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Probabilistic representations of solutions to the heat equation

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Abstract. In this paper we provide a new (probabilistic) proof of a classical result in partial differential equations, viz. if ϕ is a tempered distribution, then the solution of the heat equation for the Laplacian, with initial condition ϕ, is given by the convolution of ϕ with the heat kernel (Gaussian density). Our results also extend the probabilistic representation of solutions of the heat equation to initial conditions that are arbitrary tempered distributions.

Keywords. Brownian motion; heat equation; translation operators; infinite dimensional stochastic differential equations.

1. Introduction

Let $(X_t)_{t\geq 0}$ be a d-dimensional Brownian motion, with $X_0 \equiv 0$. Let $\varphi \in \mathcal{S}'(\mathbb{R}^d)$, the space of tempered distributions. Let φ_t represent the unique solution to the heat equation with initial value φ, viz.

$$\partial_t \varphi_t = \frac{1}{2}\Delta\varphi_t \quad 0 \leq t \leq T; \qquad \varphi_0 = \varphi.$$

It is well-known that $\varphi_t = \varphi * p_t$, where $p_t(x) = \frac{1}{(2\pi t)^{d/2}}\mathrm{e}^{-(|x|^2/2t)}$ and '$*$' denotes convolution. When φ is smooth, say $\varphi \in \mathcal{S}$, the space of rapidly decreasing smooth functions, then the probabilisitc representation of the solution is given by the equality $\varphi(t,x) = E\varphi(X_t + x)$ and is obtained by taking expectations in the Ito formula

$$\varphi(X_t + x) = \varphi(x) + \int_0^t \nabla\varphi(X_s + x)\cdot \mathrm{d}X_s + \frac{1}{2}\int_0^t \Delta\varphi(X_s + x)\mathrm{d}s.$$

Such representations are well-known (see [1–4]) and extend to a large class of initial value problems, with the Laplacian Δ replaced by a suitable (elliptic) differential operator L and (X_t) being replaced by the diffusion generated by L. A basic problem here is to extend the representation to situations where φ is not smooth.

The main contribution of this paper is to give a probabilistic representation of solutions to the initial value problem for the Laplacian with an arbitrary initial value $\varphi \in \mathcal{S}'$. This representation follows from the Ito formula developed in [9], for the $\mathcal{S}'$-valued process $(\tau_{X_t}\varphi)$, where $\tau_x\varphi$ is the translation of φ by $x \in \mathbb{R}^d$. Our representation (Theorem 2.4) then reads, $\varphi_t = E\tau_{X_t}\varphi$ where of course φ_t is the solution of the initial value problem for the Laplacian, with initial value $\varphi \in \mathcal{S}'$. In particular, the fundamental solution $p_t(x - \cdot)$

has the representation, $p_t(x-\cdot) = E\tau_{X_t}\delta_x$. However, the results of [9] only show that if $\varphi \in \mathcal{S}'_p$, then there exists $q > p$ such that the process $(\tau_{X_t}\varphi)$ takes values in $\mathcal{S}'_q$. Here for each real p, the $\mathcal{S}_p$s are the 'Sobolev spaces' associated with the spectral decomposition of the operator $|x|^2 - \Delta$ or equivalently they are the Hilbert spaces defining the countable Hilbertian structure of $\mathcal{S}'$ (see [6]). $\mathcal{S}'_p$, the dual of $\mathcal{S}_p$, is the same as $\mathcal{S}_{-p}$. Clearly it would be desirable to have the process $(\tau_{X_t}\varphi)$ take values in $\mathcal{S}'_p$, whenever $\varphi \in \mathcal{S}'_p$. Such a result also has implications for the semi-martingale structure of the process (τ_{X_t}) – it is a semi-martingale in $\mathcal{S}'_{p+1}$ (Corollary 2.2) and fails to have this property in $\mathcal{S}'_q$ for $q < p+1$ (see Remark 5.2 of [5]).

Given the above remarks and the results of [9], the properties of the translation operators become significant. We show in Theorem 2.1 that the operators $\tau_x : \mathcal{S}_p \to \mathcal{S}_p$ for $x \in \mathbb{R}^d$, are indeed bounded operators, for any real p, with the operator norms being bounded above by a polynomial in $|x|$. The proof uses interpolation techniques well-known to analysts. Theorem 2.4 then gives a comprehensive treatment of the initial value problem for the Laplacian from a probabilistic point of view.

2. Statements of the main results

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ be a filtered probability space with a filtration $(\mathcal{F}_t)$ satisfying usual conditions: $\mathcal{F}_t = \bigcap_{s>t} \mathcal{F}_s$ and $\mathcal{F}_0$ contains all P-null sets. Let $(X_t)_{t\geq 0}$ be a d-dimensional, $(\mathcal{F}_t)$-Brownian motion with $X_0 \equiv 0$.

