ON THE GEOMETRY OF DUPIN CYCLIDES

V. Chandru
D. Dutta
Christoph M. Hoffmann

CSD-TR-818
November 1988

ON THE GEOMETRY OF DUPIN CYCLIDES

V. Chandru[*], D. Dutta[††] and C.M. Hoffmann[†]

School of Industrial Engineering[*]
School of Industrial Engineering[††]
Computer Sciences Department[†]
Purdue University
CSD-TR-818
November 1988

Abstract

In the 19th century, the French geometer Charles Pierre Dupin discovered a nonspherical surface with circular lines of curvature. He called it a cyclide in his book, *Applications de Geometrie* published in **1822**. Recently, cyclides have been revived for use as surface patches in computer aided geometric design (CAGD). Other applications of cyclides in CAGD are possible (e.g., variable radius blending) and require a deep understanding of the geometry of the cyclide. We resurrect the geometric descriptions of the cyclide found in the classical papers of James Clerk Maxwell and Arthur Cayley. We present a unified perspective of their results and use them to devise effective algorithms for synthesizing cyclides. We also discuss the morphology of cyclides and present a new classification scheme.

1. INTRODUCTION

Early last century the French geometer C. Dupin discovered a non-spherical surface with the property that all its lines of curvature were circular. In his book, *Applications de Geometrie* published in Paris in 1822, he called this surface a **cyclide.** Mathematicians, including [Casey 1871] and [Darboux 1887], have analyzed and generalized Dupin's cyclide in various ways. Until the early part of this century, most books on analytical geometry contained material on Dupin's cyclides and their generalizations [Salmon 1915] [Woods 1922]. However, the generalized cyclides have properties quite different from those discovered by Dupin. In this paper, *Dupin cyclide* and *cyclide* will be used interchangeably, and will always refer to the cyclides of Dupin.

After a lapse of more than fifty years from their discovery, a paper of James Clerk Maxwell [Maxwell 1868], revived the interest in Dupin cyclides. Maxwell was interested in finding two curves such that the congruence of lines meeting the curves can be cut orthogonally by a family of surfaces. He found that Dupin cyclides were such surfaces if the two curves were conics in perpendicular planes, with vertices of one passing through the foci of the other. A few years later, Cayley wrote about his investigations of the cyclide [Cayley 1873]. He was interested in the mathematics of cyclides and in his paper, Cayley simplified the earlier definitions by Dupin. Since then, to our knowledge, the interest in cyclides waned, gradually leading to its omission from most geometry books of this century.

Interest in cyclides revived again in the 1980's. This time it was motivated by research in Computer Aided Geometric Design (CAGD). In 1982 at Cambridge University, U.K., Martin introduced *principal patches* — surface patches bounded by their lines of curvature — in surface modeling and proposed using cyclides to generate such surface patches [Martin 1983]. Since then there has been continued interest in cyclides at Cambridge [Martin et al. 1986]. About the same time at Chrysler Corporation, U.S., McLean proposed a different technique for composing cyclide patches to model automobile surfaces [McLean 1984]. Finally, our own interest in cyclides arose from its applicability as a variable radius blending surface [Hoffmann 1988]. In a recent book, the authors Nutbourne and Martin describe the use of cyclidal patches in surface modeling and briefly trace the history of cyclides [Nutbourne and Martin 1988].

The recent Cambridge/Chrysler interest in cyclides as surface patches focuses on properties "*in the small*" rather than properties "*in the large*". This approach reflects the intended use of a cyclide by designers who are not required to know about its global properties. It is our thesis that the realization of the true potential of cyclides in CAGD applications can be facilitated if we understand and utilize its global properties. In that case, one is faced with the formidable task of visualizing the cyclide and its various forms in the classical framework, i.e. in terms of envelopes, cones, spheres and conics. While the papers of Maxwell [Maxwell 1867] and Cayley [Cayley 1871] are a starting point for such a project, their style and lack of appropriate figures can be dissuading [cf. page 223, Nutbourne and Martin 1988].

We attempt to fill this gap by translating the classical results into the modern context. In doing so, we unify the classical definitions (§2), give complete proofs of some of the key properties (§3), formulate algorithms (constructions) for computer aided synthesis (§4), and present a new classification of the morphology of cyclides (§5). We conclude with a short catalogue of cyclide properties (§6) we deem useful for geometric modeling. In the appendix, we briefly describe our implementation of algorithms for drawing cyclides along with a sample of generated pictures.

2. DEFINITIONS

The following definitions of the cyclide can be found in the classical literature [Dupin 1822], [Maxwell 1868], [Cayley 1873]. We give a brief explanation for each and provide relevant figures. All references to Maxwell and Cayley in this paper, refers to [Maxwell 1868] and [Cayley 1873]. As will be evident later, cyclides are symmetric about a pair of orthogonal planes — the planes of their conics. Thus, illustrations become much simpler on the planes of symmetry as all spheres can be replaced by their circles of intersection. All figures in this paper are on the planes of symmetry. Let F_1, F_2 and F_3 denote three fixed spheres and $\varnothing$ denote a null sphere.

Definition 1 [Dupin]: A cyclide is the envelope of a variable sphere that touches three fixed spheres in a continuous manner.

Given the fixed spheres F_1, F_2, F_3, a variable sphere V tangent to all three of them is in one of eight possible topological positions. With the variable sphere represented by a parenthesis pair, i.e. ( ), we denote the eight positions symbolically as follows:

$F_1F_2F_3(\varnothing)$: The fixed spheres touch V from the outside of V.

$\varnothing(F_1F_2F_3)$: The fixed spheres touch V from the inside of V.

$F_1F_2(F_3)$: V contains F_3 but neither F_1 nor F_2.

$F_3(F_1F_2)$: V contains F_1 and F_2 but not F_3.

$F_1F_3(F_2)$: V contains F_2 but neither F_1 nor F_3.

$F_2(F_1F_3)$: V contains F_1 and F_3 but not F_2.

$F_2F_3(F_1)$: V contains F_1 but neither F_2 nor F_3.

$F_1(F_2F_3)$: V contains F_2 and F_3 but not F_1.

[Figs. 1a & 1b here]

We consider these positions in more detail. Let P be the plane of symmetry defined by the centers of the fixed spheres. Let V be a sphere in position $F_1F_2F_3(\varnothing)$ with its center on P. The intersection of this configuration is shown in Fig. 1a. We imagine that the radius of V increases while its center rises above P as needed to maintain tangency with the fixed spheres. As V enlarges, the points of contact with the fixed spheres F_i move above P towards the north poles of the F_i. If the radius of V is increased indefinitely, its center moves to infinity and V becomes a plane tangent to the F_i in points above P. At that moment, V can also be considered as a sphere of infinite radius with its center *below* P, i.e. we may exchange the inside and outside of V at that position. We do so, and now diminish the radius of V. Then, the center of V approaches P from below and the contact points with the F_i move across the respective north poles reapproaching the equators of the F_i. When the center of V reaches P, we have the configuration shown in the Fig. 1b, and V is now in position $\varnothing(F_1F_2F_3)$. We continue increasing the radius of V while raising its center above P. This time the contact points move below P towards the south poles of the F_i. In the limit, V becomes a second plane touching the F_i below P. Once again we switch the inside and outside, reduce the radius of V, and approach P with the center of V from below until we reach the starting configuration of Fig. 1a.

These considerations show that all spheres in positions $F_1F_2F_3(\varnothing)$ and $\varnothing(F_1F_2F_3)$ belong to the same series. The envelope of this series of spheres is a cyclide. The cyclide corresponding to this series (i.e. Figs. 1a and 1b) is a *ring* cyclide, as explained later.

Similar considerations show that the spheres in positions $F_1F_2(F_3)$ and $F_3(F_1F_2)$ form a series; the spheres in positions $F_1F_3(F_2)$ and $F_2(F_1F_3)$ form another series; and the spheres in positions $F_2F_3(F_1)$ and $F_1(F_2F_3)$ form yet another series. These are the series being referred to by Cayley, in the next definition of a cyclide.

Definition 2 [Cayley]: A cyclide is the envelope of a variable sphere belonging to one of the four series of spheres which touch three given spheres.

Consider any one of the four series of variable spheres. Their circles of intersection V_1 and V_2 on the plane of centers of $F_1F_2\ F_3$ is shown in Fig. 2a. If the role of fixed and variable spheres are now reversed i.e. the two variable spheres of Fig. 2a are the fixed spheres F_1 and F_2 in Fig. 2b and the three fixed spheres of Fig. 2a are the variable spheres $V_1\ V_2$ and V_3 in Fig. 2b, we again obtain a cyclide. This cyclide is defined in terms of only two fixed spheres but with an additional constraint that all variable spheres have their centers on a plane namely, the one defined by $V_1V_2V_3$ in Fig. 2b.

[Fig. 2 here]

Definition 3 [Cayley]: A cyclide is the envelope of a variable sphere having its center on a given plane and touching two given spheres.

The envelope of a series of spheres whose centers lie on a fixed curve has been called a *canal surface* and is attributed to the 19th century French mathematician Gaspard Monge [Hilbert and Cohn-Vossen 1932]. Such an envelope can be thought of as a collection of all circles of intersection between adjacent spheres of the series. These have been referred to as the *characteristic circles* of a canal surface. All surface normals of a canal surface pass through its characteristic circles and hence through the fixed curve. *Lines of curvature* on a surface are defined to be curves such that normals to the surface at two consecutive points on the curve intersect. Thus, all characteristic circles on a canal surface are its lines of curvature. Each sphere that forms the canal surface is tangent to it along a line of curvature.

The envelope of variable spheres in each of the definitions 1, 2 and 3 is a canal surface. Now consider the envelope as obtained by any one definition (e.g. definition 1). If three spheres of this envelope are fixed and the definition 1 reapplied, a second envelope is obtained. All spheres of the second envelope are tangent to the fixed spheres of the first envelope. Since the choice of fixed spheres from the first envelope is arbitrary, *all* spheres of the second envelope are tangent to *all* spheres of the first. Both envelopes are canal surfaces and, furthermore, they are complements of each other in the sense that the space swept by the spheres of the first envelope is the *outside* of the space swept by the spheres of the second envelope, and vice versa. Thus, they share a common surface which is by definition 1, a cyclide. The curvature lines of each canal surface form the curvature lines of the cyclide. Hence, every cyclide can be thought of having a pair of canal surfaces associated with it. The surface normals of these canal surfaces by definition pass through two fixed curves. Thus, we arrive at a new definition for the cyclide.

Definition 4 [Maxwell]: The cyclide is a surface, all normals of which pass through two fixed curves.

In general, the normal sections at any point on a surface yield curves through the point. The centers of the osculating circles of these curves lie on the normal through the point. The two centers farthest apart on the normal are referred to as the *centers of curvature* corresponding to the principal directions of the surface, at that point. In general, the centers of curvature of the points on a surface, form a pair of surfaces. This pair of surfaces is referred to as the *surface of centers* of the original surface. By definition 4, the surface of centers of a cyclide are two fixed curves. The nature of these curves is revealed by the following theorem.

Theorem 1: The fixed curves of a cyclide are conics.

Proof: Let $\acute{C}_1$ and $\acute{C}_2$ denote the pair of canal surfaces common to a cyclide. Consider any sphere S of the first canal surface $\acute{C}_1$. By definition 1, S is tangent to all spheres of the second canal surface $\acute{C}_2$. Let (p_1, p_2, p_3, p_4) denote the points of tangency between S and any four spheres of $\acute{C}_2$.

If $(p_1,\ldots,p_4)$ are non-coplanar $\acute{C}_1$ consists of a single sphere, namely S, since four non-coplanar points uniquely define a sphere. Thus the cyclide is a sphere and its spine is a degenerate conic. If $(p_1,\ldots,p_4)$ are coplanar but not cocircular, $\acute{C}_1$ is the plane since $(p_1,\ldots,p_4)$ cannot lie on a sphere of finite radius. Once again the spine is a degenerate conic. Finally, if $(p_1,\ldots,p_4)$ are cocircular, $\acute{C}_1$ is no longer a singleton since there are an infinite number of spheres that pass through a given circle. Thus, for the general cyclide, all spheres of $\acute{C}_2$ must be tangent to S along a circle M of S.

