Probabilistic Representations of Solutions of the Forward Equations

By

B. Rajeev,
Statistics and Mathematics Unit,
Indian Statistical Institute,
Bangalore - 560 059.
brajeev@isibang.ac.in

S. Thangavelu,
Department of Mathematics,
Indian Institute of Science,
Bangalore - 560 012.
veluma@math.iisc.ernet.in

Abstract: In this paper we prove a stochastic representation for solutions of the evolution equation

$$\partial_t \psi_t \;=\; \frac{1}{2} L^* \psi_t$$

where L^* is the formal adjoint of a second order elliptic differential operator L, with smooth coefficients, corresponding to the infinitesimal generator of a finite dimensional diffusion (X_t). Given $\psi_0 = \psi$, a distribution with compact support, this representation has the form $\psi_t = E(Y_t(\psi))$ where the process $(Y_t(\psi))$ is the solution of a stochastic partial differential equation connected with the stochastic differential equation for (X_t) via Ito's formula.

Key words : Stochastic differential equation, Stochastic partial differential equation, evolution equation, stochastic flows, Ito's formula, stochastic representation, adjoints, diffusion processes, second order elliptic partial differential equation, monotonicity inequality .

1 Introduction

The first motivation for the results of this paper is that they extend the results in [17] for Brownian motion, to more general diffusions. To recall, there we had recast the classical relationship between Brownian motion and the heat equation in the language of distribution theory. To be more precise, the solutions of the initial value problem,

$$\begin{aligned} \partial_t \psi_t &= \frac{1}{2}\Delta \psi_t, \\ \psi_0 &= \psi \end{aligned}$$

for any distribution $\psi \in \mathcal{S}'(I\!R^d)$ were represented in terms of a standard d-dimensional Brownian motion $(X_t)_{t\geq 0}$, as $\psi_t = E\tau_{X_t}(\psi)$. Here $\tau_x : I\!R^d \to I\!R^d$ is the translation operator and the expectation is taken in a Hilbert space $S_p \subset \mathcal{S}'$ in which the process takes values.

The main result of this paper is that the above result extends to the solutions of the initial value problem

$$\begin{aligned} \partial_t \psi_t &= \frac{1}{2} L^* \psi_t, \\ \psi_0 &= \psi \end{aligned} \tag{1.1}$$

where $\psi \in \mathcal{E}'$, i.e it is a distribution on $I\!R^d$ with compact support and L^* is the formal adjoint of L, a second order elliptic differential operator with smooth coefficients given as the first component of the pair (L, A), where

$$\begin{aligned} L &= \frac{1}{2}\sum_{i,j} (\sigma\sigma^t)_{ij}(x)\ \partial^2_{ij} + \sum_i b_i(x)\ \partial_i, \\ A &= (A_1, \cdots, A_d), \\ A_k &= -\sum_{i=1}^d \sigma_{ij}(x)\partial_i. \end{aligned}$$

To describe the stochastic representation of solutions of (1.1), let $(X(t,x))$ denote the solutions of the stochastic differential equation

$$\begin{aligned} dX_t &= \sigma(X_t)\cdot dB_t + b(X_t)\ dt, \\ X_0 &= x. \end{aligned}$$

Then it is well known that a.s, $x \to X(t,x)$ is smooth and induces a map $X_t : C^\infty \to C^\infty$, namely $X_t(\phi)(x) = \phi(X(t,x))$ for $\phi \in C^\infty$. Here and in the rest of the paper C^∞ denotes the space of all smooth functions on $I\!R^d$. Let $Y_t := X_t^* : \mathcal{E}' \to \mathcal{E}'$ be the adjoint of the map $X_t : C^\infty \to C^\infty$. If $\psi \in \mathcal{E}' \subset \mathcal{S}'$

where $\mathcal{S}'$ is the space of tempered distributions, then we can show that the process $(Y_t(\psi))$ takes values in one of the Hilbert spaces $S_{-p}, p > 0$ that define the countable Hilbertian structure of $\mathcal{S}'$ (Proposition 3.1). Our stochastic representation now reads, $\psi_t = E(Y_t(\psi))$(Theorem 4.3). These results extend the well known results connecting diffusion processes and PDE (see [1], [5], [19], [6], [3]). Moreover, they establish a natural link with the subject of stochastic partial differential equations (see [13],[9], [12],[11] [22],[4]), viz. the process $(Y_t(\psi))$ for $\psi \in \mathcal{E}'$, is the solution of a stochastic partial differential equation (eqn (3.7) below) associated naturally with the equation for (X_t) via Ito's formula. This stochastic partial differential equation is different from the one satisfied by the process (δ_{X_t}) in [16],[7],[8] - the former is associated with the operators (L^*, A^*) as above, the latter being associated with the random operators $(L(t,\omega), A(t,\omega))$, $A(t,\omega) := (A_1(t,\omega), \cdots A_d(t,\omega))$,

$$\begin{aligned} A_k(t,\omega) &= -\sum_{i=1}^{d} \sigma_{ij}(X_t(\omega)\partial_i, \\ L(t,\omega) &= \frac{1}{2}\sum_{i,j}(\sigma\sigma^t)_{ij}(X_t(\omega))\ \partial^2_{ij} - \sum_i b_i(X_t(\omega))\ \partial_i \end{aligned}$$

(see [16],[8]). However it is easily seen that when $\psi = \delta_x$, the process $(Y_t(\psi))$ is the same as the process (δ_{X_t}). We also note that solutions (ψ_t) of equation (1.1) are obtained by averaging out the diffusion term in the stochastic partial differential equation satisfied by $(Y_t(\psi))$ (see Theorem 4.3), a result that corresponds quite well with the original motivation for studying stochastic partial differential equations, viz. 'Stochastic PDE = PDE + noise' (see [22] for example).