$\mathcal{S}$ denotes the space of rapidly decreasing smooth functions on $\mathbb{R}^d$ (real valued) and $\mathcal{S}'$ its dual, the space of tempered distributions. We refer to [11] for formal definitions. For $x \in \mathbb{R}^d$, $\delta_x \in \mathcal{S}'$ will denote the Dirac distribution at x. Let $\{\tau_x : x \in \mathbb{R}^d\}$ denote the translation operators defined on functions by the formula $\tau_x f(y) = f(y-x)$ and let $\tau_x : \mathcal{S}' \to \mathcal{S}'$ act on distributions by

$$\langle \tau_x\varphi, f\rangle = \langle \varphi, \tau_{-x} f\rangle.$$

The nuclear space structure of $\mathcal{S}'$ is given by the family of Hilbert spaces $\mathcal{S}_p$, $p \in \mathbb{R}$, obtained as the completion of $\mathcal{S}$ under the Hilbertian norms $\{\|\cdot\|_p\}_{p\in\mathbb{R}}$ defined by

$$\|\varphi\|_p^2 = \sum_k (2|k|+d)^{2p}\langle \varphi, h_k\rangle^2,$$

where $\varphi \in \mathcal{S}$, and the sum is taken over $k = (k_1, \dots, k_d) \in \mathbb{Z}_+^d$, $|k| = (k_1 + \cdots + k_d)$, $\langle \varphi, h_k\rangle$ denotes the inner product in $L^2(\mathbb{R}^d)$ and $\{h_k, k \in \mathbb{Z}_+^d\}$ is the ONB in $L^2(\mathbb{R}^d)$, constructed as follows: for $x = (x_1, \dots, x_d)$, $h_k(x) = h_{k_1}(x_1)\dots h_{k_d}(x_d)$. The one-dimensional Hermite functions are given by $h_\ell(s) = \frac{1}{(\sqrt{\pi}2^\ell \ell!)^{1/2}} \mathrm{e}^{-(s^2/2)} H_\ell(s)$, where $H_\ell(s) = (-1)^\ell \mathrm{e}^{s^2} \frac{\mathrm{d}^\ell}{\mathrm{d}s^\ell} \mathrm{e}^{-s^2}$ are the Hermite polynomials. While we mainly deal with real valued functions, at times we need to use complex valued functions. In such cases, the spaces $\mathcal{S}_p$ are defined in a similar fashion as above, i.e. as the completion of $\mathcal{S}$ with respect to $\|\cdot\|_p$. However, in the definition of $\|\varphi\|_p^2$ above we need to replace the real L^2 inner product $\langle \varphi, h_k\rangle$ by the one for complex valued functions, viz. $\langle \varphi, \psi\rangle = \int_{\mathbb{R}^d} \varphi(x)\bar{\psi}(x)\mathrm{d}x$ and $\langle \varphi, h_k\rangle^2$ is replaced by $|\langle \varphi, h_k\rangle|^2$. It is well-known (see [6, 7]) that $\mathcal{S} = \bigcap_p \mathcal{S}_p$, $\mathcal{S}' = \bigcup_p \mathcal{S}_p$ and $\mathcal{S}'_p =$: dual of $\mathcal{S}_p = \mathcal{S}_{-p}$. We will denote by $\langle\cdot,\cdot\rangle_p$, the inner product corresponding to the norm $\|\cdot\|_p$.

Let $(Y_t)_{t\geq 0}$ be an $\mathcal{S}_p$-valued, locally bounded, previsible process, for some $p \in \mathbb{R}$. Let $\partial_i : \mathcal{S}_p \to \mathcal{S}_{p-1/2}$ be the partial derivatives, $1 \leq i \leq d$, in the sense of distributions. Then since ∂_i, $1 \leq i \leq d$ are bounded linear operators it follows that $(\partial_i Y_t)_{t\geq 0}$ is an $\mathcal{S}_{p-1/2}$-valued, locally bounded, previsible process. From the theory of stochastic integration in Hilbert spaces [8], it follows that the processes

$$\left(\int_0^t Y_s \mathrm{d}X_s^i\right)_{t\geq 0}, \left(\int_0^t \partial_i Y_s \mathrm{d}X_s^i\right)_{t\geq 0}$$

are continuous $\mathcal{F}_t$ local martingales for $1 \leq i \leq d$, with values in $\mathcal{S}_p$ and $\mathcal{S}_{p-1/2}$ respectively. If $X_t = (X_t^1, \ldots, X_t^d)$ is a continuous $\mathbb{R}^d$-valued, $\mathcal{F}_t$-semi-martingale, it follows from the general theory that the above processes too are continuous $\mathcal{F}_t$-semi-martingales with values in $\mathcal{S}_p$ and $\mathcal{S}_{p-1/2}$ respectively.

Theorem 2.1. *Let $p \in \mathbb{R}$. There exists a polynomial $P_k(\cdot)$ of degree $k = 2([|p|]+1)$ such that the following holds*: *For $x \in \mathbb{R}^d$, $\tau_x : \mathcal{S}_p \to \mathcal{S}_p$ is a bounded linear map and we have*

$$\|\tau_x\varphi\|_p \leq P_k(|x|)\|\varphi\|_p$$

for all $\varphi \in \mathcal{S}_p$.

In ([9], Theorem 2.3) we showed that if $(X_t)_{t\geq 0}$ is a continuous, d-dimensional, $\mathcal{F}_t$-semi-martingale and $\varphi \in \mathcal{S}_p \subset \mathcal{S}'$, then the process $(\tau_{X_t}\varphi)_{t\geq 0}$ is an $\mathcal{S}_q$-valued continuous semi-martingale for some $q < p$. Corollary 2.2 below says that we can take $q = p - 1$.