The circle M along with the center of S define a right circular cone. *The centers of all spheres of $\acute{C}_2$ lie on this cone.* If S is the smallest sphere of $\acute{C}_1$ then M has to be a great circle of S. This implies the right circular cone is now a plane through the center of S. Thus, *the centers of all spheres of $\acute{C}_2$ also lie on a plane.* Hence the spine curve of $\acute{C}_2$ is a conic. Similarly it can be shown that the spine curve of $\acute{C}_1$ is also a conic. ■

Corollary 1: Viewed from any point on one conic, along the tangent, the other conic appears as a circle.

Proof: From Theorem 1 it follows that each conic spine of a cyclide is the locus of vertices of all right circular cones that pass through the other. This implies that each conic is the envelope of the axes of all right circular cones that pass through the other. The tangent at any point on one conic is the axis of the corresponding right circular cone. Clearly, a conic when viewed along the axis of any cone that passes through it will appear as a circle ■.

[Fig. 3 here]

An ellipse and a hyperbola on mutually perpendicular planes, oriented such that the vertices of one are the foci of the other, are called *anticonics* (see Fig 3). Anticonics are also referred to as the focal conics of an ellipsoid since they serve the same purpose in its thread construction as do the foci in the thread construction of an ellipse [Hilbert and Cohn-Vossen 1932]. The terms "anticonics" and "a pair of anticonics" will be used interchangeably in this paper, to refer to a pair of conics positioned as defined above. It follows from Corollary 1 that the spine curves of a cyclide are anticonics. It is a property of the anticonics that, if two points be fixed on the hyperbola then the *sum* of distances between a variable point on the ellipse to the two fixed points is a constant if the fixed points lie on two branches of the hyperbola, and the *difference* between the distances is a constant if the fixed points are on the same branch of the hyperbola. Similarly, if the two fixed points be on the ellipse and the variable point on the hyperbola, then the absolute value of the difference between the distances of variable to fixed points is a constant. Thus, in definition 3, one of the two fixed spheres is redundant in view of the fact that their centers lie on anticonics. So, definition 3 can be further simplified as follows.

Definition 5 [Cayley]: Considering any two anticonics, the cyclide is the envelope of a variable sphere on the first anticonic and touching a given sphere whose center is on the second anticonic.

Visualizing a cyclide by definiton 5 is easy (e.g. one form of the cyclide resembles a squashed torus and has been referred to as a ring cyclide of Figs. 1a & 1b). It follows from the definition that planes of symmetry of a cyclide are the planes of its anticonics. In general, sections by these planes will yield a pair of circles that "bound" the cyclide. We shall refer to these as the *extreme circles* on the plane of symmetry. In definiton 5, consider the variable circles generated by intersections of the variable spheres. The centers of these circles lie on a plane of symmetry, whereas the circles themselves lie on planes that are perpendicular to it. Therefore, each variable circle intersects the plane of symmetry at two points. These are the end points of the diameter of the variable circle since they are collinear with its center. Thus, a section of the cyclide by an anticonic plane yields (see Fig. 4a): the conic which is the locus of centers of all variable spheres (e.g. the ellipse E); the curve K which is the locus of centers of all variable circles generated by these variable spheres; the extreme circles C_1 and C_2, which are the locus of the two diametral end points of these variable circles. Hence, the cyclide can also be defined in terms of the variable circles.

[Figs. 4a & 4b here]

As shown in Fig. 4b, the *centers of symmetry* (or *centers of similitude*) of two circles on a plane are the two points T and S, that cut the line through their centers in the ratio of their radii [Hilbert and Cohn-Vossen 1932]. The diameter of the variable circle of a cyclide, on the plane of its anticonic, is given by the segment between the extreme circles, of a line joining the center of the variable circle to either center of symmetry of the two extreme circles. The end points of the diameter on the extreme circles are so chosen that the tangents to the extreme circles at those points are not parallel. In the following definiton of a cyclide by variable circles, Cayley refers to the diametral endpoints as anti-parallel points.

Definition 6 [Cayley]: Consider in a plane any two circles, and through either of the centres of symmetry draw a secant cutting the two circles, in the perpendicular plane through the secant, having for their diameter the chords formed by two pairs of anti-parallel points on the secant (viz. each pair consists of two points, one on each circle, such that the tangents at the two points are not parallel to each other): the locus of the variable circles is the cyclide.

Definition 5 might be easier for the purpose of visualizing cyclides. However, definition 6 is useful for computer implementation and gives better access to analyzing geometric properties of cyclides.

3. GEOMETRICAL INSIGHTS

It is known that each of the conics namely, ellipse, parabola and hyperbola, is the locus of all points whose distance from a fixed point is in a constant ratio to the distance from a fixed line. This ratio is called the *eccentricity* of the conic. The fixed point and fixed line are referred to as the *focus* and *directrix,* respectively. The eccentricity of the ellipse is < 1; of the hyperbola >1; and for the parabola it is 1. All conics can also be generated by plane sections of a right circular cone. (Henceforth by cones we shall always mean a right circular cone). The eccentricity of a conic by this mode of generation can be expressed as the ratio of the angle of inclination of the plane of the conic and the apex angle of the cone i.e., $\frac{\cos(\gamma)}{\cos(\beta)}$ in Fig 5. All planes perpendicular to the axis of a cone intersect it in circles. If the cone passes through a conic, these planes intersect the plane of the conic, in lines parallel to its directrix. These are special lines and we shall refer to them as *characteristic lines* of the conic (e.g. D_e is a characteristic line in Fig 5). Each characteristic line of a conic can be thought of as generating a unique circle on every cone that passes through the conic. If a characteristic line intersects the conic, so does its generated circle at the same points on the conic. It is evident that a pair of characteristic lines for anticonics are mutually perpendicular.

[Fig. 5 here]

Let ellipse E on the XY plane and hyperbola H on the YZ plane be a pair of anticonics. From Theorem 1, we know that each conic spine of a cyclide is the locus of vertices of all right circular cones that pass through the other conic. Let C_E denote the family of cones with vertices on H and passing through E. Similarly, let C_H denote the family of cones with vertices on E and passing through H. The XY plane is a degenerate cone of C_E, just as the XZ plane is one of C_H. We shall now describe the cyclide with respect to these two families of cones.

On the plane of the ellipse, choose a characteristic line D_e. If D_e remains fixed for all cones of C_E, a family of circles is obtained (see Fig 6). These are circles of intersection on the planes that pass through D_e and are perpendicular to the axes of all cones of C_E. Thus corresponding to E and D_e a family of circles is obtained that lie on planes orthogonal to the plane of H. These circles form the family of *latitudinal curvature lines* on the cyclide. If D_e intersects the ellipse so do all circles, at the same points on the ellipse. Otherwise, depending on the position of D_e with respect to E, the radii of the circles might either be positive always, or diminish to zero and increase again. If it is the latter, in general, there are two circles of zero radii, located on opposite points on any one branch of the hyperbola.

[Fig. 6 here]

By a similar process a second family of circles, corresponding to the cones of C_H, is obtained once characteristic line D_h is fixed. These circles are on planes orthogonal to the plane of the ellipse and form the family of *longitudinal curvature lines* on the cyclide. As before, if D_h intersects a branch of the hyperbola, so do all circles of this family, at the same points on the hyperbola. Otherwise, depending on the position of D_h with respect to H, the radii of the circles might either be positive always, or diminish to zero and increase again. If it is the latter, there are two circles of zero radii at opposite points on either side of the major axis of E. Therefore, associated with each cyclide is a pair of characteristic lines D_e and D_h that are parallel to the Y and Z axes respectively.

Theorem 2: The family of circles obtained from all right cones passing through one conic, by fixing the position of its characteristic line, forms a family of curvature lines on the cyclide.†

Proof: Let $\acute{C}_1$ and $\acute{C}_2$ with spine curves c_1 and c_2 respectively, be the pair of canal surfaces defining the cyclide. Suppose characteristic line D_1 corresponding to conic c_1 is fixed. All cones passing through c_1 belong to spheres of $\acute{C}_2$. It suffices to show that every circle generated with respect to D_1 is a line of curvature on $\acute{C}_2$.

From Theorem 1, the circle m_i defined by D_1 on any sphere $S_i \in \acute{C}_2$ is the locus of points of tangency between S_i and all spheres of $\acute{C}_1$. We know that the lines of curvature on a canal surface are the circles of intersection of adjacent spheres. Thus spheres S_i and S_{i+1}, of $\acute{C}_2$ intersect in a circle which is a line of curvature on $\acute{C}_2$. But S_{i+1} is also tangent to all spheres of $\acute{C}_1$. Thus it must intersect S_i at circle m_i, which implies m_i is a line of curvature on $\acute{C}_2$. ■

From differential geometry it is known that lines of curvature on any surface form an orthogonal net. Thus, given one family of curvature lines the other family is determined. Consequently for the cyclide, fixing one characteristic line automatically fixes the other. These lines intersect only at the origin, yielding circles of curvature which we refer to as the *reference circles* of each cone. Otherwise, when one characteristic line is at a distance λ from the origin, all latitudinal and longitudinal circles of curvature are at a distance r from their respective reference circles. This distance r, is a function of λ and the focal lengths of the anticonics and is measured along the generating lines of the cones. One can either choose r or λ as the one parameter to define a cyclide, the other two being the focal lengths of the anticonics.

In his formulation of the implicit equation for a cyclide, Maxwell chose the parameter r and the focal lengths of the anticonics. The reason we adhere to Maxwell's choice is because in the context of CAGD, the parameter r can be used directly to generate offsets of the cyclide. Recall that the cones are the collection of surface normals at any circle of curvature on the cyclide. Thus parallel circles on each such cone represent offsets of the corresponding curvature lines and hence can be used to form the

† This theorem can be viewed as a direct consequence of a more general theorem due to Monge, which states that the necessary and sufficient condition for a curve on a surface to be a line of curvature is that the surface normals along this curve form a developable surface [Struik 1961].

offset of an entire cyclide. As such, any circle of curvature on a cyclide can be viewed as a positive or negative magnification of its reference circle parameterised by r.

4. CYCLIDE CONSTRUCTIONS

Maxwell's Method: This method of construction is due to Maxwell and is the procedure described in section 3 for generating all lines of curvature on the cyclide. The three parameters used in this construction provides a basis for an analysis of the various forms of the cyclide.

Let ellipse E and hyperbola H form a pair of anticonics, with E on the XY plane and H on the XZ plane, both centered at the origin. Let the eccentricity of E be $e_E = \frac{f}{a}$. Thus the eccentricity of H is $e_H = \frac{a}{f}$. Thus the general equation for E is

$$\frac{x^2}{a^2} + \frac{y^2}{a^2-f^2} = 1$$

If α is the eccentric angle of a point P on E (see Fig. 7a), the parametric equations for P is given by

$$x(P) = a\cos(\alpha), \quad y(P) = \sqrt{a^2-f^2}\sin(\alpha), \quad z(P) = 0$$

Similarly for the hyperbola H we get

$$\frac{x^2}{f^2} - \frac{z^2}{a^2-f^2} = 1$$

and the parametric equations for point Q on H are

$$x(Q) = f\sin(\beta), \quad y(Q) = 0, \quad z(Q) = \sqrt{a^2-f^2}\tan(\beta)$$

where β is the auxiliary angle of H (see Fig. 7b).

[Figs. 7a & 7b here]

C_E and C_H denote the families of cones passing through E and H respectively. For each family the reference circles are those special circles generated when characteristic lines D_e and D_h are the Y and Z axes respectively. As shown in Fig. 8a, for any cone passing through the ellipse E, the distance between a variable point on the ellipse and the corresponding point on the reference circle, along a generating line of the cone, is $(-f\ \cos\alpha)$. Similarly as shown in Fig. 8b, for any cone passing through the hyperbola H, the distance between a variable point on the hyperbola and the corresponding point on the reference circle, along a generating line of the cone, is $(-a\ sec\beta)$. Let M be any circle distinct from its reference circle, on a cone of C_E or C_H. Let r be the distance of M from the reference circle along a generating line of the cone. If the cone belongs to C_E then the distance from a variable point on the ellipse to corresponding points on M along generating lines of the cone is given by $(r - f\ \cos\alpha)$. If the cone belongs to C_H then the same distance is given by $(r - a\ sec\beta)$.