The definition of Y_t as the adjoint of the map $X_t : C^\infty \to C^\infty$ induced by the flow $(X(t,x,\omega))$ does not automatically lead to good path properties for the process $(Y_t(\psi)), \psi \in \mathcal{E}'$. To get these we generalise the representation

$$Y_t(\psi)(\omega) = \int \delta_{X(t,x,\omega)}\ d\psi(x)$$

which is easily verified when ψ is a measure with compact support. Indeed,

$$\begin{aligned} \langle Y_t(\psi)(\omega), \varphi\rangle &= \langle \psi, X_t(\omega) \circ \varphi\rangle \\ &= \int X_t(\omega) \circ \varphi(x)\ d\psi(x) \end{aligned}$$

$$
\begin{aligned}
&= \int \varphi(X(t,x,\omega))\ d\psi(x) \\
&= \int \langle \delta_{X(t,x,\omega)}, \varphi\rangle\ d\psi(x) \\
&= \langle \int \delta_{X(t,x,\omega)}\ d\psi(x), \varphi\rangle.
\end{aligned}
$$

Here the integral $\int \delta_{X(t,x,\omega)} d\psi(x)$ is understood in the sense of Bochner and takes values in a suitable Hilbert space $S_{-p} \subseteq \mathcal{S}'$. In Section 3, we define the process $(Y_t(\psi))$, for $\psi \in \mathcal{E}'$, via a representation such as the one above and verify that indeed $Y_t(\psi) = X_t^*(\psi)$. In Theorem (3.3) we show that $(Y_t(\psi))$ satisfies a stochastic partial differential equation.

In Section 2, we state some well known results from the theory of stochastic flows [13] in a form convenient for our purposes. These are used in Section 3 for constructing the process $(Y_t(\psi))$ and proving its properties. In Section 4, we prove the representation result for solutions of equation (1.1). Our results amount to a proof of existence of solutions for equation (1.1). Our proofs require that the coefficients be smooth. However, we do not require that the diffusion matrix be non degenerate. The uniqueness of solutions of (1.1) can be shown to hold if the so called 'monotonicity inequality' (see (4.2) below) for the pair of operators (L^*, A^*) is satisfied (Theorem 4.4). It may be mentioned here that the 'monotonicity inequality' is known to hold even when the diffusion matrix is degenerate (see [8]). As mentioned above, when $\psi = \delta_x, x \in \mathbb{R}^d, Y_t(\psi) = \delta_{X_t}$ where $X_0 = x$. As discussed above, for $\psi \in \mathcal{E}'$, the solutions $(Y_t(\psi))$ of the stochastic partial differential equation (3.7) can be constructed out of the particular solutions (δ_{X_t}) corresponding to $\psi = \delta_x$. In other words the processes $(\delta_{X_t}), X_0 = x$, can be regarded as the 'fundamental solution' of the stochastic partial differential equation (3.7) that $(Y_t(\psi))$ satisfies. This property is preserved on taking expectations: in other words $E\delta_{X_t}, X_0 = x$ is the fundamental solution of the partial differential equation (1.1) satisfied by $\psi_t = EY_t(\psi)$ - a well known result for probabilists, if one notes that $E\delta_{X_t} = P(t, x, \cdot)$ the transition probability measure of the diffusion (X_t) starting at x (Theorem 4.5). Assuming that this diffusion has a density $p(t, x, y)$ satisfying some mild integrability conditions, we deduce some well known results. If $L^* = L$ then the density is symmetric (Theorem (4.6)), and in the constant coefficient case we further have $p(t, x, y) = p(t, 0, y - x)$. Finally if $T_t : C^\infty \to C^\infty$ denotes the semigroup corresponding to the diffusion (X_t), and $S_t : \mathcal{E}' \to \mathcal{E}'$ is the adjoint, then S_t, given by $S_t(\psi) = EY_t(\psi)$, is a uniformly bounded (in t) operator when restricted to the Hilbert spaces S_{-p} (Theorem (4.8)).