COROLLARY 2.2

Let $(X_t)_{t\geq 0}$ be a continuous d-dimensional, $\mathcal{F}_t$-semi-martingale. Let $\varphi \in \mathcal{S}_p$, $p \in \mathbb{R}$. Then $(\tau_{X_t}\varphi)_{t\geq 0}$ is an $\mathcal{S}_p$-valued, continuous adapted process. Moreover it is an $\mathcal{S}_{p-1}$-valued, continuous $\mathcal{F}_t$-semi-martingale and the following Ito formula holds in $\mathcal{S}_{p-1}$: a.s., $\forall\, t \geq 0$,

$$\begin{aligned}\tau_{X_t}\varphi = \tau_{X_0}\varphi &- \sum_{i=1}^d \int_0^t \partial_i(\tau_{X_s}\varphi)\mathrm{d}X_s^i \\ &+ \frac{1}{2}\sum_{i,j=1}^d \int_0^t \partial_{ij}^2(\tau_{X_s}\varphi)\mathrm{d}\langle X^i, X^j\rangle_s,\end{aligned} \tag{2.1}$$

where $X_t = (X_t^1, \ldots, X_t^d)$ and $(\langle X^i, X^j\rangle_t)$ is the quadratic variation process between (X_t^i) and (X_t^j), $1 \leq i, j \leq d$.

Proof. From Theorem 2.1, it follows that $(\tau_{X_t}\varphi)$ is an $\mathcal{S}_p$-valued continuous adapted process. By Theorem 2.3 of [9], $\exists\, q < p$, such that $(\tau_{X_t}\varphi)$ is an $\mathcal{S}_q$ semi-martingale and the above equation holds in $\mathcal{S}_q$. Clearly each of the terms in the above equation is in $\mathcal{S}_{p-1}$ and the result follows. $\square$

The next corollary pertains to the case when $(X_t) = (X_t^1, \ldots, X_t^d)$ is a d-dimensional Brownian motion, $X_0 \equiv 0$. In ([5], Definition 3.1), we introduced the notion of an $\mathcal{S}_p'(= \mathcal{S}_{-p}, p > 0)$-valued strong solution of the SDE

$$\begin{aligned}\mathrm{d}Y_t &= \frac{1}{2}\Delta(Y_t)\mathrm{d}t + \nabla Y_t \cdot \mathrm{d}X_t, \\ Y_0 &= \varphi,\end{aligned} \tag{2.2}$$

where $\nabla = (\partial_1, \ldots, \partial_d)$ and $\Delta = \sum_{i=1}^d \partial_i^2$. There we showed that if $\varphi \in \mathcal{S}'_p$, then the above equation has a unique $\mathcal{S}'_q$-valued strong solution, $q \geq p+2$. Theorem 2.1 implies that we indeed have an (unique) $\mathcal{S}'_p$-valued strong solution.

COROLLARY 2.3

Let $\varphi \in \mathcal{S}'_p$. Then, eq. (2.2) *has a unique $\mathcal{S}'_p$-valued strong solution on* $0 \leq t \leq T$.

Proof. By Corollary 2.2, the process $(\tau_{X_t}\varphi)$, where (X_t) is a d-dimensional Brownian motion, $X_0 \equiv 0$, satisfies eq. (2.1). Further,

$$E \int_0^T \|\tau_{X_t}\varphi\|_{-p}^2 \mathrm{d}t = \int_0^T \int_{\mathbb{R}^d} \|\tau_x \varphi\|_{-p}^2 \frac{\mathrm{e}^{-(|x|^2/2t)}}{(2\pi t)^{d/2}} \mathrm{d}x \; \mathrm{d}t < \infty.$$

Uniqueness follows as in Theorem 3.3 of [5]. □

We now consider the heat equation for the Laplacian with initial condition $\varphi \in \mathcal{S}_p$, for some $p \in \mathbb{R}$.

$$\begin{aligned} &\partial_t \varphi_t = \frac{1}{2}\Delta\varphi_t \quad 0 < t \leq T, \\ &\varphi_0 = \varphi. \end{aligned} \tag{2.3}$$

By an $\mathcal{S}_p$-valued solution of (2.3), we mean a continuous map $t \to \varphi_t : [0, T] \to S_p$ such that the following equation holds in $\mathcal{S}_{p-1}$:

$$\varphi_t = \varphi + \int_0^t \frac{1}{2}\Delta\varphi_s \mathrm{d}s. \tag{2.4}$$