[Fig. 8a & 8b here]

The distance between the apex of any cone of C_E or C_H and a variable point on the conic it passes through, is simply the distance between the variable points, P on the ellipse and Q on the hyperbola. Thus,

$$[PQ]^2 = [x(P) - x(Q)]^2 + [y(P) - y(Q)]^2 + [z(P) - z(Q)]^2$$

which upon simplification yields

$$PQ = a\ sec\beta - f\ \cos\alpha$$

Thus, for all cones of C_E with apex Q on the hyperbola and variable point P on the ellipse, distance QR is a constant if

$$PR = r - f\ \cos\alpha$$

and R generates a latitudinal circle of curvature on the cone. Similarly, for all cones of C_H with apex P on the ellipse and variable point Q on the hyperbola, distance PR is a constant if

$$QR = r - a\ sec\beta$$

and again R generates a longitudinal circle of curvature on the cone. Therefore, as points P and Q traverse the ellipse and hyperbola respectively, the point R traces out all latitudinal and longitudinal lines of curvature on the cyclide. For a cyclide, the parameter r could be chosen to be a positive or negative constant and represents the fixed distance of each circle of curvature from its reference circle. The following steps can be outlined for a computer implementation of Maxwell's construction.

INPUT : Parameters $[f, a, r]$

STEP 1: Longitudinal circles – for each **fixed** point $[P(\alpha) \mid 0 \leq \alpha \leq 360^o]$ on the ellipse, take **variable** points $[Q(\beta) \mid 0 \leq \beta \leq 360^o]$ on the hyperbola. On line segment PQ, the point $[R(\alpha,\beta) \mid QR = r - a \sec \beta]$, traces out a longitudinal circle of curvature.

STEP 3: Latitudinal circles – for each **fixed** point $[Q(\beta) \mid 0 \leq \beta \leq 360^o]$ on the hyperbola, take **variable** point $[P(\alpha) \mid 0 \leq \alpha \leq 360^o]$ on the ellipse. On line segment QP, the point $[R(\beta,\alpha) \mid PR = r - f \cos\alpha]$ traces out a latitudinal circle of curvature.

Cayley's Method: This method of construction of a cyclide is due to Cayley. It is essentially a procedure for generating one family of curvature lines on the cyclide, as mentioned in definition 6. Here the cyclide is viewed as an envelope of variable circles. The parameters required for this construction are a pair of extreme circles of the cyclide, on a plane of symmetry. The cyclide is then constructed such that it is bounded by the extreme circles and symmetric about the given plane.

INPUT: Extreme circles C_1 and C_2 on a plane P and the center of symmetry of C_1 and C_2 (*inner* or *outer*).

STEP 1: Locate inner and outer centers of symmetry of C_1 and C_2. Take pencil of lines L through the chosen center of symmetry.

STEP 2: On each $l_i \in L$ draw two circles perpendicular to P, with diameters, the segments of l_i terminated by circles C_1 and C_2. The end points of every diameter belong one each, to C_1 and C_2 such that, the tangents to the circles at these points are not parallel.

To obtain the second family of curvature lines the two extreme circles of the first family can be used as parameters. This procedure is simple and amenable for computer implementation. The choice between Maxwell's and Cayley's method for cyclide construction depends on the context of the intended application. For example, if a cyclide (or a portion of it) is to be constructed on a given spine, Maxwell's method is more suitable. But if the cyclide is to be used for a variable radius rolling-ball blend of a cylinder and inclined-plane intersection [Hoffmann 1988], it might be easier to visualize (and specify) it by the extreme circles which then represent the minimum and maximum diameter of the rolling ball (see Fig. 9).

[Fig. 9 here]

Implicit Forms: For a cyclide with anticonic parameters a and f and a magnification parameter r, the implicit equation has been shown by Maxwell to be

$$(x^2+y^2+z^2-r^2)^2 - 2(x^2+r^2)(f^2+a^2) - 2(y^2-z^2)(a^2-f^2) + 8afrx + (a^2-f^2)^2 = 0$$

When f and a are increased to infinity, in the limit, the anticonics become a pair of parabolae in perpendicular planes, the focus of one coinciding with the vertex of the other. Thus, parameters f, a, r are no longer valid. This form of the cyclide is a cubic surface and its implicit equation can be derived using Cayley's definition 5 as follows.

[Fig. 10 here]

In Fig. 10, let P_1 be a parabola on the horizontal plane with directrix D_{P1} and focus M, and let P_2 be another parabola on the vertical plane, with directrix D_{P2} and focus N. Furthermore, if M and N are the vertices of P_2 and P_1 respectively, then the parabolae are a pair of anticonics. Assume a fixed sphere F of radius r to be centered at the vertex M of P_2. The cyclide is the envelope of the variable spheres U, centered on P_1, tangent to F. Points on P_1 have coordinates (x_1, y_1) where,

$$y_1 = 2pt$$

$$x_1 = 2pt^2 - \frac{p}{4}$$

where p is the distance between the focus and directrix of P_1 and P_2, and, t is a parameter indentified geometrically with the gradient of the tangent to the parabola at that point. The variable radius r_t of the spheres U is now given by

$$r_t = 2pt^2 - \frac{p}{4} + \frac{p}{2} + \frac{p}{4}$$

Thus, the spheres of U are given by

$$S:\ (x - 2pt^2 + \frac{p}{4})^2 + (y - 2pt)^2 + z^2 - (2pt^2 + \frac{p}{2} - r)^2$$

The envelope of U is obtained by eliminating the parameter t, between S and $\frac{dS}{dt}$. Using resultants to do so and then simplifying, the equation for a parabolic cyclide becomes

$$z^2(x-g') + y^2(x-h') + (x-g')(x-h')(x-f') = 0$$

where,

$$g' = r + \frac{p}{4}$$

$$h' = r - \frac{3}{4}p$$

$$f' = -r + \frac{p}{4}$$

Furthermore, the characteristic lines of this cyclide are the lines

$$x = h'; \quad z = 0$$

$$x = g'; \quad y = 0$$

and are on the surface of the cyclide. The meridian and equatorial circles are given by

$$z = 0: \quad y^2 + (x - \frac{p}{4})^2 - r^2 = 0$$

$$y = 0: \quad z^2 + (x + \frac{p}{4})^2 - (r - \frac{p}{2})^2 = 0$$

5. MORPHOLOGY OF CYCLIDES

Let cyclide $[f,a,r]$ denote a cyclide with the parameters f,a,r and let parameters in **bold** indicate their values fixed (i.e. the cyclide $[\mathbf{f}, \mathbf{a}, r]$ represents a particular family of cyclides with fixed anticonics). The forms of a cyclide$[f,a,r]$ can be classified at two levels. A primary classification is based on the first two parameters f and a. Once these parameters are fixed, a secondary classification can be made based on the third parameter r. Thus the basic form of a cyclide depends on the form of its anticonics, while the subform depends on the magnification of the lines of curvature There are four pairs of conics that satisfy the anticonic property viz., (ellipse/hyperbola), (parabola/parabola), (circle/straight line) and (degenerate conics) i.e., points, double lines, and intersecting lines. Since the ellipse and hyperbola are often referred to as the central conics, we might call the associated family of cylides, **central cyclides**. Cyclides having the parabolae as anticonics have been referred to as **parabolic cyclides** . The cyclides with a circle and a straight line as anticonics always generate surfaces of revolution and so we might refer to them as **revolute cyclides** and finally the ones with degenerate conics might simply be called **degenerate cyclides** . Fixing the anticonics and their parameters, f and a yields a one parameter family of cyclides that Maxwell refers to as *confocal cyclides* (analogous to *confocal quadrics* and *confocal conics* in the classical literature [Hilbert and Cohn-Vossen 1932]).

Within each confocal system of a primary category are three subforms that depend on the value of r in relation to the values of f and a. Positive and negative values of r yield symmetric subforms with reference to the plane orthogonal to the planes of the anticonics and so it suffices to consider the positive values of r. The **central cyclides** are unique in that, their anticonics are devoid of any degeneracies. As a result, their subforms are distinct and have been portrayed in drawings and plaster models as typical examples of cyclides [p. 218, Hibert and Cohn-Vossen 1932]. Nevertheless, each primary category of the cyclides admits a further classification of its subforms based on the value of parameter r. Basically the value of r, in relation to a and f, determines the existence of pinch points (or nodal points) on the surface of the cyclide.

When $(0 \leq r < f)$ there are two pinch points on the surface of the cyclide. These points lie on the ellipse on a line parallel to its minor axis. The shape of the cyclide resembles a pair of cresents touching each other at their ends. When $(r=0)$ the cresents are equal. As r increases one cresent becomes smaller while the other becomes larger.

Maxwell has referred to this subform as a *horned cyclide* since it can also be visualized thus. When $(r=f)$ the smaller cresent completely vanishes and the larger cresent meets itself at the ends, in what appears to be one pinch point on the surface. This point is now at the vertex of the ellipse (or, focus of the hyperbola).

When $(f<r<a)$ there are no visible pinch points on the surface anymore and the cyclide resembles a squashed torus, the minimum diameter being around the last pinch point. Maxwell refers to this subform as the *ring cyclide*. When $(r=a)$, the inner circle of the annular ring diminishes to a point and again a single pinch point appears on the surface at the vertex of the hyperbola (or, focus of the ellipse). The surface of the cyclide now resembles that of an inflated spherical balloon, held by the thumb and index fingers, meeting at a point away from the center.

When $(a<r)$ the new pinch point becomes a pair which move away from the vertex of the hyperbola on a line perpendicular to its transverse axis. The creation of the new pinch point gives rise to a spindle inside the cyclide. Maxwell has referred to this form of the cyclide as a *spindle cyclide*. The process of creation of the spindle can be visualized as the inverse of vanishing of one of the cresents of the horned cyclide. We classify the various forms and subforms of the cyclide in Table-I. The associated pictures are indicated by the respective plate numbers.

DUPIN CYCLIDES

	Horn	*Ring*	*Spindle*
Central Cyclides	Plate-I (a)	Plate -I (b)	Plate-I (c)
Revolute Cyclides	Circle	Torus	Plate-II (b)
Parabolic Cyclides	Plate-III (a)	Plate-III (b)	Plate-III (a)
Degenerate Cyclides	Cone	Cylinder	Cone

TABLE - I

By definition, each cyclide can be viewed as the envelope of two distinct families of spheres. For the purposes of visualization, the envelope of one family of spheres is what an observer might actually view the cyclide to be, while the second family of spheres forms an envelope that *avoids* the first. It might be stated, without a discourse on the process of visualization, that for a **parabolic cyclide,** the *horned* and *spindle* subforms are mirror images of each other, rotated 90 degrees about the transverse axis of either parabola. Similarly a cone of revolution if viewed as a **degenerate cyclide** is a *horn* and *spindle* cyclide with a double line as the anticonic.

Cayley's construction (and definition 6) provides an alternate method for visualizing *all* primary and subforms of the cyclide. By an appropriate choice of diameters and positions for the extreme circles, all the forms described earlier can be obtained. In fact, by this method the **parabolic cyclide,** the most complicated form to visualize otherwise, becomes very simple as follows. The extreme circles are a straight line and and a circle, non-intersecting for a ring cyclide (Fig. 11a), intersecting otherwise (Fig. 11b). The centers of symmetry for such a pair is defined to be the points of intersection of the circle and a perpendicular through its center to the straight line. The cylinder and cone can also be obtained if the extreme circles are a pair of parallel and intersecting lines respectively.

[Fig. 11 here]

Offset surfaces are known to be important in CAGD. The cyclides have the property that they are closed under offsetting. Recall from section 3 that parameter r is directly related to the offset distance of a cyclide. But there is an exception to this rule for one subform of the **central cyclide** namely, the ***horn cyclide.*** As mentioned, in this subform there are two cresents, call them positive and negative cresents, which meet at two pinch points on the ellipse. As r increases, the positive cresent increases while the negative cresent decreases which when viewed in terms of offsets, implies a positive offset of the positive cresent and a negative offset of the negative cresent. The reverse happens when r is decreased. The reason for this phenomenon is evident since at the pinch points of the ellipse the plane defined by the characteristic line D_h crosses over the cone apex and the parameter r begins to have an inverse effect. This inherent problem can be handled by offsetting each cresent appropriately, i.e. by $(+r)$ and $(-r)$.

6. PROPERTIES OF THE CYCLIDE

We now summarize our findings with a short catalogue of the key properties of the cyclide. While most properties mentioned below have been described in sections 2 and 3, the others can be derived from them. Most of these properties provide insights that are helpful in understanding the cyclide in its entirety for applications in CAGD.

