n_2 = (q-1)$$

$$\lambda_1 = 1,\ \lambda_2 = 0$$

$$p^1_{ij} = \begin{pmatrix} q(p-2) & q-1 \\ q-1 & 0 \end{pmatrix} \quad p^2_{ij} = \begin{pmatrix} q(p-1) & 0 \\ 0 & q-2 \end{pmatrix}$$

It may be seen that this is also a group divisible design with p groups each containing q varieties. Two varieties from the same group do not occur in any block while they occur in just one block when they belong to two different groups. We shall not list these designs as the actual plans are given, by Bose, Clatworthy and Shrikhande (1954).

3.2. *Series derivable by the method of joining.* Let us write down the $p\,q$ numbers representing varieties in the form of $p \times q$ rectangular lattice with p rows and q columns. Suppose that balanced incomplete block designs exist for the parameters

$$v = p,\ \ b = b_1,\ r = r_1,\ k, \lambda_1$$

$$v = q,\ \ b = b_2,\ r = r_2, k, \lambda_2$$

then by forming separate designs for varieties in each row and column and combining them we get a quasifactorial design for pq varieties with parameters

$$b = pb_2+qb_1,\ r = r_1+r_2,\ k$$

$$\lambda_{11} = 0,\ \lambda_{01} = \lambda_1,\ \lambda_{10} = \lambda_2.$$

This design is resolvable if the balanced designs used for each row and column are resolvable. Only special cases are of interest.

$$v = p^2,\ b = 2p,\ r = 2,\ k = p \quad \dots \quad (3.2.1)$$

$$\lambda_{11} = 0,\ \lambda_{01} = \lambda_{10} = 1.$$

This is Yates' two dimensional square lattice obtained by considering the rows and columns as complete randomised blocks. The partially balanced design has only two associates.

$$v = pq,\ b = pb_2,\ \ k = p,\ r = 1+r_2 \quad (3.2.2)$$

$$\lambda_{11} = 0,\ \lambda_{10} = \lambda_2,\ \lambda_{01} = 1.$$

In this series the design used for the columns is the randomised block with one replication. The list of useful designs together with the parameters of the balanced design used in the construction is given in Table 4.

TABLE 4. DESIGNS OF THE SERIES (3.2.2)

sl. no.	parameters of the balanced design				parameters of the quasifactorial design			
+	$v=q$	b_2	r_2	λ_2	$v=pq$	b	r	$k=p$
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
1	4	4	3	2	12	16	4	3
2	6	10	5	2	18	36	6	3
3	7	7	3	1	21	28	4	3
4	7	14	6	2	21	49	7	3
*5	9	12	4	1	27	45	5	3
6	9	24	8	2	27	81	9	3
7	10	30	9	2	30	100	10	3
8	13	26	6	1	39	91	7	3
9	4	2	2	2	16	12	3	4
10	7	7	4	2	28	35	5	4
11	10	15	6	2	40	70	7	4
12	13	13	4	1	52	65	5	4
*13	16	20	5	1	64	96	6	4
*14	8	14	7	3	32	64	8	4
15	4	4	3	3	16	20	4	4
16	5	5	4	3	20	25	5	4
17	11	11	5	2	55	66	6	5
18	5	3	3	3	25	20	4	5

The designs marked with* are resolvable.

More general forms of designs obtained by the method of joining are represented by the followiag system of parameters.

$$v = pq,\ b = p\,b_2 + qb_1, \qquad r = r_1 + r_2,\ k \qquad \dots \quad (3.2.3)$$
$$\lambda_{11} = 0,\ \lambda_{01} = \lambda_1,\ \lambda_{10} = \lambda_2.$$

The useful designs of this general class are given in Table 5. All of them are partially balanced with three associate classes except the one corresponding to serial no. 1, which has only two associates.

TABLE 5. DESIGNS OF THE SERIES (3.2.3)

sl. no.	parameters of the balanced design								parameters of the quasifactorial design			
	rows				columns							
	$v=p$	b	r_1	λ_1	$v=q$	b_2	r_2	λ_2	v	b	r	k
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)
1	4	4	3	2	4	4	3	2	16	32	6	3
2	4	4	3	2	6	10	5	2	24	64	8	3
3	4	4	3	2	7	7	3	1	28	56	6	3
4	4	4	3	2	7	14	6	2	28	84	9	3
5	4	4	3	2	9	12	4	1	36	84	7	3
6	4	4	2	2	7	7	4	2	28	42	6	4
7	4	4	2	2	10	15	6	2	40	80	8	4
8	4	4	2	2	4	4	3	3	16	32	5	4
9	4	4	3	3	5	5	4	3	20	40	7	4
10	5	5	4	3	5	5	4	3	25	50	8	4
11	8	14	7	3	4	4	2	2	32	88	9	4