Let $\{h_k^{p-1}\}$ be the ONB in $\mathcal{S}_{p-1}$ given by $h_k^{p-1} = (2|k| + d)^{-(p-1)} h_k$. We then have for $p < 0$ and $t \leq T$:

$$\begin{aligned} \|\varphi_t\|_{p-1}^2 &= \sum_{|k|=0}^{\infty} \langle \varphi_t, h_k^{p-1}\rangle_{p-1}^2 \\ &= \sum_{|k|=0}^{\infty} \left\{ \langle \varphi, h_k^{p-1}\rangle_{p-1}^2 + 2\int_0^t \langle \varphi_s, h_k^{p-1}\rangle_{p-1} \mathrm{d}\langle \varphi_s, h_k^{p-1}\rangle_{p-1} \right\} \\ &= \|\varphi\|_{p-1}^2 + \sum_{|k|=0}^{\infty} 2\int_0^t \langle \varphi_s, h_k^{p-1}\rangle_{p-1} \left\langle \frac{1}{2}\Delta\varphi_s, h_k^{p-1} \right\rangle_{p-1} \mathrm{d}s \\ &= \|\varphi\|_{p-1}^2 + 2\int_0^t \left\langle \frac{1}{2}\Delta\varphi_s, \varphi_s \right\rangle_{p-1} \mathrm{d}s. \end{aligned}$$

It follows from the results of [5] (the monotonicity condition) that for $p < 0$,

$$2\left\langle \frac{1}{2}\Delta\varphi, \varphi \right\rangle_{p-1} + \sum_{i=1}^d \|\partial_i \varphi\|_{p-1}^2 \leq C \; \|\varphi\|_{p-1}^2$$

for some constant $C > 0$ for all $\varphi \in \mathcal{S}_p$. We then get

$$\|\varphi_t\|_{p-1}^2 \leq \|\varphi\|_{p-1}^2 + C\int_0^t \|\varphi_s\|_{p-1}^2 \mathrm{d}s.$$

Hence for the case $p < 0$, uniqueness follows from the Gronwall lemma. Uniqueness for the case $p \geq 0$, follows from uniqueness for the case $p < 0$ and the inclusion $\mathcal{S}_p \subset \mathcal{S}_q$ for $q < p$. It is well-known that the solutions of the initial value problem (2.3) in $\mathcal{S}'(\mathbb{R}^d)$ are given by convolution of φ and $p_t(x)$, the heat kernel. That these coincide (as they should) with the $\mathcal{S}_p$-valued solutions follows from the 'probabilistic representation' given by Theorem 2.4 below. Define the Brownian semi-group $(T_t)_{t\geq 0}$ on $\mathcal{S}$ in the usual manner:

$$T_t\varphi(x) = \varphi * p_t(x) \;\; t > 0, \qquad T_0\varphi = \varphi$$

where $p_t(x) = \frac{1}{(2\pi t)^{d/2}} \mathrm{e}^{(-|x|^2/2t)}, t > 0$ and '$*$' denotes convolution: $f * g(x) = \int_{\mathbb{R}^d} f(y)g(x-y)\mathrm{d}y$. In the next theorem we consider standard Brownian motion (X_t).

Theorem 2.4. (a) *Let $\varphi \in \mathcal{S}_p$. Then for $t \geq 0$, the $\mathcal{S}_p$-valued random variable $\tau_{X_t}\varphi$ is Bochner integrable and we have*

$$E\,\tau_{X_t}\varphi = \varphi * p_t = T_t\varphi.$$

In particular, for every $p \in \mathbb{R}$, and $T > 0$, $\sup_{t\leq T} \|T_t\| < \infty$ where $\|T_t\|$ is the operator norm of $T_t : \mathcal{S}_p \to \mathcal{S}_p$.
(b) *For $\varphi \in \mathcal{S}_p$, the initial value problem* (2.3) *has a unique $\mathcal{S}_p$-valued solution φ_t given by*

$$\varphi_t = E\tau_{X_t}\varphi.$$

Further $\varphi_t \to \varphi$ strongly in $\mathcal{S}_p$ as $t \to 0$.

3. Proofs of Theorems 2.1 and 2.4

The spaces $\mathcal{S}_p$ can be described in terms of the spectral properties of the operator H defined as follows:

$$Hf = (|x|^2 - \Delta)f, \quad f \in \mathcal{S}.$$

If $\{h_k\}$ is the ONB in $L^2(\mathbb{R}^d)$ consisting of Hermite functions (defined in §2), then it is well-known (see [10]) that

$$Hh_k = (2|k| + d)h_k.$$

For $f \in \mathcal{S}$, define the operator H^p as follows:

$$H^p f = \sum_k (2|k| + d)^p \langle f, h_k\rangle h_k.$$

Here p is any real number. For $f \in \mathcal{S}$ and $z = x + iy \in \mathbb{C}$ define $H^z f = \sum_k (2|k| + d)^z \langle f, h_k\rangle h_k$ and note that, $H^z f = H^x(H^{iy} f) = H^{iy}(H^x f)$ and $H^{iy} : L^2 \to L^2$ is an isometry. Further,

$$\begin{aligned}\|H^z f\|_0^2 &= \sum_k (2|k| + d)^{2x} \langle f, h_k\rangle^2 \\ &= \|f\|_x^2.\end{aligned}$$

The following propositions (3.1, 3.2 and 3.3) may be well-known. We include the proofs for completeness.

PROPOSITION 3.1

For any p and q, $\|H^p\varphi\|_{q-p} = \|\varphi\|_q$ for $\varphi \in \mathcal{S}$. Consequently, $H^p : \mathcal{S}_q \to \mathcal{S}_{q-p}$ extends as a linear isometry. Moreover, this isometry is onto.