P1. The cyclide has three degrees of freedom namely, a and f, the focal lengths of its anticonics, and r, the magnification of its lines of curvature. It is the only quartic surface with circular lines of curvature.

P2. All forms of the cyclide lie between two tangent planes which meet them along two circles. The section by planes of its anticonics always yield a pair of extreme circles of the cyclide.

P3. Fixing the anticonics yield a one parameter family of confocal cyclides which can further be classified as *horn*, *ring* or *spindle*. Changing parameters of the anticonics leads to varying shapes *within* a primary category. Changing the anticonics generates cyclides of different primary categories.

P4. Given a cyclide [**f**,**a**,r] of a confocal system, its longitudinal characteristic D_H is parallel to the Z-axis and intersects the X-axis at a point P' distant $(\frac{f}{a}r)$ from the origin. Its latitudinal characteristic D_E is parallel to the Y-axis and intersects the X-axis at a point Q' distant $(\frac{a}{f}r)$ from the origin. The characteristics D_H and D_E are polars of each other with respect to a sphere of radius r centered at the origin.

P5. For a cyclide[f,a,r], the plane U containing a longitudinal circle of curvature corresponding to point P on the ellipse makes angle θ_1 with the XZ plane where α is the eccentric angle of P and,

$$\tan\theta_1 = \frac{a}{\sqrt{a^2 - f^2}} \tan\alpha$$

Similarly the plane V containing a latitudinal circle of curvature corresponding to point Q on the hyperbola makes angle θ_2 with the XY plane where β is the auxiliary angle of Q and

$$\tan\theta_2 = \frac{f}{\sqrt{a^2 - f^2}} \sin\beta$$

Other than the planes of the anticonics, U and V are planes which yield a pair of circles when intersected with a cyclide.

P6. Cyclides admit a rational parametrization [Martin et al. 1986].

P7. The offset of a cyclide[**f**,**a**,**r**] by a distance **d** along its surface normals, is the cyclide [**f**,**a**,**r**+**d**].†

† Offset distance adjusted for a *horn cyclide*.

P8. A cyclide can be defined either in terms of variable spheres, or in terms of variable circles. With each circle of curvature of the cyclide [**f,a,r**] are associated *two* radii namely, the radius of the sphere which contains the circle, and, the radius of the circle itself.

P9. Cyclides can always be obtained by inversion in a cone or cylinder, with respect to a sphere. Cyclides are anallagmatic surfaces, i.e. inverses of themselves [Woods 1922].

7. CONCLUSIONS

Cyclides are being revived by their use in CAGD. They have been proposed for use as surface patches and blending surfaces. Further applications of the cyclides in CAGD can arise from knowledge about their overall geometry. We believe that the exercise of visualizing cyclides by spheres, envelopes and conics (i.e. in the classical framework) is useful in this context. This geometric approach, as opposed to a purely analytic or algebraic approach, yields intuitive insights on constructive and non-constructive properties of cyclides. The constructive properties led to simple algorithms for synthesizing cyclides and to a precise classification of the morphology of cyclides. Several non-constructive properties were also detailed. For example, offsets of cyclides were easily shown to be cyclides and the radius variation in a cyclide was also characterized.

References

Casey J. (1871) On Cyclides and Spheroquartics. Philosophical Transactions of the Royal Society of Dublin, 585-721.

Cayley A. (1873) On the cyclide. Quarterly Journal of Pure and Applied Mathematics, 12, 148-163.

Darboux G. (1887) La Theorie Generale des Surfaces, Gauthier-Villars, Paris.

Dupin C. (1822) Applications de Geometrie et de Mechanique, Bachelier, Paris.

Hilbert D. and Cohn-Vossen S. (1932) Geometry and the Imagination, Chelsea Publishing Co., New York, reprinted 1983.

Hoffmann C.M. (1988) Variable radius blending using cyclides, NSF-IFIP Workshop on Geometric Modeling, Renssalaerville, N.Y., September 1988.

Maxwell J.C. (1868) On the cyclide, Quarterly Journal of Pure and Applied Mathematics, 9, 111-126.

Martin R.R. (1983) Principal patches – a new class of surface patch, Proceedings of Eurographics 1983, 47, North Holland, Amsterdam.

Martin R.R., de Pont J.J., and Sharrock T.J. (1986) Cyclide surfaces for computer-aided design. In: Gregory J. (ed.) Mathematics of Surfaces, Conference Proceedings, Institute of Mathematics and its Applications, 1984, Oxford University Press, Oxford.

McLean D. (1984) A method for generating surfaces as a composite of cyclide patches, The Computer Journal, 28, 4, 433-438

Nutbourne A.W., and Martin R.R., (1988) Differential Geometry Applied To Curve And Surface Design, Volume 1: Foundations, Ellis Horwood Publishers.

Salmon G. (1915) A Treatise on the Analytic Geometry of Three Dimensions, Longmans Green & Co., London.

Struik D.J. (1961) Lectures on Classical Differential Geometry, 2nd. edition, Addison-Wesley, Dover reprint 1988.

Woods F.S., (1922) Higher Geometry, Ginn and Company.

Appendix

The computer program used for generating the cyclides shown in the color plates is based on a hybrid scheme. For the central cyclides with input parameters a, f, r the extreme circles on the plane of symmtery of the ellipse are determined. They have radii $(a+r)$ and $(a-r)$ and are centered at $(-f, 0, 0)$ and $(f, 0, 0)$ respectively. Similarly, the extreme circles on the plane of symmetry of the hyperbola have radii $(f-r)$ and $(f+r)$, and are centered at $(a,0,0)$ and $(-a,0,0)$ respectively. The characteristic lines are also determined (i.e. D_h on the xz plane is at $x=\frac{fr}{a}$ and D_e on the xy plane is at $x=\frac{ar}{f}$). The latitudinal and longitudinal lines of curvature are now drawn using Cayley's definition 6. The process is simpler for the **revolute cyclides** since $f=0$ and the latitudinal curvature lines are contained in parallel planes $z=c$, where $-r \leq c \leq +r$.

For **parabolic cyclides** the approach is the same, but differs somewhat in detail, since one of the extreme circles in each plane of symmetry is now a line. Hence there is only one pair of antiparallel points, one of them on a circle, the other on a line. Note that the parabolic cyclide is a cubic surface.