2 Stochastic Flows

Let $\Omega = C([0,\infty), I\!R^r)$ be the set of continuous functions on $[0,\infty)$ with values in $I\!R^r$. Let $\mathcal{F}$ denote the Borel σ-field on Ω and let P denote the Wiener measure. We denote $B_t(\omega) := \omega(t),\ \omega \in \Omega,\ t \geq 0$ and recall that under P, (B_t) is a standard r dimensional Brownian motion. Let $(X_t)_{t\geq 0}$ be a strong solution on $(\Omega, \mathcal{F}, P)$ of the stochastic differential equation

$$\left.\begin{array}{lcl} dX_t &=& \sigma(X_t)\cdot dB_t + b(X_t)dt \\ X_0 &=& x \end{array}\right\} \qquad (2.1)$$

with $\sigma = (\sigma^i_j)$, $i = 1\ldots d, j = 1\ldots r$ and $b = (b^1, \ldots b^d)$, where σ^i_j and b^i are given by C^∞ functions on $I\!R^d$ with bounded derivatives satisfying

$$\|\sigma(x)\| + \|b(x)\| = \left(\sum_{i=1}^d \sum_{j=1}^r |\sigma^i_j(x)|^2\right)^{1/2} + \left(\sum_{i=1}^d |b^i(x)|^2\right)^{1/2} \leq K(1+|x|)$$

for some $K > 0$. Under the above assumptions on σ and b, it is well known that a unique, non-explosive strong solution $(X(t,x,\omega))_{t\geq 0, x\in I\!R^d}$ exists on $(\Omega, \mathcal{F}, P)$ (see [10]). We also have the following theorem (see [2], [13] and [10], p.251).

Theorem 2.1 *For $x \in I\!R^d$ and $t \geq 0$, let $(X(t,x,\omega))$ be the unique strong solution of equation (2.1) above. Then there exists a process $(\tilde{X}(t,x,\omega))_{t\geq 0, x\in I\!R^d}$ such that*
(1) For all $x \in I\!R^d, P\{\tilde{X}(t,x,\omega) = X(t,x,\omega),\ \forall\ t \geq 0\} = 1$.
(2) For a.e. $\omega(P), x \to \tilde{X}(t,x,\omega)$ is a diffeomorphism for all $t \geq 0$.
(3) Let $\theta_t : \Omega \to \Omega$ be the shift operator i.e. $\theta_t\omega(s) = \omega(s+t)$; then for $s,t \geq 0$, we have

$$\tilde{X}(t+s,x,\omega) = \tilde{X}(s, \tilde{X}(t,x,\omega), \theta_t\omega)$$

for all $x \in I\!R^d$, a.e. ω (P).

Denote by $\partial X(t,x,\omega)$ the $d\times d$ matrix valued process, given for a.e. ω by $(\partial X)^i_j(t,\omega,x) = \frac{\partial X^i}{\partial x^j}(t,x,\omega)$ for all $t \geq 0$ and $x \in I\!R^d$. Denote by $\partial\sigma_\alpha(x)$ the $d\times d$ matrix $(\partial\sigma_\alpha(x))^i_j = \frac{\partial\sigma^i_\alpha(x)}{\partial x^j}$ and by $\partial b(x)$ the $d\times d$ matrix $(\partial b(x))^i_j = \frac{\partial b^i(x)}{\partial x^j}$. Then it is well known (see [13]) that almost surely, $\partial X(t,\omega,x)$ is invertible for all t and x and the inverse satisfies the SDE

$$\begin{aligned} dJ_t &= -\sum_{\alpha=1}^r J_t\cdot\partial\sigma_\alpha(X_t)\ dB^\alpha_t \\ &\quad -J_t\cdot\left[\partial b(X_t) - \sum_{\alpha=1}^r(\partial\sigma_\alpha)\cdot(\partial\sigma_\alpha)(X_t)\right]dt, \\ J_0 &= I \end{aligned}$$

where I is the identity matrix and $J_t\cdot\partial\sigma_\alpha(X_t)$ etc. denote the product of $d\times d$ matrices. In proving our results, we will need to show that $\sup_{x\in K}|\partial^r X(t,x)|^q$ and $\sup_{x\in K}|(\partial X)^{-1}(t,x)|$ (here $\partial^r := \partial_1^{r_1}\ldots\partial_d^{r_d}$ and $|\cdot|$ denotes the Euclidean norm on $I\!R^d$ in the first case and on $I\!R^{d^2}$ in the second case) have finite expectation for $q\geq 1$ and $K\subseteq I\!R^d$ a compact set. To do this we will use the results of section 4.6 of [13], as also the notation there. First, we note that the stochastic differential equations for (X_t) and $(\partial X(t))$ can be combined into a single stochastic differential equation in $I\!R^{d+d^2}$, which in the language of [13], can be based on a spatial semi-martingale $F(x,t) = (F^1(x,t),\ldots F^{d+d^2}(x,t))$. Having done this and having verified the regularity hypothesis on the local characteristics of $F(x,t)$ we can apply Corollary 4.6.7 of [13] to get our results.