Proof. Let $h_k^p = (2|k| + d)^{-p} h_k$. Then from the relation $\langle\varphi, h_k\rangle_p = (2|k| + d)^{2p}\langle\varphi, h_k\rangle$ it follows that $\{h_k^p\}$ is an ONB for $\mathcal{S}_p$. Let $\varphi \in \mathcal{S}$. Since

$$H^p\varphi = \sum_k \langle\varphi, h_k\rangle(2|k| + d)^p h_k$$
$$= \sum_k \langle\varphi, h_k\rangle(2|k| + d)^q h_k^{q-p},$$

we get $\|H^p\varphi\|_{q-p}^2 = \|\varphi\|_q^2$.

To show that H^p is onto, consider $\psi \in \mathcal{S}_{q-p}$,

$$\psi = \sum_k \langle\psi, h_k^{q-p}\rangle_{q-p} h_k^{q-p}.$$

Defining $\varphi =: \sum_k \langle\psi, h_k^{q-p}\rangle_{q-p} h_k^q$, we see that $\varphi \in \mathcal{S}_q$. Also,

$$H^p\varphi = \sum_k \langle\varphi, h_k^q\rangle_q h_k^{q-p} = \sum_k \langle\psi, h_k^{q-p}\rangle_{q-p} h_k^{q-p} = \psi. \qquad \square$$

Let $A_j = x_j + \partial_j$ and $A_j^+ = x_j - \partial_j$, $1 \le j \le d$. Then it is easy to see that

$$H = \frac{1}{2}\sum_{j=1}^d (A_j A_j^+ + A_j^+ A_j).$$

For multi-indices $\alpha = (\alpha_1, \ldots, \alpha_d)$ and $\beta = (\beta_1, \ldots, \beta_d)$ we define

$$A^\alpha =: A_1^{\alpha_1} \ldots A_d^{\alpha_d}, \qquad (A^+)^\beta =: (A_1^+)^{\beta_1} \ldots (A_d^+)^{\beta_d}.$$

For an integer $\ell \ge 0$ and $x \in \mathbb{R}$, recall that

$$h_\ell(x) = \frac{1}{(\sqrt{\pi}2^\ell \ell!)^{1/2}} \mathrm{e}^{-(x^2/2)} H_\ell(x),$$

where H_ℓ is the Hermite polynomial defined by

$$H_\ell(x) = (-1)^\ell \mathrm{e}^{x^2} \frac{\mathrm{d}^\ell}{\mathrm{d}x^\ell} \mathrm{e}^{-x^2}.$$

It is easily verified that

$$\left(x + \frac{\mathrm{d}}{\mathrm{dx}}\right)\left(\mathrm{e}^{-(x^2/2)} H_\ell(x)\right) = 2\ell\left(\mathrm{e}^{-(x^2/2)} H_{\ell-1}(x)\right),$$
$$\left(x - \frac{\mathrm{d}}{\mathrm{dx}}\right)\left(\mathrm{e}^{-(x^2/2)} H_\ell(x)\right) = \mathrm{e}^{-(x^2/2)} H_{\ell+1}(x).$$

It then follows that

$$A_j^+ h_{k_j}(x_j) = \sqrt{2(k_j + 1)}\, h_{k_j+1}(x_j),$$
$$A_j h_{k_j}(x_j) = \sqrt{2k_j}\, h_{k_j-1}(x_j).$$

Iterating these two formulas we get the following:

PROPOSITION 3.2

Let k, β and α be multi-indices such that $k_j \geq \alpha_j$, $j = 1, \dots, d$. Then

$$(A^+)^\beta h_k(x) = 2^{|\beta|/2} \left(\frac{(k+\beta)!}{k!} \right)^{1/2} h_{k+\beta}(x),$$
$$A^\alpha h_k(x) = 2^{|\alpha|/2} \left(\frac{k!}{(k-\alpha)!} \right)^{1/2} h_{k-\alpha}(x),$$

where $k! = k_1! \dots k_d!$.

PROPOSITION 3.3

For all $m \geq 0$, $\exists$ constants $C_1 = C_1(m)$ and $C_2 = C_2(m)$ such that the following hold:
(a) *For all $f \in \mathcal{S}$,*

$$\|f\|_m \leq C_1 \sum_{|\alpha|+|\beta| \leq 2m} \|A^\alpha (A^+)^\beta f\|_0 \leq C_2 \|f\|_m.$$

(b) *For all $f \in \mathcal{S}$,*

$$\|f\|_m \leq C_1 \sum_{|\alpha|+|\beta| \leq 2m} \|x^\alpha \partial^\beta f\|_0 \leq C_2 \|f\|_m.$$