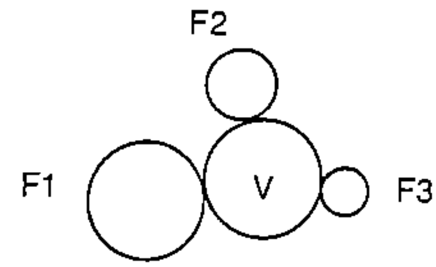


Fig. 1a

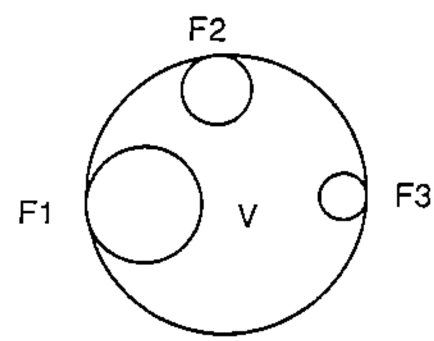


Fig. 1b

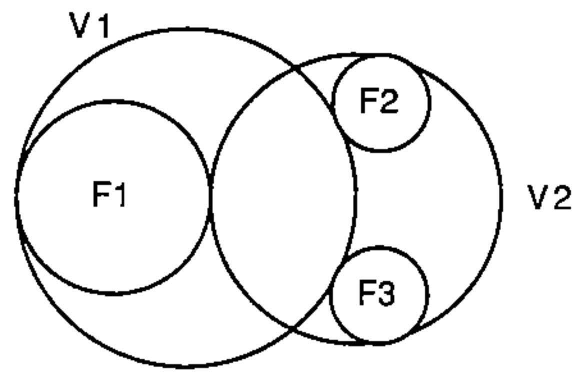


Fig. 2a

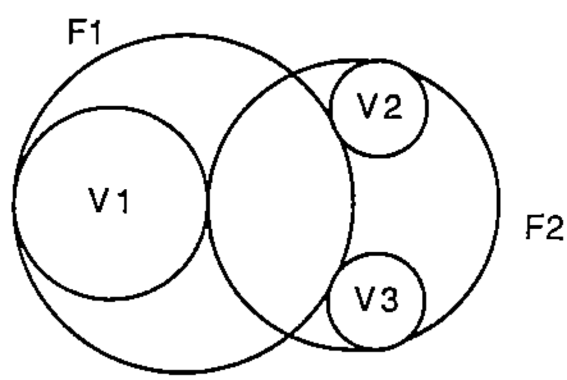


Fig. 2b

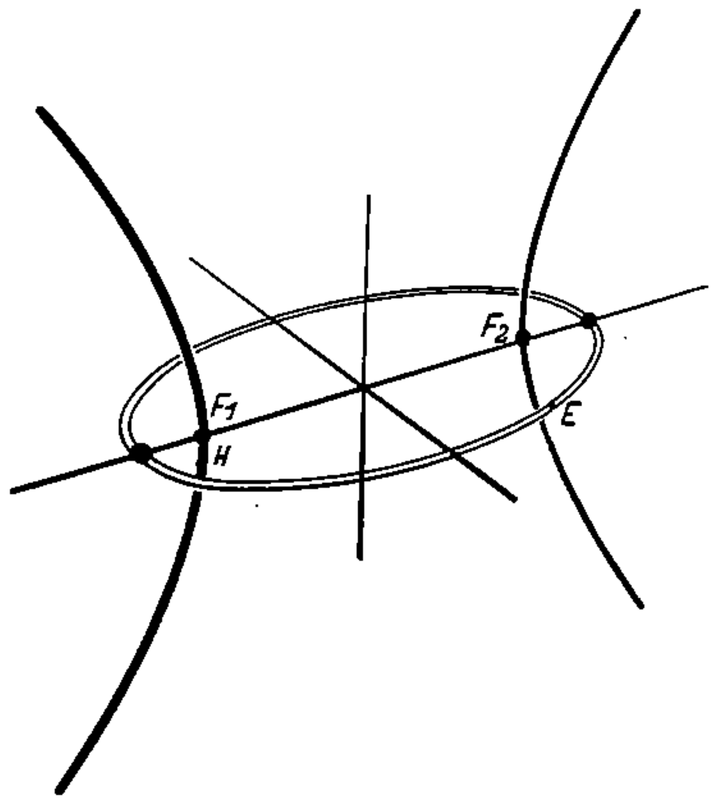


Fig. 3

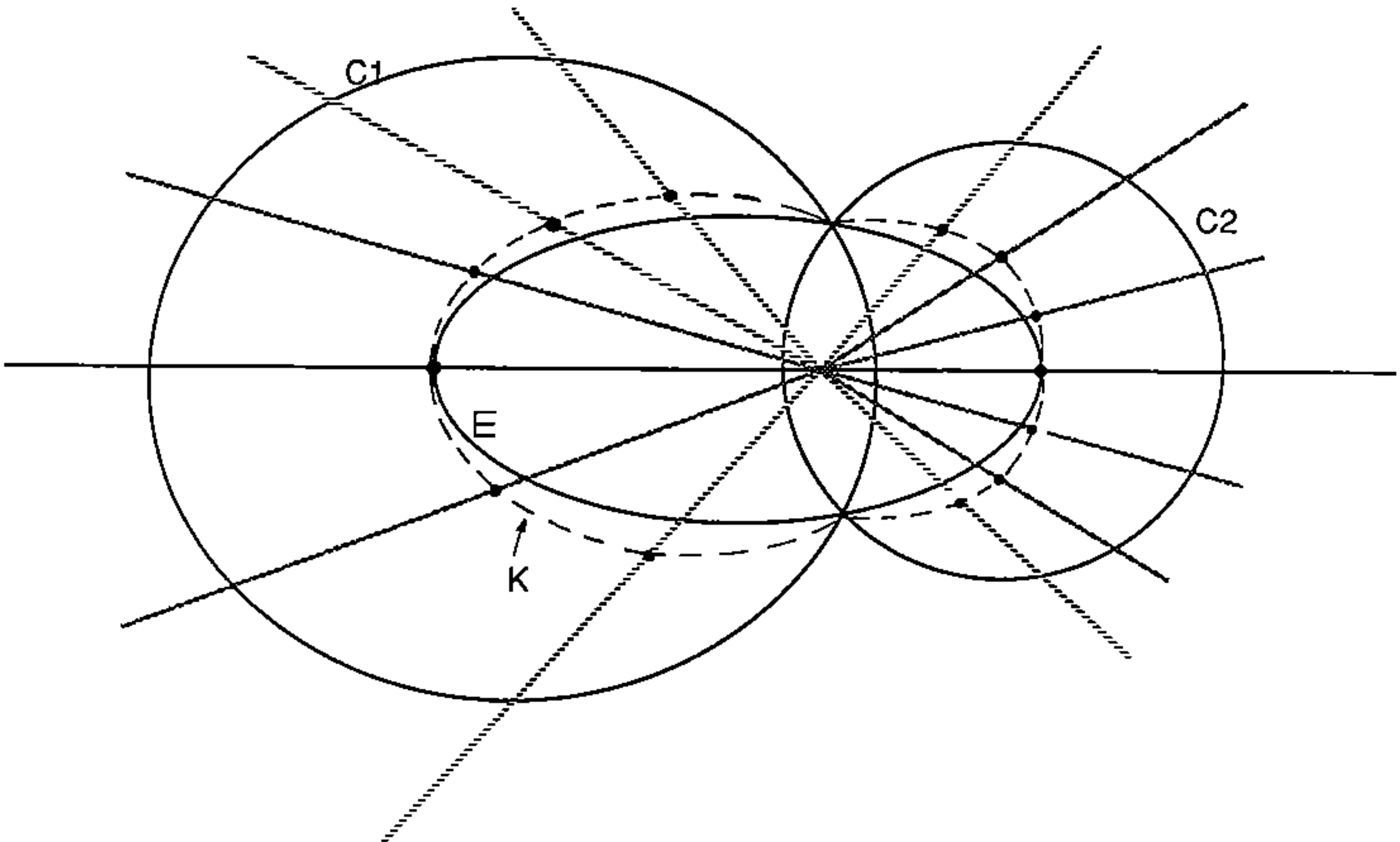


Fig. 4a

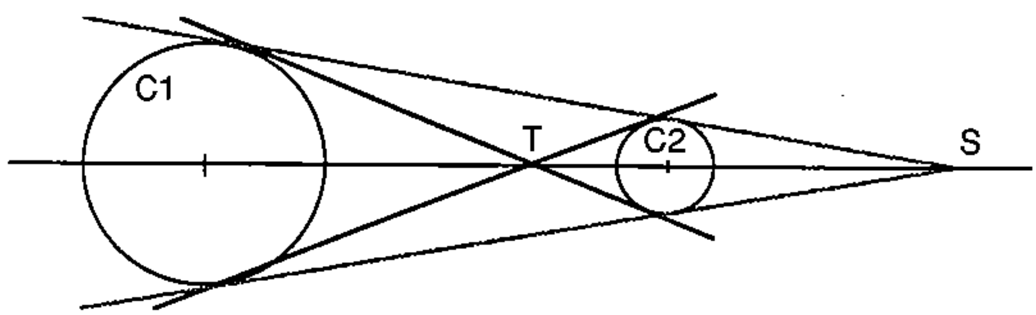