To fix notation we note that the set $\{k : d < k \leq d^2+d\}$ is in 1-1 correspondence with $\{(i,j) : 1\leq i,j\leq d\}$. We fix such a correspondence and write $k\leftrightarrow(i,j)$ for $d<k\leq d+d^2$ and $1\leq i,j\leq d$. If $x\in I\!R^{d+d^2}$ we will write $x=(x_1,x_2)$ where $x_1\in I\!R^d$ and $x_2\in I\!R^{d^2}$. For $1\leq k\leq d$, let

$$F^k(x,t) = \sum_{\alpha=1}^r \sigma^k_\alpha(x_1)B^\alpha_t + b^k(x_1)t.$$

For $d+1\leq k\leq d+d^2$, let

$$F^k(x,t) \;=\; -\sum_{\alpha=1}^r (x_2\cdot\partial\sigma_\alpha(x_1))^i_j\ B^\alpha_t$$

$$- \quad t\left(x_2 \cdot \left[\partial b(x_1) - \sum_{\alpha=1}^{r}(\partial\sigma_\alpha)\cdot(\partial\sigma_\alpha)(x_1)\right]\right)^i_j$$

where $k \leftrightarrow (i,j), x_2 \cdot \partial\sigma_\alpha(x_1)$ is the product of $d \times d$ matrices x_2 and $\partial\sigma_x(x_1)$ etc. The local characteristics of $F(x,t)$ are then given by $(\alpha(x,y,t), \beta(x,t), t)$ where:

$$\begin{aligned}
\beta^k(x,t) &= b^k(x_1), \quad 1 \le k \le d \\
&= x_2 \cdot \partial b(x_1) - \sum_{\alpha=1}^{r}(\partial\sigma_\alpha)\cdot(\partial\sigma_\alpha)(x_1), \quad d+1 \le k \le d+d^2 \\
\beta(x,t) &= (\beta^1(x,t), \ldots \beta^{d+d^2}(x,t)), \quad x = (x_1,x_2) \in I\!R^{d+d^2}.
\end{aligned}$$

Further, for $x = (x_1,x_2), y = (y_1,y_2)$,$\alpha(x,y,t) = \alpha^{k\ell}(x,y,t),\ 1 \le k,\ \ell \le d + d^2$ where

$$\begin{aligned}
\alpha^{k\ell}(x,y,t) &= (\sigma(x_1)\cdot\sigma^t(y_1))^k_\ell, \quad 1 \le k,\ell \le d \\
&= \sum_{\alpha=1}^{r}(x_2\cdot\partial\sigma_\alpha(x_1))^i_j(y_2\cdot\partial\sigma_\alpha(y_1))^{i'}_{j'} \\
&\quad d+1 \le k,\ell \le d+d^2, k \leftrightarrow (i,j), \ell \leftrightarrow (i',j') \\
&= -\sum_{\alpha=1}^{r}\sigma^k_\alpha(x_1)(y_2\cdot\partial\sigma_\alpha(y_1))^i_j \\
&\quad 1 \le k \le d, d+1 \le \ell \le d+d^2 \text{ and } \ell \leftrightarrow (i,j) \\
&= -\sum_{\alpha=1}^{r}(x_2\cdot\partial\sigma_\alpha(x_1))^i_j\sigma^\ell_\alpha(y_1) \\
&\quad 1 \le \ell \le d, d+1 \le k \le d+d^2 \text{ and } k \leftrightarrow (i,j).
\end{aligned}$$

Let $n = d + d^2$ and $f : I\!R^n \to I\!R^n, g : I\!R^n \times I\!R^n \to I\!R^{n^2}$. Consider for $m \ge 1$ and $\delta > 0$, the following semi-norms (see [13]),

$$\begin{aligned}
\|f\|_{m,\delta} &= \sup_{x\in I\!R^n}\frac{|f(x)|}{(1+|x|)} + \sum_{1\le|\alpha|\le m}\ \sup_{x\in I\!R^n}|\partial^\alpha f(x)| \\
&\quad + \sum_{|\alpha|=m}\ \sup_{\substack{x,y\in I\!R^n\\ x\ne y}}\frac{|\partial^\alpha f(x) - \partial^\alpha f(y)|}{|x-y|^\delta}
\end{aligned}$$

$$\|g\|^{\sim}_{m,\delta} = \sup_{x\in \mathbb{R}^n} \frac{|g(x,y)|}{(1+|x|)(1+|y|)} + \sum_{1\le|\alpha|\le m} \sup_{x,y\in \mathbb{R}^n} |\partial_x^\alpha \partial_y^\alpha g(x,y)|$$

$$+ \sum_{|\alpha|=m} \sup_{\substack{x,x',y,y'\in \mathbb{R}^n\\ x\neq x', y\neq y'}} \frac{|\partial_x^\alpha\partial_y^\alpha g(x,y) - \partial_x^\alpha\partial_y^\alpha g(x',y) - \partial_x^\alpha\partial_y^\alpha g(x,y') + \partial_x^\alpha\partial_y^\alpha g(x',y')|}{|x-x'|^\delta |y-y'|^\delta}.$$

Let for $0 < s < T$, $\{\varphi_{s,t}(x), s \le t \le T\}$ be the solution of Ito's SDE based on the semi-martingale $F(x,t), x \in \mathbb{R}^{d+d^2}$, i.e.

$$\varphi_{s,t}(x) = x + \int_s^t F(\varphi_{s,r}(x), dr).$$

Note that $\varphi^k_{0,t}(x) = X^k(t,x_1) \quad 1 \le k \le d$ and $\varphi^k_{0,t}(x) = (\partial X^{-1})^i_j(t,x_1)$, $d+1 \le k \le d+d^2$ and $k \leftrightarrow (i,j), x = (x_1,x_2), x_1 \in \mathbb{R}^d$ and $x_2 \leftrightarrow I$ the identity matrix. We then have the following theorem.