Proof. (a) We can write

$$H^m = \sum_{|\alpha|+|\beta| \leq 2m} C_{\alpha\beta} A^\alpha (A^+)^\beta,$$

where $C_{\alpha\beta}$ are constants. Since $\|f\|_m = \|H^m f\|_0$, the first part of the inequality follows. To show the second half of the inequality it is sufficient to show that for $f \in \mathcal{S}$ and $|\alpha| + |\beta| \leq 2m$, $\|A^\alpha (A^+)^\beta H^{-m} f\|_0 \leq C_{\alpha\beta} \|f\|_0$. Now,

$$\begin{aligned}
\|A^\alpha (A^+)^\beta H^{-m} f\|_0^2 &= \sum_\ell \langle A^\alpha (A^+)^\beta H^{-m} f, h_\ell \rangle^2 \\
&= \sum_\ell \left[\sum_k (2|k| + d)^{-m} \langle f, h_k \rangle \langle A^\alpha (A^+)^\beta h_k, h_\ell \rangle \right]^2 \\
&= \sum_\ell \left[\sum_k (2|k| + d)^{-m} \langle f, h_k \rangle C_{k,\beta,\alpha} \langle h_{k+\beta-\alpha}, h_\ell \rangle \right]^2 \\
&= \sum_\ell (2|\ell + \alpha - \beta| + d)^{-2m} C^2_{\ell+\alpha-\beta,\beta,\alpha} \langle f, h_{\ell+\alpha-\beta} \rangle^2,
\end{aligned}$$

where the sum is taken over $\ell = (\ell_1, \dots, \ell_d)$ such that $\ell_j + \alpha_j - \beta_j \geq 0$ for $1 \leq j \leq d$ and where we have used Proposition 3.2 in the last but one equality above. From the same proposition, it follows that

$$(2|\alpha + \ell - \beta| + d)^{-2m} C^2_{\ell+\alpha-\beta,\beta,\alpha}$$

are uniformly bounded in ℓ for $|\alpha| + |\beta| \leq 2m$ and the second inequality in (a) follows.

(b) Since $\|f\|_m = \|H^m f\|_0$ and clearly $H^m = \sum_{|\alpha|+|\beta|\le 2m} C_{\alpha\beta} x^\alpha \partial^\beta$, the first inequality follows. To prove the second inequality, note that

$$x_j = \frac{1}{2}(A_j + A_j^+), \qquad \partial_j = \frac{1}{2}(A_j - A_j^+).$$

Hence, using $[A_j, A_k^+] = \delta_{jk} I$,

$$x^\alpha \partial^\beta = \sum_{|k|+|\ell|\le|\alpha|+|\beta|} C_{k,\ell} A^k (A^+)^\ell$$

and hence by part (a) we get

$$\sum_{|\alpha|+|\beta|\le 2m} \|x^\alpha \partial^\beta f\|_0 \le C_1 \sum_{|k|+|\ell|\le 2m} \|A^k (A^+)^\ell f\|_0 \le C_2 \|H^m f\|_0.$$

□

Proof of Theorem 2.1. We first show that for an integer $m \ge 0$,

$$\|\tau_x \varphi\|_m \le P_{2m}(|x|)\|\varphi\|_m,$$

where $P_{2m}(t)$ is a polynomial in $t \in \mathbb{R}$ of degree $2m$ with non-negative coefficients. This follows from Proposition 3.3:

$$\begin{aligned} \|\tau_x f\|_m &\le C_1 \sum_{|\alpha|+|\beta|\le 2m} \|y^\alpha \partial^\beta \tau_x f\|_0 \\ &\le C_1 \sum_{|\alpha|+|\beta|\le 2m} \|(y+x)^\alpha \partial^\beta f\|_0. \end{aligned}$$

The last sum is clearly dominated by $P_{2m}(|x|)\|f\|_m$ for some polynomial P_{2m}. If $m < p < m+1$, where $m \ge 0$ is an integer, we prove the result using the 3-line lemma: for $f, g \in \mathcal{S}$, let

$$F(z) = \langle H^{\bar{z}} \tau_x H^{-z} f, g\rangle_0.$$

Then from the expansion in L^2 for the RHS it is verified that $F(z)$ is analytic in $m < \mathrm{Re}\, z < m+1$ and continuous in $m \le \mathrm{Re}\, z \le m+1$. We will show that

$$\begin{aligned} &|F(m+iy)| \le P_{2m}(|x|)\|f\|_0\|g\|_0, \\ &|F(m+1+iy)| \le P_{2(m+1)}(|x|)\|f\|_0\|g\|_0 \end{aligned} \tag{3.1}$$

for $-\infty < y < \infty$. Hence from the 3-line lemma [12], it follows that

$$\begin{aligned} |F(p+iy)| &\le (P_{2m}(|x|)\|f\|_0\|g\|_0)^{m+1-p}(P_{2(m+1)}(|x|)\|f\|_0\|g\|_0)^{p-m} \\ &\le P_k(|x|)\|f\|_0\|g\|_0, \end{aligned}$$

where $P_k(t)$ is a polynomial in t of degree $k = 2([p]+1)$. It follows that

$$\|\tau_x f\|_p \le P_k(|x|)\|f\|_p.$$

Using the fact that $\mathcal{S}_{-p} = \mathcal{S}'_p$ we get $\|\tau_x f\|_{-p} \leq P_k(|x|)\|f\|_{-p}$ for $m \leq p \leq m+1$.