Fig. 4b

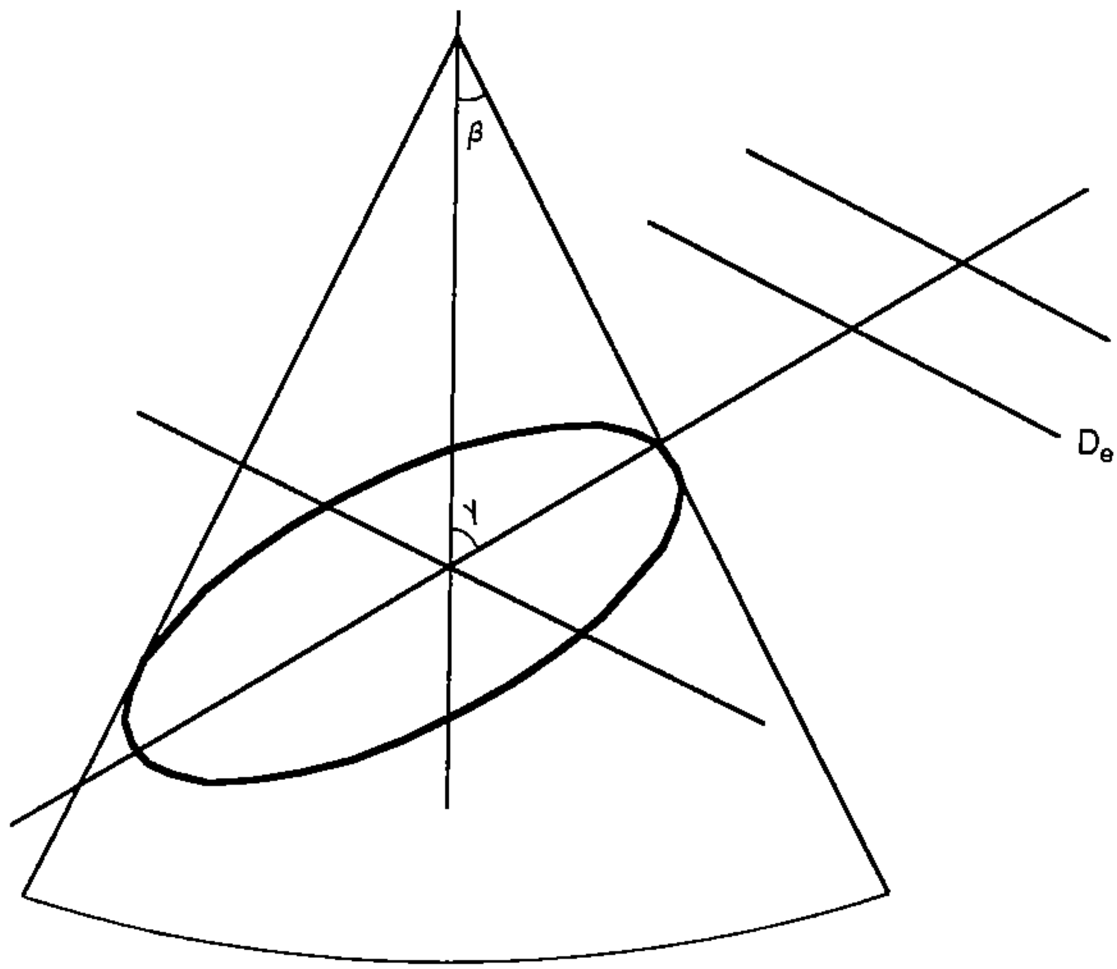


Fig. 5

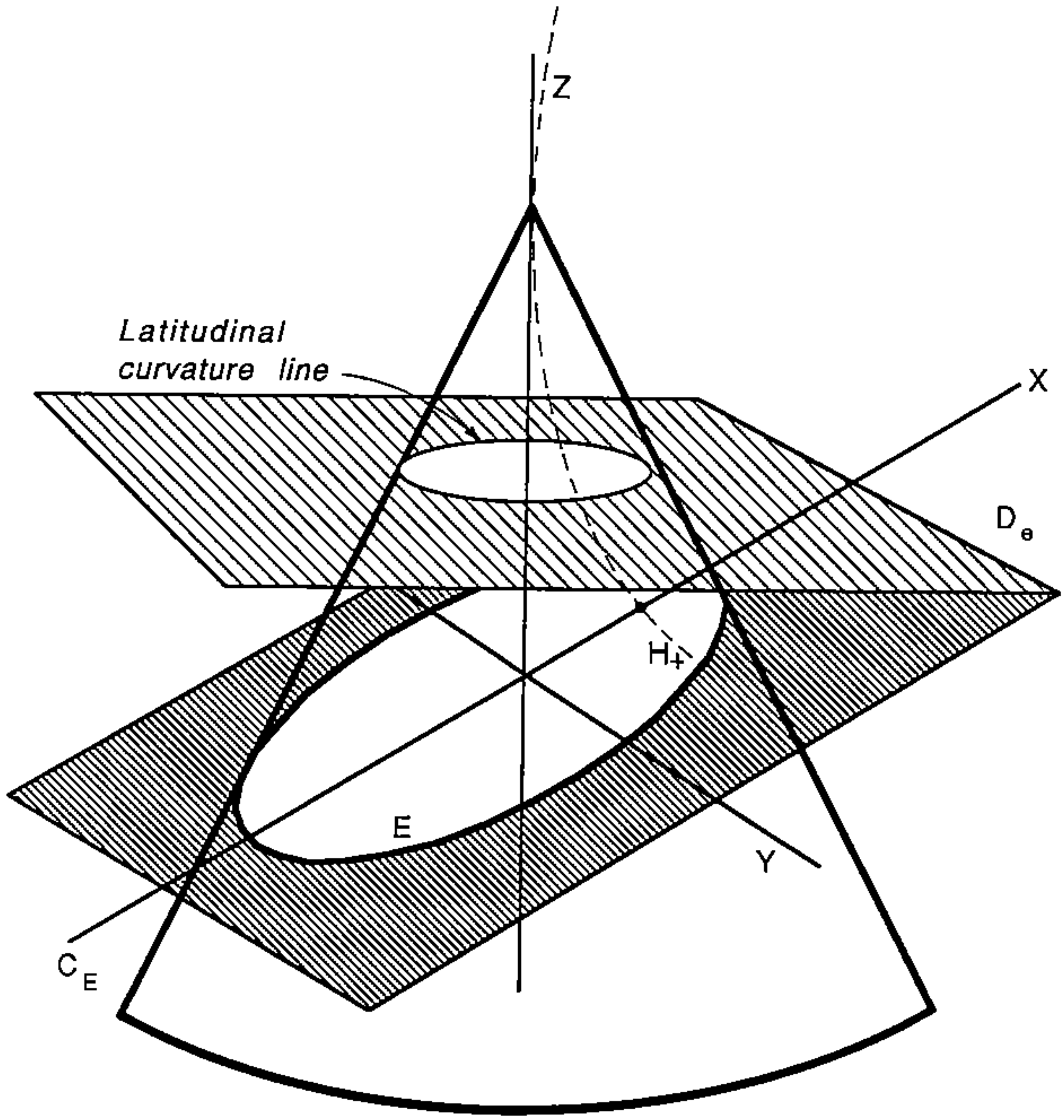


Fig. 6

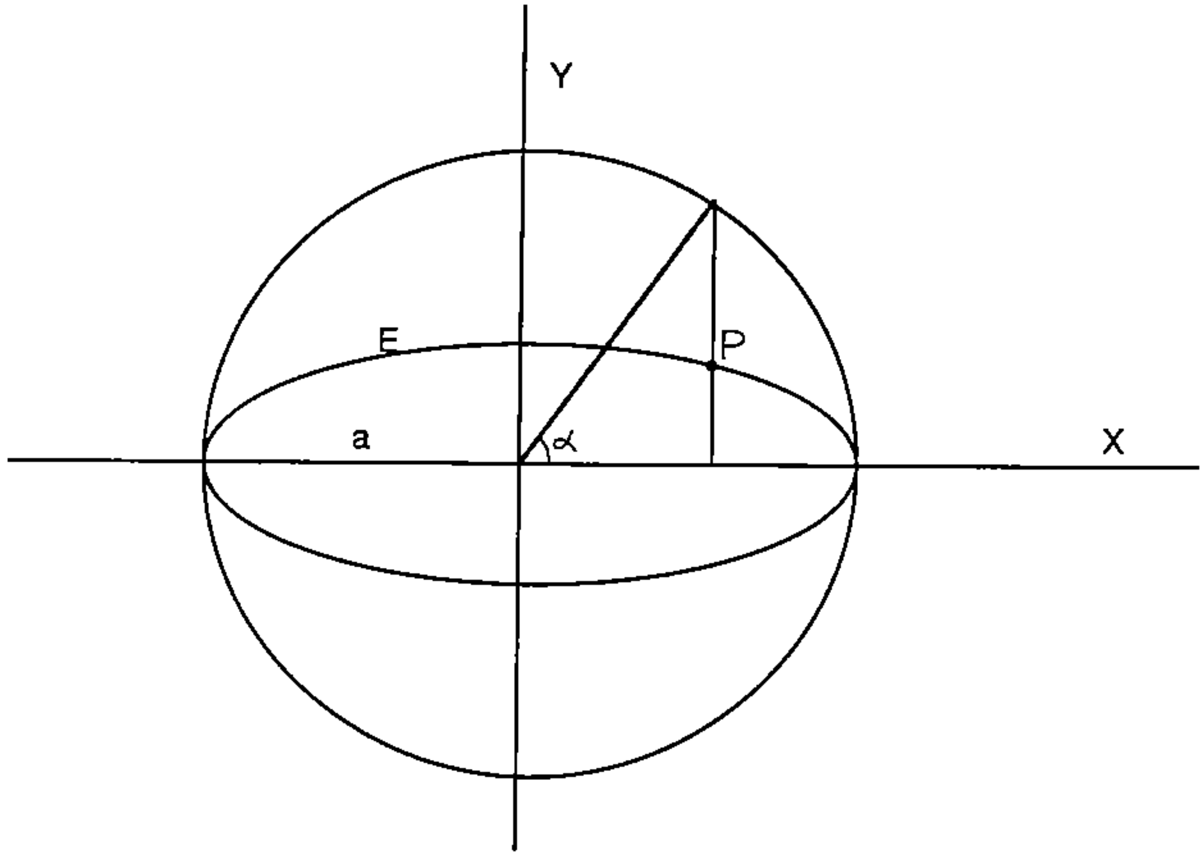


Fig. 7a

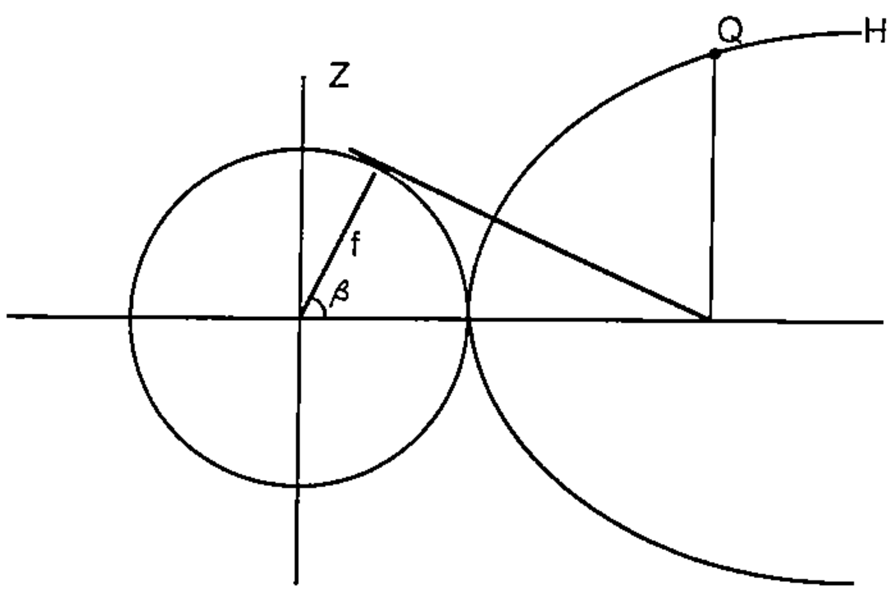
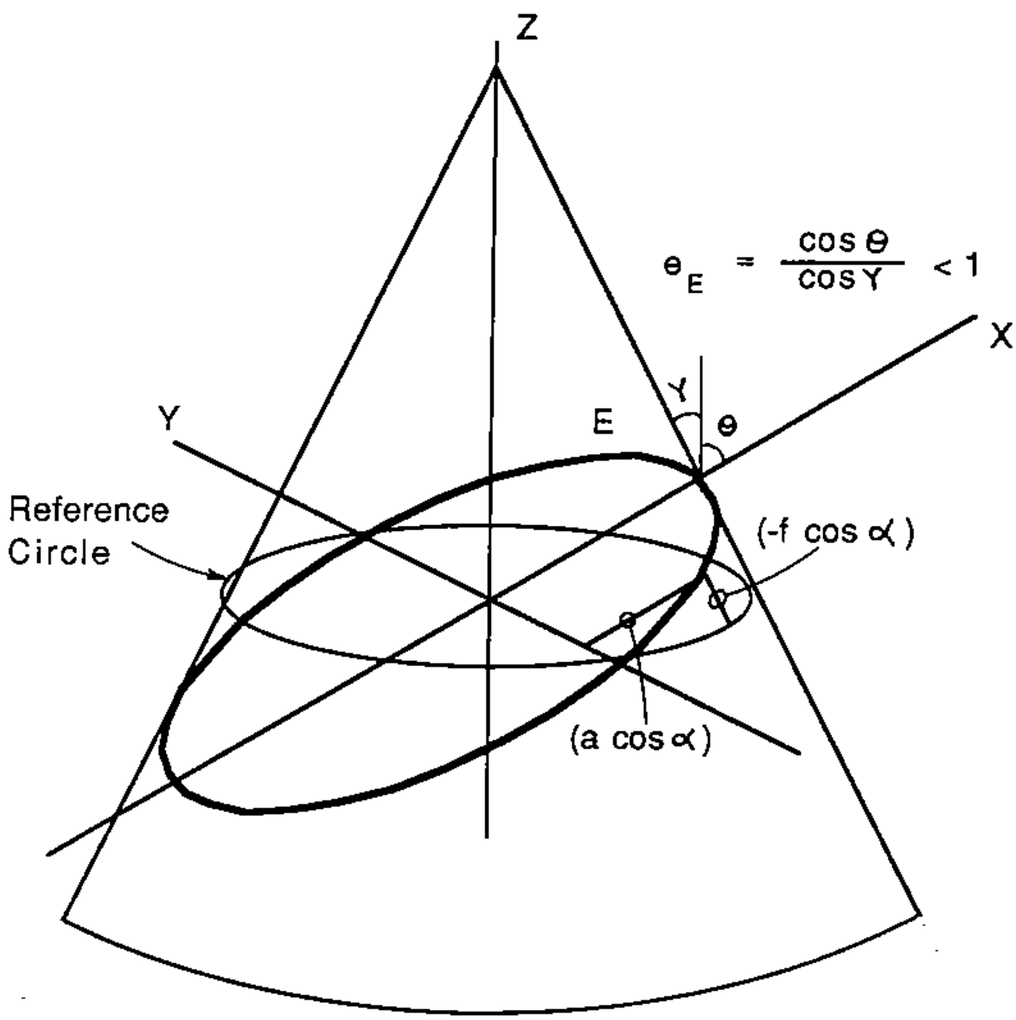


Fig. 7b



Z
$e_E = \frac{\cos\Theta}{\cos\Upsilon} < 1$
X
Y
E
Y
Θ
Reference Circle
(-f cos α)
(a cos α)