Theorem 2.2 *Given $0 \le s \le T, \alpha = (\alpha_1 \dots \alpha_d)$ a multi-index, $N > 0$, and $q \ge 1$, there exists $C = C(s,T,\alpha,N,q) > 0$ such that*

$$E \sup_{|x|\le N} |\partial^\alpha \varphi_{s,t}(x)|^q < C$$

for any t satisfying $s \le t \le T$. In particular, for any compact $K \subseteq \mathbb{R}^d, q \ge 1, \alpha$ a multi-index, there exists $C > 0$

$$E \sup_{x\in K} |\partial^\alpha X(t,x)|^q < C$$

and

$$E \sup_{x\in K} |\partial X^{-1}(t,x)^i_j|^q < C$$

for $0 \le t \le T$.

Proof: It is easily verified that the local characteristics (α, β, t) verify for $m \ge 1,\ \delta = 1$:

$$\sup_{t\le T} \|\alpha(t)\|^{\sim}_{m,1} < \infty \text{ and } \sup_{t\le T} \|\beta(t)\|_{m,1} < \infty.$$

In the language of [13], the local characteristics (α, β, t) belong to the class $B_{ub}^{m,1}$ for all $m \geq 1$. Thus the hypothesis of Corollary 4.6.7 in [13] is satisfied. Hence for $p > 1, \alpha$ a multi-index, $N > 0$ and $0 < s < T$, $\exists\ C = C(p, \alpha, N, s, T)$

$$E \sup_{|x| \leq N} |\partial^\alpha \varphi_{s,t}(x)|^{2p} < C.$$

The result for $q = 2p > 2$ and hence for $q \geq 1$ follows. □

3 The Induced Flow on Distributions with compact support

We will denote the modification obtained in Theorem 2.1 again by $(X(t, x, \omega))$. For ω outside a null set $\tilde{N}$, the flow of diffeomorphisms induces, for each $t \geq 0$ a continuous linear map, denoted by $X_t(\omega)$ on $C^\infty(I\!R^d)$. $X_t(\omega) : C^\infty(I\!R^d) \rightarrow C^\infty(I\!R^d)$ is given by $(X_t(\omega)(\varphi)(x) = \varphi(X(t, x, \omega))$. This map is linear and continuous with respect to the topology on $C^\infty(I\!R^d)$ given by the following family of semi-norms: For $K \subseteq I\!R^d$ a compact set, let $\|\varphi\|_{n,K} = \max_{|\alpha| \leq n} \sup_{x \in K} |D^\alpha f(x)|$ where $\varphi \in C^\infty(I\!R^d)$ and $n \geq 1$ an integer and $\alpha = (\alpha_1, \ldots \alpha_d)$ and $|\alpha| = \alpha_1 + \ldots + \alpha_d$. Let $K_{t,\omega}$ denote the image of K under the map $x \rightarrow X(t, x, \omega)$. Then using the 'chain rule' it is easy to see that there exists a constant $C(t, \omega) > 0$ such that

$$\|X_t(\omega)(\varphi)\|_{n,K} \leq C(t, \omega) \|\varphi\|_{n,K_{t,\omega}}.$$

Let $X_t(\omega)^*$ denote the transpose of the map $X_t(\omega) : C^\infty(I\!R^d) \rightarrow C^\infty(I\!R^d)$. Then if $\mathcal{E}'$ denotes the space of distributions with compact support we have $X_t(\omega)^* : \mathcal{E}' \rightarrow \mathcal{E}'$ is given by

$$\langle X_t(\omega)^* \psi, \varphi \rangle = \langle \psi, X_t(\omega) \varphi \rangle$$

for all $\varphi \in C^\infty$ and $\psi \in \mathcal{E}'$. Let $\psi \in \mathcal{E}'$. Let supp $\psi \subseteq K$ and let $N = \text{order}(\psi) + 2d$. Then there exist continuous functions $g_\alpha, |\alpha| \leq N, \text{supp } g_\alpha \subseteq V$ where V is an open set having compact closure, containing K, such that

$$\psi = \sum_{|\alpha| \leq N} \partial^\alpha g_\alpha. \qquad (3.1)$$

See [21]. Let $\varphi \in C^\infty(I\!R^d)$. Let $f_i \in C^\infty(I\!R^d)$ and $f = (f_1, \dots f_d)$. Then, using the chain rule, it is easy to see that there exist polynomials $P_\gamma(x_1 \dots x_d)$, one for each multi-index γ, with $|\gamma| \leq |\alpha|$ and $deg\ P_\gamma = |\gamma|$, and such that

$$\partial^\alpha(\varphi \circ f)(x) = \sum_{\substack{|\gamma|\leq|\alpha| \\ |\beta|\leq|\alpha|}} (-1)^{|\gamma|} P_\gamma(\partial^{\beta_1} f_1, \dots \partial^{\beta_d} f_d)(x)\ \langle \varphi, \partial^\gamma \delta_{f(x)} \rangle. \quad (\ 3.2\ )$$