The following chain of inequalities establish the inequalities (3.1):

$$\begin{aligned}
|F(m+iy)| &\leq \|H^{m-iy}\tau_x H^{-(m+iy)} f\|_0 \|g\|_0 \\
&\leq \|H^m \tau_x H^{-(m+iy)} f\|_0 \|g\|_0 \\
&= \|\tau_x H^{-(m+iy)} f\|_m \|g\|_0 \\
&\leq P_{2m}(|x|)\|H^{-(m+iy)} f\|_m \|g\|_0 \\
&= P_{2m}(|x|)\|H^{-iy} f\|_0 \|g\|_0 \\
&= P_{2m}(|x|)\|f\|_0 \|g\|_0.
\end{aligned}$$

This completes the proof of Theorem 2.1. □

Proof of Theorem 2.4. (a) Let $\varphi \in \mathcal{S}_p$, $p \in \mathbb{R}$. From Theorem 2.1 we have

$$\|\tau_{X_t}\varphi\|_p \leq P_k(|X_t|)\|\varphi\|_p,$$

where P_k is a polynomial. Since $EP_k(|X_t|) < \infty$, Bochner integrability follows. For $\psi \in \mathcal{S}, \varphi \in \mathcal{S}$,

$$\begin{aligned}
\left\langle \psi, \int \tau_x\varphi \; p_t(x)\mathrm{d}x \right\rangle &= \int \langle \psi, \tau_x\varphi\rangle p_t(x)\mathrm{d}x \\
&= \int p_t(x)\mathrm{d}x \int \psi(y)\varphi(y-x)\mathrm{d}y \\
&= \int \psi(y)\mathrm{d}y \int \varphi(y-x)p_t(x)\mathrm{d}x \\
&= \int \psi(y)\varphi * p_t(y)\mathrm{d}y \\
&= \langle \psi, \varphi * p_t\rangle.
\end{aligned}$$

The result for $\varphi \in \mathcal{S}_p$ follows by a continuity argument: Let $\varphi_n \in \mathcal{S}, \varphi_n \to \varphi$ in $\mathcal{S}_p$. Hence $\varphi_n * p_t \to \varphi * p_t$ weakly in $\mathcal{S}'$. Hence,

$$\begin{aligned}
\langle \psi, \varphi * p_t\rangle &= \lim_{n\to\infty} \langle \psi, \varphi_n * p_t\rangle \\
&= \lim_{n\to\infty} \int \psi(y)\varphi_n * p_t(y)\mathrm{d}y \\
&= \lim_{n\to\infty} \int \langle \psi, \tau_x\varphi_n\rangle p_t(x)\mathrm{d}x \\
&= \int \langle \psi, \tau_x\varphi\rangle p_t(x)\mathrm{d}x \\
&= \left\langle \psi, \int \tau_x\varphi \; p_t(x)\mathrm{d}x \right\rangle,
\end{aligned}$$

where we have used DCT in the last but one equality. That $T_t : \mathcal{S}_p \to \mathcal{S}_p$ is a (uniformly) bounded operator follows:

$$\begin{aligned}\|T_t\varphi\|_p &= \|\varphi * p_t\|_p = \|E\tau_{X_t}\varphi\|_p \\ &= \left\| \int \tau_x\varphi\ p_t(x)\mathrm{d}x \right\|_p \leq \int \|\tau_x\varphi\|_p p_t(x)\mathrm{d}x \\ &\leq \|\varphi\|_p \int P_k(|x|)p_t(x)\mathrm{d}x \leq C\ \|\varphi\|_p,\end{aligned}$$

where $C = \sup_{s\leq T} \int P_k(|x|)p_s(x)\mathrm{d}x < \infty$.

(b) Let (X_t) be the standard Brownian motion so that $\langle X^i, X^j\rangle \equiv 0$ for $i \neq j$. Equation (2.1) then reads, for $\varphi \in \mathcal{S}_p$, $p \in \mathbb{R}$,

$$\tau_{X_t}\varphi = \varphi - \int_0^t \nabla(\tau_{X_s}\varphi)\cdot \mathrm{d}X_s + \frac{1}{2}\int_0^t \Delta(\tau_{X_s}\varphi)\mathrm{d}s. \tag{3.2}$$

The stochastic integral is a martingale in $\mathcal{S}_{p-1}$:

$$\begin{aligned}E\ \left\| \int_0^t \partial_i(\tau_{X_s}\varphi)\mathrm{d}X_s^i \right\|_{p-1}^2 &\leq C_1\ E\int_0^t \|\partial_i(\tau_{X_s}\varphi)\|_{p-1}^2 \mathrm{d}s \\ &= C_1 \int_0^t \left(\int \|\partial_i(\tau_x\varphi)\|_{p-1}^2 p_s(x)\mathrm{d}x \right)\mathrm{d}s \\ &\leq C_2 \int_0^t \left(\int \|\tau_x\varphi\|_p^2\ p_s(x)\mathrm{d}x \right)\mathrm{d}s \\ &\leq C_3\|\varphi\|_p \int_0^t \left(\int P_k(|x|)p_s(x)\mathrm{d}x \right)\mathrm{d}s \\ &< \infty.\end{aligned}$$

Let $\varphi_t = E\tau_{X_t}\varphi$. Taking expected values in (3.2) we get eq. (2.4). Hence φ_t is the solution to the heat equation with initial value $\varphi \in \mathcal{S}_p$. The uniqueness of the solution is well-known and also follows from the remarks preceeding the statement of Theorem 2.4.