Fig. 8a

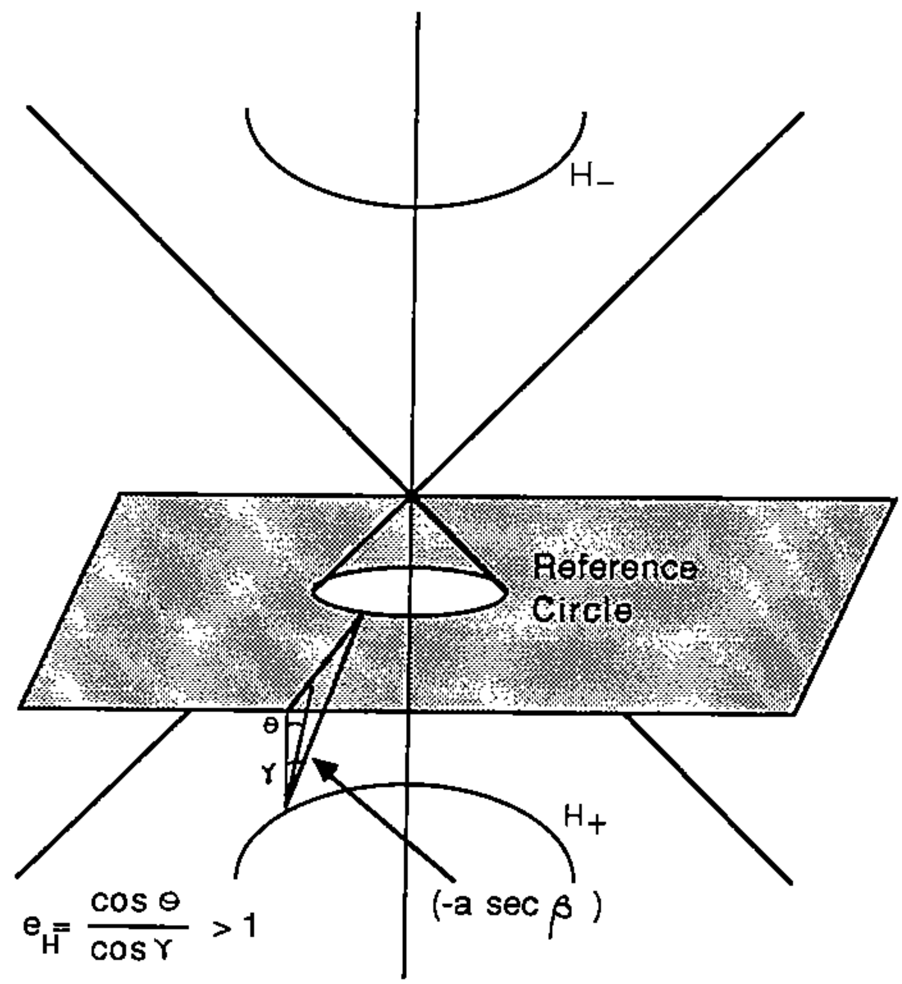


Fig. 8b

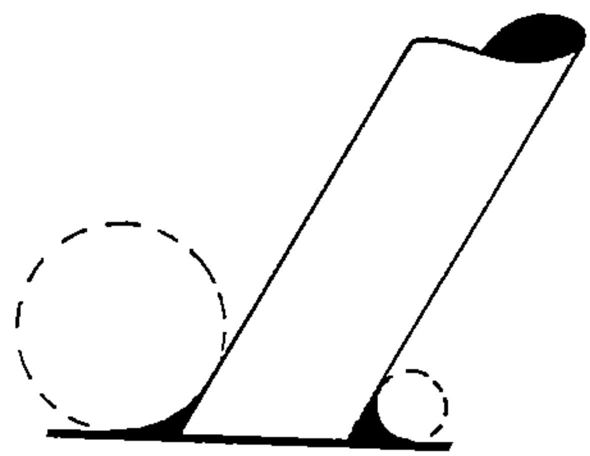


Fig. 9

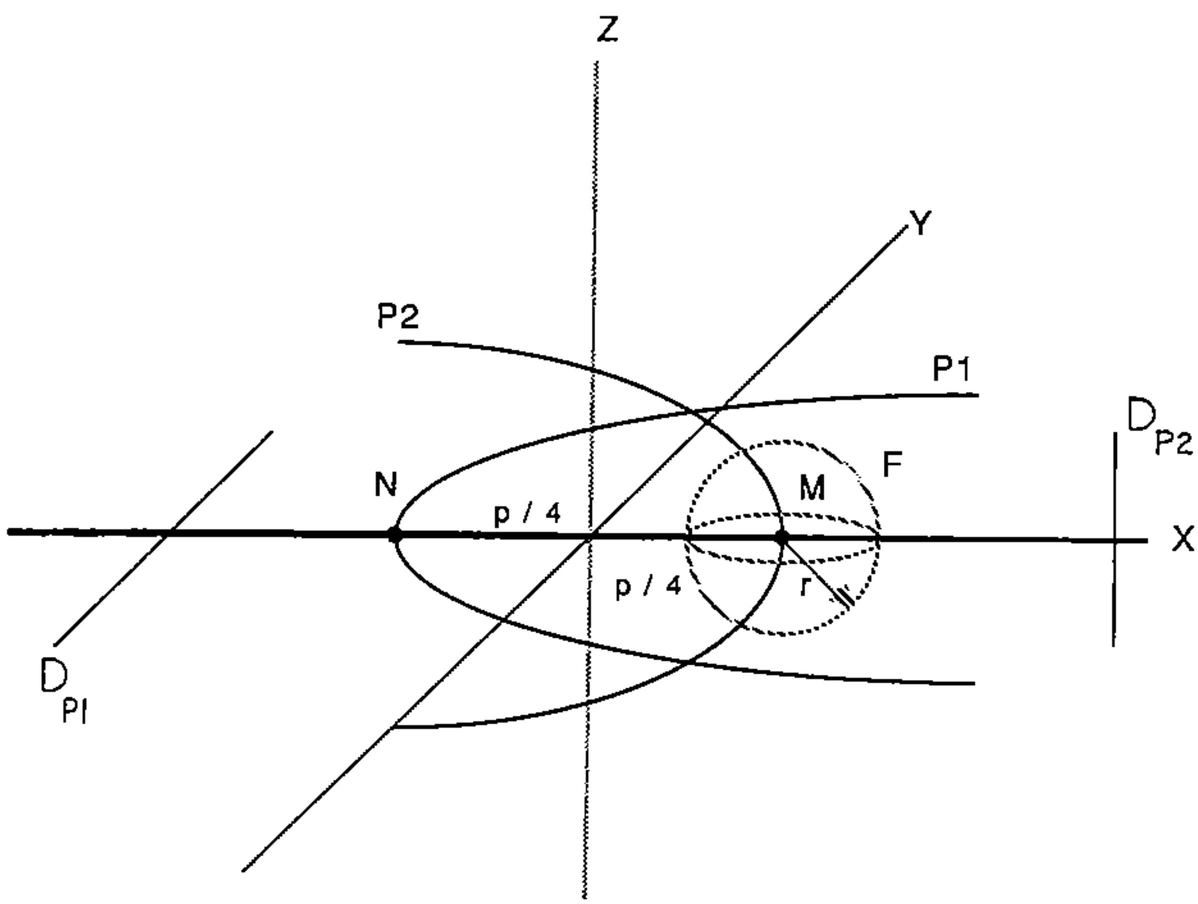


Fig. 10

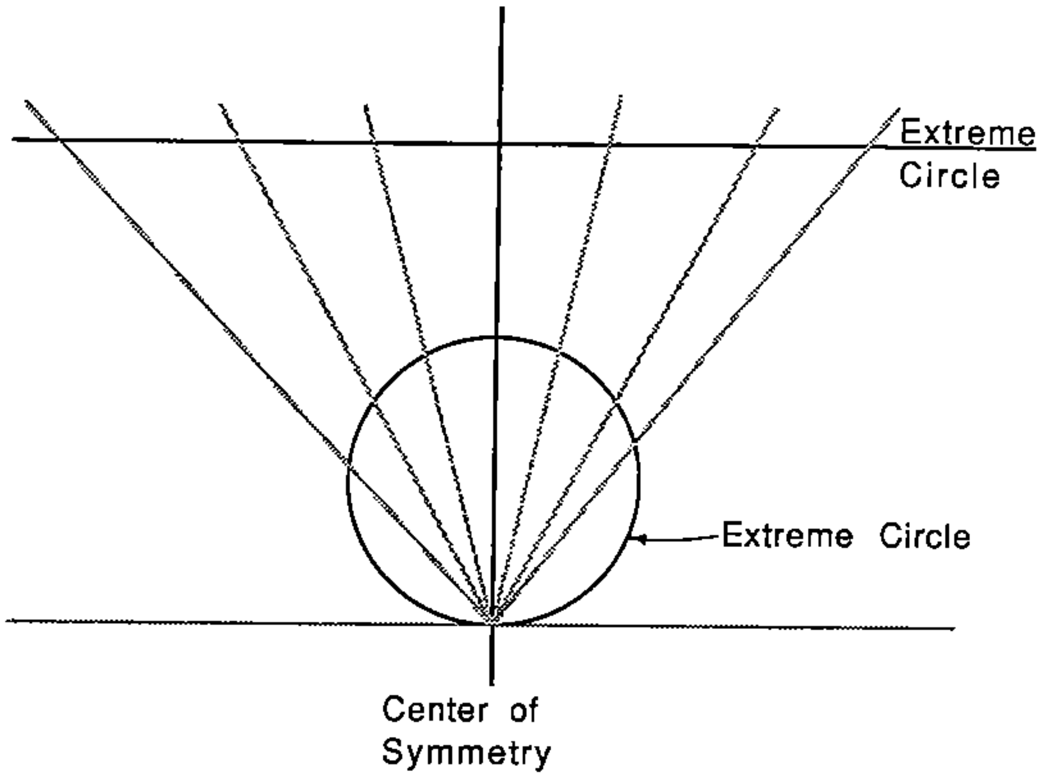


Fig. 11a

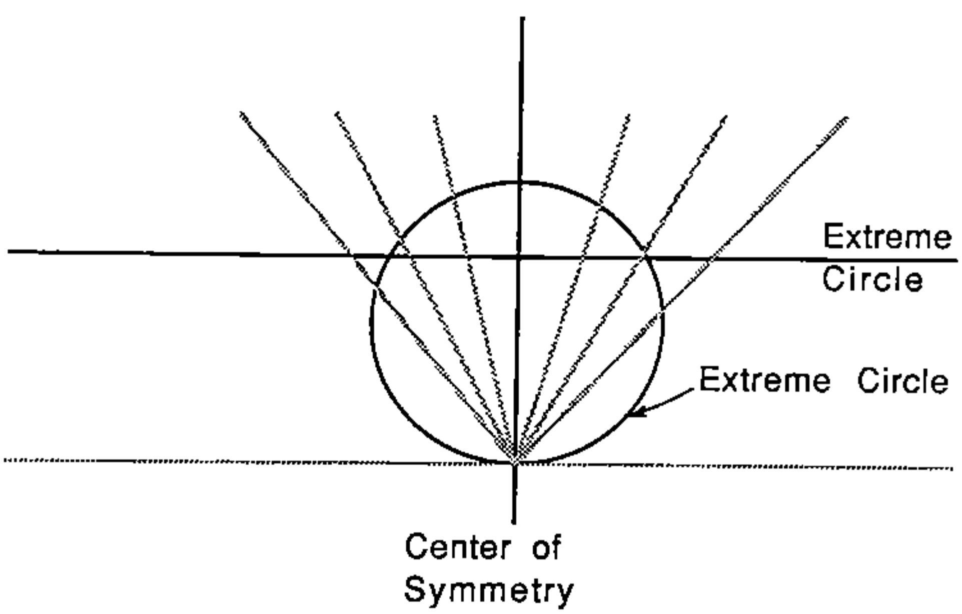


Fig 11b

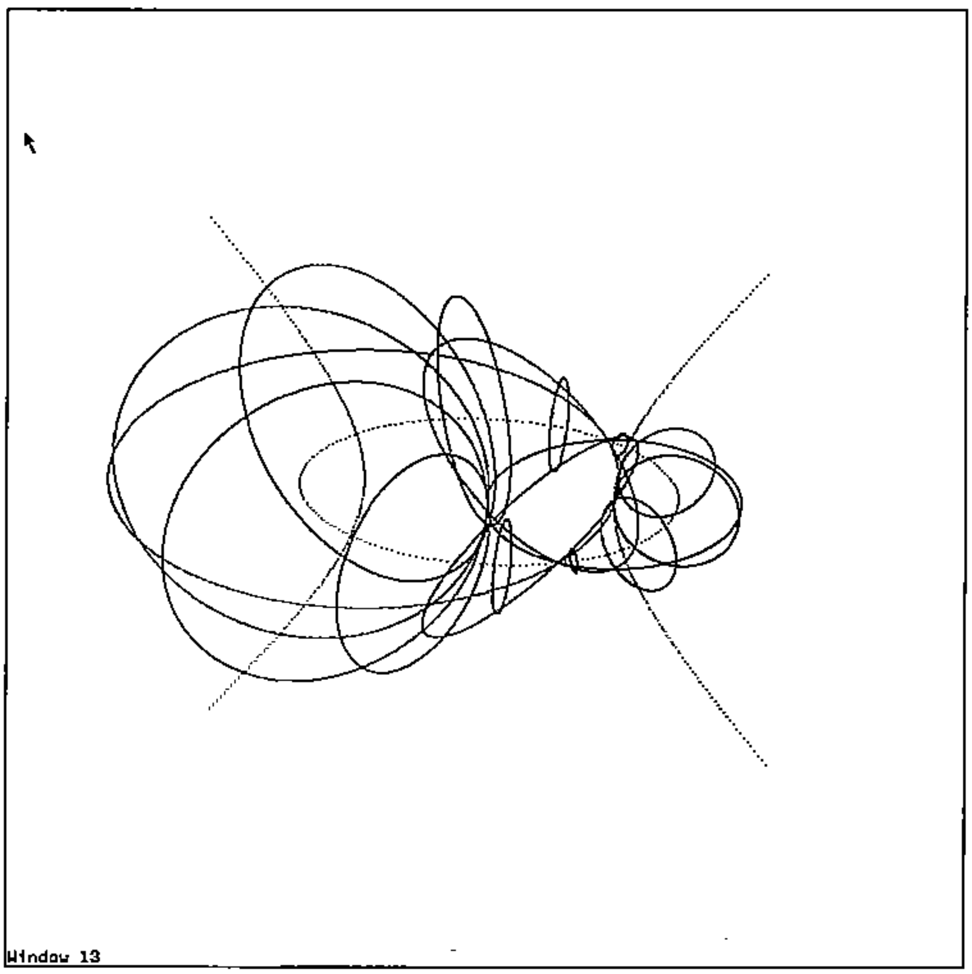


Plate Ia
Central Cyclide (Horn)
$a = 3,\ f = 2,\ r = 1$

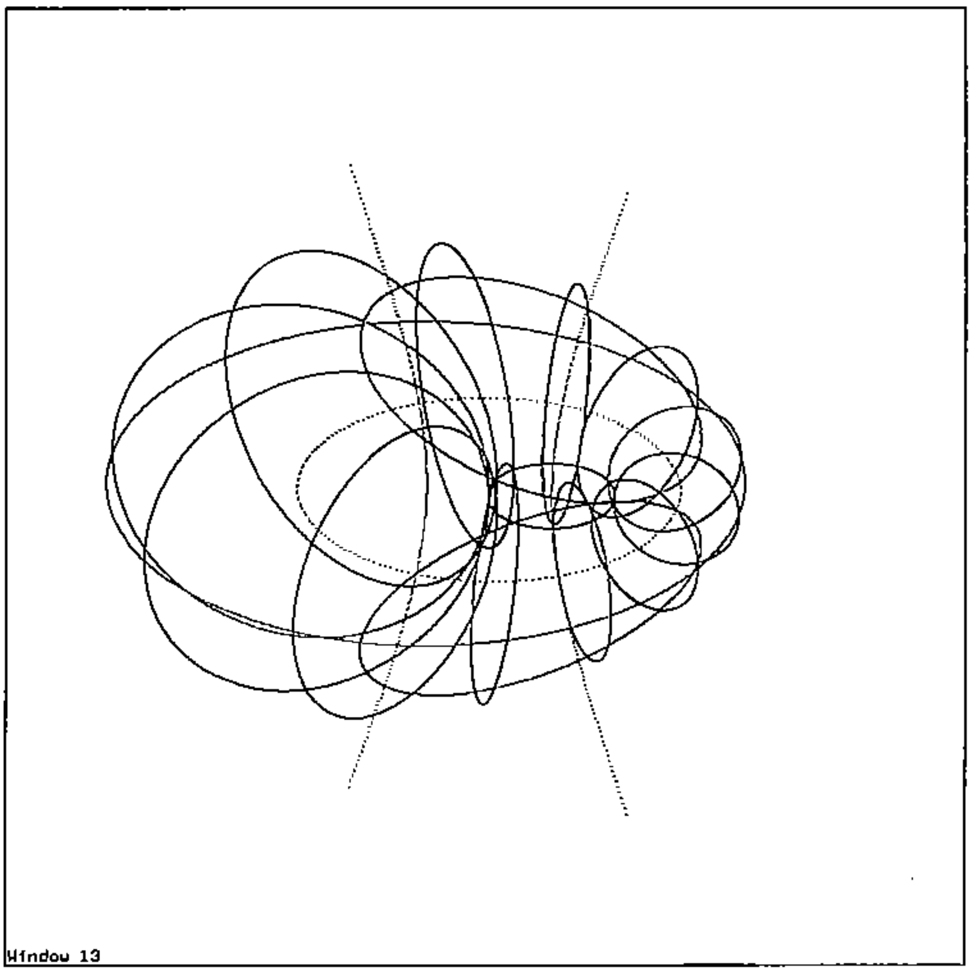


Plate Ib
Central Cyclide (Ring)