For $\omega \notin \tilde{N}$, define $Y_t(\omega) : \mathcal{E}' \to \mathcal{E}'$ by

$$\begin{aligned} Y_t(\omega)(\psi) &= \sum_{|\alpha|\leq N} (-1)^{|\alpha|} \sum_{\substack{|\gamma|\leq|\alpha| \\ |\beta_i|\leq|\alpha|}} (-1)^{|\gamma|} \\ &= \int_V g_\alpha(x)\ P_\gamma(\partial^{\beta_1} X_1 \dots \partial^{\beta_d} X_d)(t,x,\omega)\ \partial^\gamma \delta_{X(t,x,\omega)}\ dx \quad (\ 3.3\ ) \end{aligned}$$

Take $Y_t(\omega) = 0$ if $\omega \in \tilde{N}$.

Let $\mathcal{S}$ be the space of smooth rapidly decreasing functions on $I\!R^d$ with dual $\mathcal{S}'$, the space of tempered distributions. It is well known ([9]), that $\mathcal{S}$ is a nuclear space, and that $\mathcal{S} = \bigcap_{p>0} (S_p, \|\cdot\|_p)$, where the Hilbert spaces S_p are equipped with increasing norms $\|\cdot\|_p$, defined by the inner products

$$\langle f, g \rangle_p = \sum_{|k|=0}^{\infty} (2|k| + d)^{2p} \langle f, h_k \rangle \langle g, h_k \rangle, \quad f, g \in \mathcal{S}$$

Above, $\{h_k\}_{|k|=0}^\infty$ is an orthonormal basis for $L^2\left(I\!R^d, dx\right)$ given by Hermite functions (for $d = 1$, $h_k(t) = (2^k k! \sqrt{\pi})^{-1/2} \exp\{-t^2/2\} H_k(t)$, with $H_k(t)$, a Hermite polynomial. see [9]), and $\langle \cdot, \cdot \rangle$ is the usual inner product in $L^2\left(\ R^d, dx\right)$. We also have $\mathcal{S}' = \bigcup_{p>0} (S_{-p}, \|\cdot\|_{-p})$. Note that for all $-\infty < p < \infty$, S_p is the completion of $\mathcal{S}$ in $\|\cdot\|_p$ and S_p' is isometrically isomorphic with S_{-p}, $p > 0$. We then have the following proposition.

Proposition 3.1 *Let ψ be a distribution with compact support having representation (3.1). Let $p > 0$ be such that $\partial^\alpha \delta_x \in S_{-p}$ for $|\alpha| \leq N$. Then $(Y_t(\psi))_{t\geq0}$ is an S_{-p} valued continuous adapted process such that for all $t \geq 0$,*

$$Y_t(\psi) = X_t^*(\psi)\ a.s.\ P.$$

Proof: From Theorem 4.6.5 of [13], it follows that for any multi index γ and compact set $K \subseteq \mathbb{R}^d$, $\sup_{s\leq T}\ \sup_{x\in K} |\partial^\gamma X_i(s,x,\omega)| < \infty$ a.s. for all $T > 0$.

From this result, the fact that $\partial_i : S_{-p-\frac{1}{2}} \to S_{-p}$ is bounded, and Theorem 2.1 of [17], it follows that for all $T > 0$ a.s.

$$\sup_{t\leq T} \int_V |g_\alpha(x)|\ |P_\gamma(\partial^{\beta_1} X_1, \ldots \partial^{\beta_d} X_d)(t,x,\omega))|\ \|\partial^\gamma \delta_{X(t,x,\omega)}\|_{-p} dx < \infty. \quad (3.4)$$

It follows that $(Y_t(\psi))$ is a well defined S_{-p}-valued process. Since for all $x \in \mathbb{R}^d, (\partial^\gamma X_i(t,x,\omega))$ is an adapted process, and jointly measurable in (t,x,ω) it follows that $(Y_t(\psi))$ is an S_{-p}-valued adapted process.