4. CIRCULAR LATTICE DESIGNS

4.1. *Construction of designs.* Let us consider n concentric circles and n diameters defining $2n^2$ lattice points on the circles. Each circle has $2n$ points on it and so also each diameter. If the circles and the diameters are taken as blocks we get a design with the following parameters:

$$v = 2n^2,\ b = 2n,\ k = 2n,\ r = 2.$$

Given any variety the rest fall into three groups, one occurring with it on a circle and on a diameter, $4(n-1)$ occurring with it either on a circle or on a diameter and $2(n-1)^2$ not occurring with it. This design, therefore, satisfies the requirements of the first set of parameters of a partially balanced design with

$$n_1 = 1,\ n_2 = 4(n-1),\ n_3 = 2(n-1)^2$$

$$\lambda_1 = 2,\ \lambda_2 = 1,\ \lambda_3 = 0.$$

It is not difficult to see that requirements of the second system of parameters are also satisfied. The actual matrices are

$$p^1_{ij} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 4(n-1) & 0 \\ 0 & 0 & 2(n-1)^2 \end{pmatrix},\quad p^2_{ij} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 2(n-2) & 2(n-1) \\ 0 & 2(n-1) & 2(n-1)(n-2) \end{pmatrix}$$

$$p^3_{ij} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 4 & 4(n-2) \\ 1 & 4(n-2) & 2(n-2)^2 \end{pmatrix}$$

There are only four useful designs in this series.

TABLE 6. CIRCULAR LATTICE DESIGNS

sl. no.	v	b	k	r
(1)	(2)	(3)	(4)	(5)
1	8	4	4	2
2	18	6	6	2
3	32	8	8	2
4	50	10	10	2

The method of construction for the design with serial no. 1 is illustrated below by drawing 2 circles and 2 diameters.

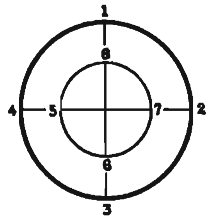

Figure 1.

The blocks are

Circles	Diameters
(1, 2, 3, 4)	(1, 8, 6, 3)
(5, 6, 7, 8)	(4, 5, 7, 2)

All the designs are resolvable into two separate replications corresponding to the circles and the diameters.

4.2. *Analysis of designs.* Since a circular lattice is partially balanced with three associate classes the general method of analysing such designs could be used. But it is simpler to use the 'P method' explained in Roy and Laha (1956), Rao (1956) as the number of blocks is very small compared to the number of varieties.

Let $$v = 2n^2,\ b = 2n = k, \quad r = 2.$$

Denoting the n circles by $c_1, \ldots, c_n$ and n diameters by $d_1, d_2, \ldots, d_n$ we define

$B(c_i)$ = the block total corresponding to the circle c_i,

$B(d_i)$ = the block total corresponding to the diameter d_i.

$P(c_i) = B(c_i)$—the sum of mean yields of varieties occurring in c_i.

$P(d_i)$ = as above for the diameter d_i.

The estimates of the block constants which need not be computed are

$$b(c_i) = \frac{P(c_i)}{n} - \frac{\Sigma P(c_i)}{v}$$

$$b(d_i) = \frac{P(d_i)}{n} - \frac{\Sigma P(d_i)}{v}.$$

The sum of squares due to blocks corrected for varieties is

$\Sigma b(c_i)\ B(c_i) + \Sigma b(d_i)\ B(d_i)$

$$= \frac{1}{n}\{\Sigma P(c_i)B(c_i) + \Sigma P(d_i)\ B(d_i)\} - \frac{1}{v}\{\Sigma P(c_i)\Sigma B(c_i) + \Sigma P(d_i)\Sigma B(d_i)\}$$

so that the only quantities need to be computed are the B and P values.

The estimate of i-th varietal effect is

$$t_i = \frac{T_i}{2} + \frac{P(c_r)+P(d_s)}{b}$$

where c_r and d_s represent the circle and diameter on which the variety t_i lies.

The variances for comparisons are

$$V(t_i - t_j) = \sigma^2, \text{ if } i, j \text{ are first associates } (\lambda = 2)$$

$$= \left(1+\frac{1}{b}\right)\sigma^2, \text{ if } i, j \text{ are second associates } (\lambda_2 = 1)$$

$$= \left(1+\frac{2}{b}\right)\sigma^2, \text{ if } i, j \text{ are third associates } (\lambda_3 = 0).$$

The average variance of all comparisons is

$$\frac{v-1}{v+k-3}\ \sigma^2$$

which may be used to test all varietal differences if the correspondence between the varieties and the integers in the plan of the design is made at random.

It may be observed that the circular lattice designs can also be obtained by considering the dual of a group divisible design with $2n$ varieties in $2n^2$ blocks of 2 plots.

$$(i, n+1)(i, n+2) \ldots (i, 2n)$$
$$(i, n+1)(i, n+2) \ldots (i, 2n) \quad i = 1, \ldots, n.$$

This suggests another type of design obtained by daulising the design with the above blocks repeated thrice instead of twice. The parameters of the new design are

$$v = 3n^2, \ b = 2n, \ k = 3n, \ r = 2$$

with three classes of associates $\lambda_1 = 2$, $\lambda_2 = 1$, $\lambda_3 = 0$. The expressions for the estimates of varietal differences and their variances can be obtained by following the method adopted in the case of circular lattice.

The circular lattice can also be deduced by considering a square lattice and replacing each variety by a pair of varieties. But what is of interest is the simplicity in the analysis of these designs. The diagramatic representation of the design as a circular lattice provides the association scheme.

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