$a = 3,\ f = 1,\ r = 2$

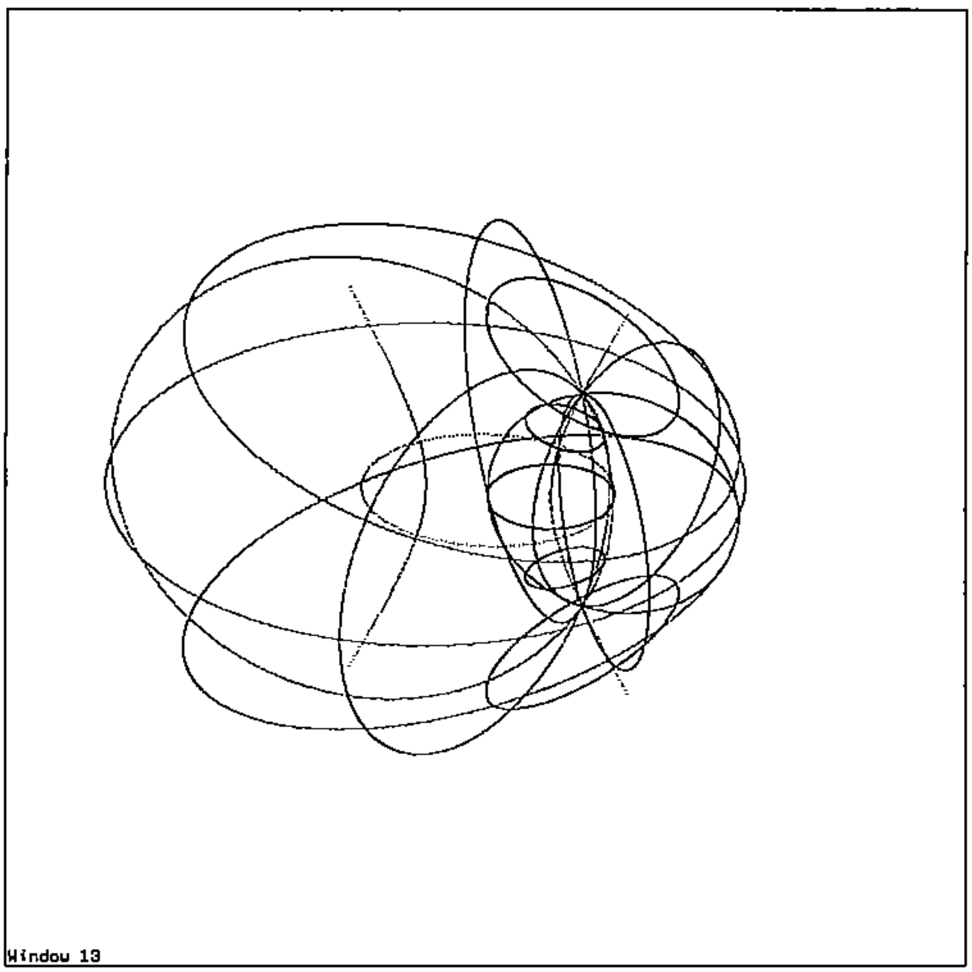


Plate Ic
Central Cyclide (Spindle)
$a = 2$, $f = 1$, $r = 3$

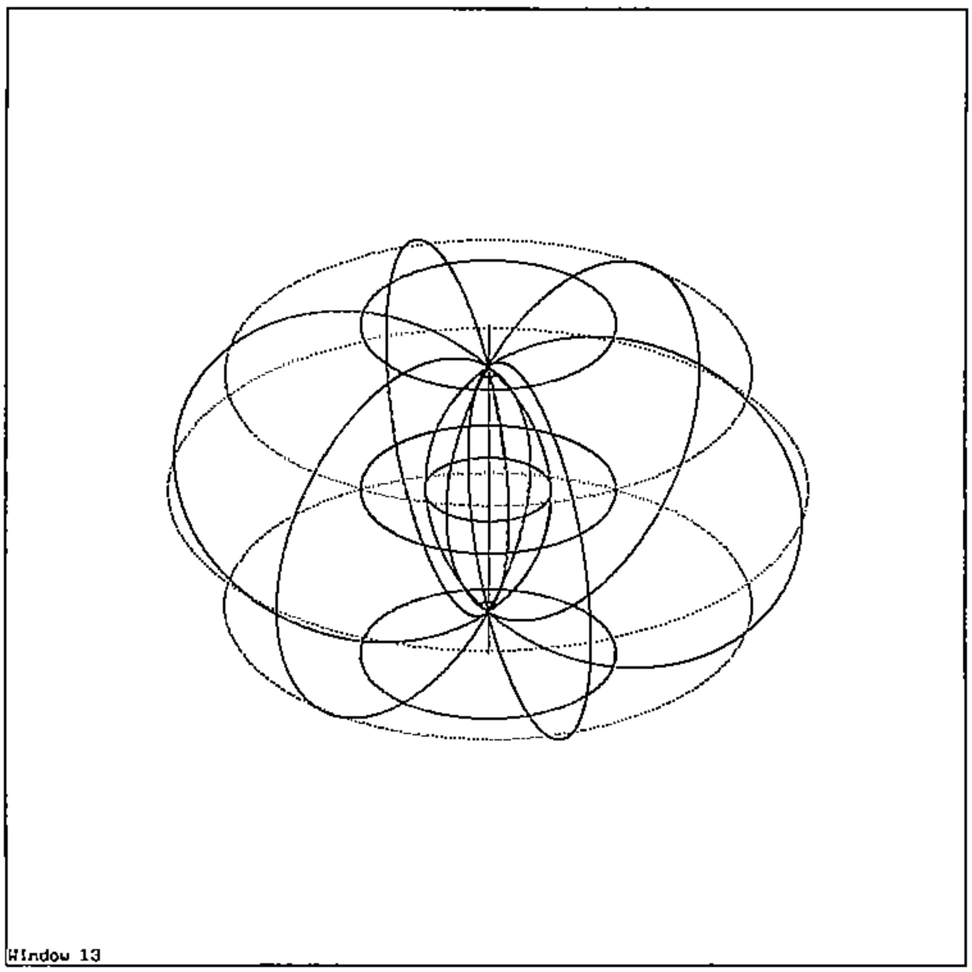


Plate IIb
Cyclide of Revolution (Spindle)
$a = 2,\ f = 0,\ r = 3$

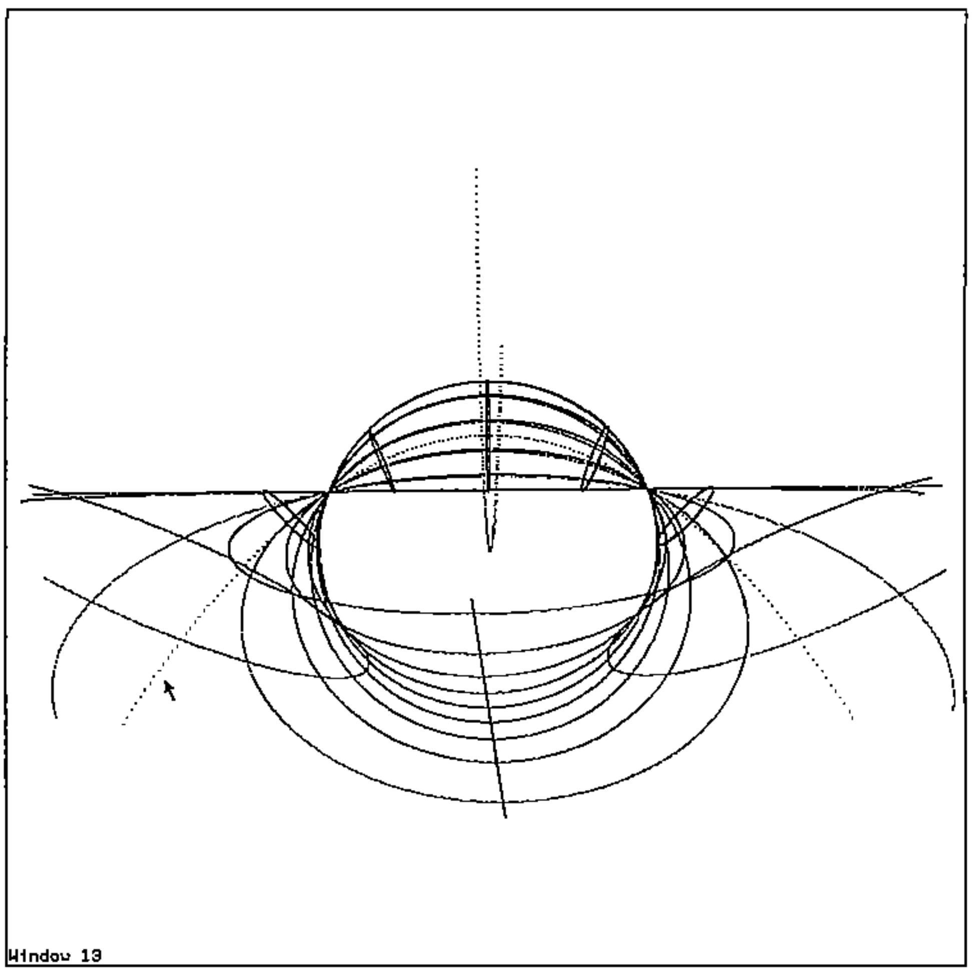


Plate IIIa
Parabolic Cyclide (Horn/Spindle)
$p = 4$, $r = 3$

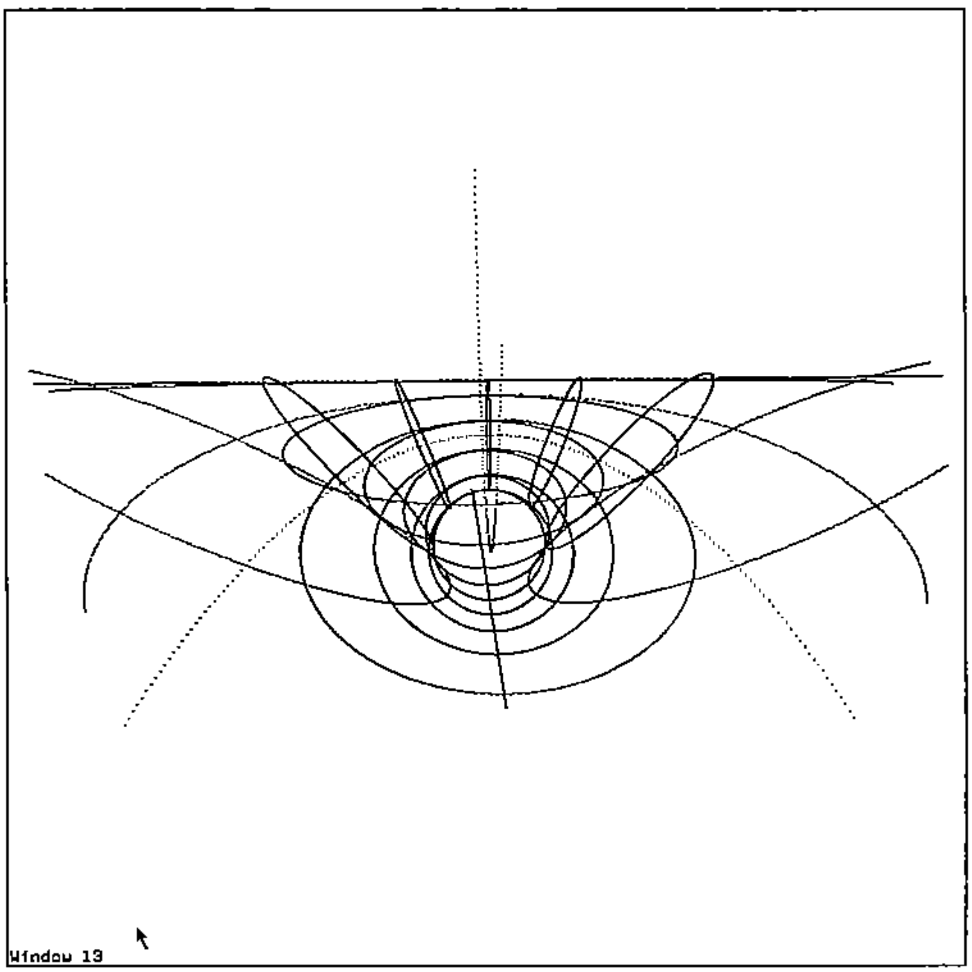


Plate IIIb
Parabolic Cyclide (Ring)
$p = 4, r = 1$