To show that the map $t \to Y_t(\psi)$ is almost surely continuous in S_{-p}, we first observe that for any $p \in \mathbb{R}$ and any $\phi \in S_p$, the map $x \to \tau_x\phi : \mathbb{R}^d \to S_p$ is continuous, where $\tau_x : S' \to S'$ is translation by $x \in \mathbb{R}^d$. To see this, let $x_n \to x \in \mathbb{R}^d$. Note that from Theorem 2 of [17], given $\epsilon > 0$, there exists $\psi \in S$ such that

$$\|\tau_x\phi - \tau_{x_n}\phi\|_p \ < \ \|\tau_x\psi - \tau_{x_n}\psi\|_p + \frac{\epsilon}{2}.$$

From the definition of $\|.\|_p$ we have

$$\|\tau_x\psi - \tau_{x_n}\psi\|_p^2 \ = \ \sum_k (2|k|+d)^{2p} \langle \tau_x\psi - \tau_{x_n}\psi, h_k\rangle^2.$$

Since $\psi \in S_q$ for every q and since the result is true for $p = 0$, the right hand side above tends to zero by dominated convergence theorem and

$$\|\tau_x\phi - \tau_{x_n}\phi\|_p \ < \ \epsilon$$

for large n thus proving the continuity of the map $x \to \tau_x\phi$. In particular if p and α are as in the statement of the theorem, the map $x \to \partial^\gamma\delta_x$ is continuous in S_{-p} for $|\gamma| \leq |\alpha|$. Now the continuity of $t \to Y_t(\psi)(\omega)$ follows from the continuity in the t variable of the processes $(\partial^\gamma X_i(t,x,\omega))$ and $(\partial^\gamma \delta_{X(t,x,\omega)})$ and the dominated convergence theorem .

We now verify that $Y_t(\psi) = X_t^*(\psi)$. Let $\varphi \in C^\infty(\mathbb{R}^d)$. From (3.3),

$$\langle Y_t(\omega)\psi, \phi\rangle \ = \ \sum_{|\alpha|\leq N} (-1)^{|\alpha|} \sum_{\substack{|\gamma|\leq|\alpha| \\ |\beta_i|\leq|\alpha|}} (-1)^{|\gamma|}$$

$$
\begin{aligned}
& \int_V g_\alpha(x) P_\gamma(\partial^{\beta_1} X_1, \dots \partial X_d^{\beta_d})(t,x,\omega)\ \langle \partial^\gamma \delta_{X(t,x,\omega)}, \phi\rangle\ dx \\
= & \sum_{|\alpha|\le N} (-1)^{|\alpha|} \int_V g_\alpha(x) \partial^\alpha (\phi \circ X(t,x,\omega)) dx \qquad \text{(by (3.2))} \\
= & \sum_{|\alpha|\le N} \langle \partial^\alpha g_\alpha, \phi \circ X(t,\cdot,\omega)\rangle \qquad \text{(by (3.1))} \\
= & \sum_{|\alpha|\le N} \langle \partial^\alpha g_\alpha, X_t(\omega)\phi\rangle \\
= & \langle \psi, X_t(\omega)\phi\rangle.
\end{aligned}
$$

□

Recall that C^∞ denotes the space of smooth functions on $I\!R^d$. Define the operators $A : C^\infty \to L(I\!R^r, C^\infty)$ and $L : C^\infty \to C^\infty$ as follows: For $\varphi \in C^\infty$, $x \in I\!R^d$,

$$
\begin{aligned}
A\varphi &= (A_1\varphi, \cdots, A_r\varphi), \\
A_i\varphi(x) &= \sum_{k=1}^d \sigma_i^k(x) \partial_k \varphi(x), \\
L\varphi(x) &= \frac{1}{2} \sum_{i,j=1}^d (\sigma\sigma^t)_j^i(x)\ \partial_{ij}^2 \varphi(x) + \sum_{i=1}^d b^i(x)\ \partial_i \varphi(x).
\end{aligned}
$$

We define the adjoint operators $A^* : \mathcal{E}' \to L(I\!R^r, \mathcal{E}')$ and $L^* : \mathcal{E}' \to \mathcal{E}'$ as follows:

$$
\begin{aligned}
A^*\varphi &= (A_1^*\varphi, \cdots, A_r^*\varphi), \\
A_i^*\psi &= -\sum_{k=1}^d \partial_k(\sigma_i^k \psi), \\
L^*\psi &= \frac{1}{2} \sum_{i,j=1}^d \partial_{ij}^2((\sigma\sigma^t)_j^i \psi) - \sum_{i=1}^d \partial_i(b^i \psi).
\end{aligned}
$$

The following proposition gives the boundedness properties of L^* and A^*. For $K \subset I\!R^d$, let $\mathcal{E}'(K) \subseteq \mathcal{E}'$ be the subspace of distributions whose support is contained in K.

Proposition 3.2 *Let $p > 0$ and $q > [p] + 2$, where $[p]$ denotes the largest integer less than or equal to p. Then, $A^* : S_{-p} \cap \mathcal{E}'(K) \to L(I\!R^r, S_{-q} \cap \mathcal{E}'(K))$*

and $L^* : S_{-p} \cap \mathcal{E}'(K) \to S_{-q} \cap \mathcal{E}'(K)$. *Moreover, there exists constants* $C_1(p) > 0, C_2(p) > 0$ *such that*

$$\|A^*\psi\|_{HS(-q)} \leq C_1(q)\ \|\psi\|_{-p}, \qquad \|L^*\psi\|_{-q} \leq C_2(q)\ \|\psi\|_{-p}$$

where

$$|A^*\psi\|^2_{HS(-q)} = \sum_{i=1}^{r} \left\| \sum_{k=1}^{d} \partial_k(\sigma_i^k \psi) \right\|^2_{-q} = \sum_{i=1}^{r} \|A_i^*\psi\|^2_{-q}.$$