To complete the proof of the theorem, we need to show that $\varphi_t \to \varphi$ in $\mathcal{S}_p$ as $t \downarrow 0$. Let $\mathcal{F}$ denote the Fourier transform, i.e. $\mathcal{F}f(\xi) = \int \mathrm{e}^{-i(x\cdot\xi)} f(x)\mathrm{d}x$ for $f \in \mathcal{S}$. Then $\mathcal{F}$ extends to $\mathcal{S}'$ by duality, where we consider $\mathcal{S}'$ as a complex vector space. Since $\mathcal{F}(h_n) = (-\sqrt{-1})^n h_n$ ([10], p. 5, Lemma 1.1.3), $\mathcal{F}$ acts as a bounded operator from $\mathcal{S}_p$ to $\mathcal{S}_p$, for all p. Let $\varphi \in \mathcal{S}_p$.

$$\varphi_t - \varphi = T_t\varphi - \varphi = \mathcal{F}^{-1}(S_t(\mathcal{F}\varphi)),$$

where

$$S_t\varphi(x) = \mathcal{F}(T_t - I)\mathcal{F}^{-1}\varphi(x) = (\mathrm{e}^{-(t/2)|x|^2} - 1)\varphi(x).$$

Clearly, $S_t : \mathcal{S}_p \to \mathcal{S}_p$ is a bounded operator and

$$\|\varphi_t - \varphi\|_p = \|S_t(\mathcal{F}\varphi)\|_p.$$

The following proposition completes the proof of the theorem.

PROPOSITION 3.4

Let $\varphi \in \mathcal{S}_p$, $p \in \mathbb{R}$. Then $\|S_t\varphi\|_p \to 0$ as $t \to 0$.

Proof. We prove the proposition by showing that (i) $S_t : \mathcal{S}_p \to \mathcal{S}_p$ are uniformly bounded, $0 < t \leq T$ and (ii) $\|S_t\varphi\|_p \to 0$ for every $\varphi \in \mathcal{S}$, as $t \to 0$. Let us assume these results for a moment and complete the proof.

Let $\epsilon > 0$ be given. By (i), there is a constant $C > 0$ such that

$$\sup_{0\leq t\leq T} \|S_t f\|_p \leq C \, \|f\|_p, \, f \in \mathcal{S}_p.$$

Choose $\varphi \in \mathcal{S}$, so that $\|f - \varphi\|_p \leq (\frac{\epsilon}{2C})$. Then,

$$\begin{aligned}\|S_t f\|_p &\leq \|S_t(f-\varphi)\|_p + \|S_t\varphi\|_p \\ &\leq \epsilon/2 + \|S_t\varphi\|_p.\end{aligned}$$

Now choose $\delta > 0$ such that $\|S_t\varphi\|_p \leq \epsilon/2$ for all $0 \leq t < \delta$, to get $\|S_t f\|_p < \epsilon$ for all $0 \leq t < \delta$.

Since $S_t = \mathcal{F}(T_t - I)\mathcal{F}^{-1}$, (i) follows from the fact that $T_t : \mathcal{S}_p \to \mathcal{S}_p$ are uniformly bounded (Theorem 2.4a) and $\mathcal{F} : \mathcal{S}_p \to \mathcal{S}_p$ is a unitary operator. The proof of (ii) is by a direct calculation when $p = m$ is a non-negative integer.

$$\|S_t\varphi\|_m = \|H^m S_t\varphi\|_0 \leq C_1 \sum_{|\alpha|+|\beta|\leq 2m} \|x^\alpha \partial^\beta S_t\varphi\|_0.$$

Since $S_t\varphi(x) = (\mathrm{e}^{-(t/2)|x|^2} - 1)\varphi(x)$, by Leibniz rule

$$\|x^\alpha \partial^\beta S_t\varphi\|_0 \leq \sum_{|\mu|+|\gamma|=|\beta|} C_{\mu\gamma} \|x^\alpha \partial^\mu (\mathrm{e}^{-(t/2)|x|^2} - 1)\partial^\gamma \varphi\|_0.$$

When $\mu \neq 0$, we have

$$\|x^\alpha \partial^\mu (\mathrm{e}^{-(t/2)|x|^2} - 1)\partial^\gamma \varphi\|_0 \leq C_2 t^{|\mu|} \|\varphi\|_m$$

and when $\mu = 0$, using the elementary inequality $|1 - \mathrm{e}^{-u}| \leq C_3 u$, $u > 0$ we get

$$\|x^\alpha (\mathrm{e}^{-(t/2)|x|^2} - 1)\partial^\gamma \varphi\|_0 \leq C_4 t \|\varphi\|_{m+1}.$$

Therefore, $\|S_t\varphi\|_m \leq C\, t\|\varphi\|_{m+1}$ for some constant C, which shows that $\|S_t\varphi\|_m \to 0$ as $t \to 0$. If p is real and m is a non-negative integer such that $p \leq m$, we have

$$\|S_t\varphi\|_p \leq \|S_t\varphi\|_m \leq Ct\|\varphi\|_{m+1}$$

and so $\|S_t\varphi\|_p \to 0$ as $t \to 0$ in this case as well. □

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