Proof Clearly $A^* : \mathcal{E}'(K) \to L(I\!R^r, \mathcal{E}'(K))$ and $L^* : \mathcal{E}'(K) \to \mathcal{E}'(K)$. We prove the bounds for A^*. The bounds for L^* follow in a similar fashion. By definition, if $q \geq p + 1/2$,

$$\begin{aligned} \|A^*\psi\|^2_{HS(-q)} &= \sum_{i=1}^{d} \| \sum_{k=1}^{d} \partial_k(\sigma_{ki}\psi)\|^2_{-q} \\ &\leq C \sum_{i=1}^{d} \sum_{k=1}^{d} \|\sigma_{ki}\psi\|^2_{-(q-\frac{1}{2})}. \end{aligned} \qquad (3.5)$$

Let σ denote a C^∞ function. We first show that the map $\psi \to \sigma\psi : \mathcal{S}_n \cap \mathcal{E}'(K) \to \mathcal{S}_n \cap \mathcal{E}'(K)$ satisfies

$$\|\sigma\psi\|_n \leq C\ \|\psi\|_n$$

where the constant C depends on σ, K and n. First assume that $\psi \in \mathcal{S} \cap \mathcal{E}'(K)$. We then have (see [17], Proposition 3.3b)

$$\|\sigma\psi\|^2_n \leq C_1 \sum_{|\alpha|+|\beta|\leq 2n} \|x^\alpha \partial^\beta(\sigma\psi)\|^2_0.$$

Clearly,

$$\|x^\alpha \partial^\beta(\sigma\psi)\|^2_0 \leq C_2 \sum_{|\gamma|\leq|\beta|} \|x^\alpha \partial^\gamma \psi\|^2_0.$$

It follows that

$$\begin{aligned} \|\sigma\psi\|^2_n &\leq C_3 \sum_{|\alpha|+|\gamma|\leq 2n} \|x^\alpha \partial^\gamma \psi\|^2_0 \\ &\leq C\|\psi\|^2_n. \end{aligned}$$

We now extend this to $\psi \in \mathcal{S}_n \cap \mathcal{E}'(K)$: Since $\mathcal{S}$ is dense in $\mathcal{S}_n$, we can get $\psi_m \in \mathcal{S}, \psi_m \to \psi$ in $\mathcal{S}_n$. By multiplying by an appropriate C^∞ function with compact support can assume, $\psi_m \in \mathcal{S} \cap \mathcal{E}'(K)$. By the above inequality applied to ψ_m, it follows that $\sigma\psi_m$ converges in $\mathcal{S}_n$, and hence weakly to a limit φ in $\mathcal{S}'$. Hence if f is C^∞ with compact support,

$$\begin{aligned} \langle \varphi, f \rangle &= \lim_{m\to\infty} \langle \sigma\psi_m, f \rangle \\ &= \lim_{m\to\infty} \langle \psi_m, \sigma f \rangle \\ &= \langle \psi, \sigma f \rangle = \langle \sigma\psi, f \rangle. \end{aligned}$$

Hence $\sigma\psi_m \to \sigma\psi$ in $\mathcal{S}_n$ and the above inequality follows for $\psi \in \mathcal{S}_n \cap \mathcal{E}'(K)$.

Now suppose $\psi \in \mathcal{S} \cap \mathcal{E}'(K)$.

$$\begin{aligned} \|\sigma\psi\|_{-n} &= \sup_{\substack{\|\varphi\|_n \leq 1 \\ \varphi \in \mathcal{S}}} |\langle \sigma\psi, \varphi \rangle| \\ &= \sup_{\substack{\|\varphi\|_n \leq 1 \\ \varphi \in \mathcal{S}}} |\langle g\sigma\psi, \varphi \rangle| \end{aligned}$$

where $g \in C^\infty, g = 1$ on K, $\text{supp}(g) \subseteq K^\epsilon$, an ϵ-neighbourhood of K. Therefore,

$$\begin{aligned} \|\sigma\psi\|_{-n} &\leq \|\psi\|_{-n} \sup_{\substack{\|\varphi\|_n \leq 1 \\ \varphi \in \mathcal{S}}} \|\sigma g \varphi\|_n \\ &\leq C\|\psi\|_{-n}. \end{aligned} \qquad (3.6)$$

In the same way as for $n \geq 0$, we can extend the above inequality to $\psi \in \mathcal{S}_{-n} \cap \mathcal{E}'(K)$. Now the proof can be completed using (3.5) and (3.6) and by choosing a $q \in I\!R$ such that $q \geq m + \frac{1}{2} > m \geq p$ for some integer m. In particular, we may take $q > [p] + 2$. □

Theorem 3.3 *Let $\psi \in \mathcal{E}'$ have the representation (3.1). Let $p > 0$ be such that $\partial^\gamma \delta_x \in S_{-p}, |\gamma| \leq N$. Let $q > p$ be as in Proposition 3.2. Then the S_{-p}-valued continuous, adapted process $(Y_t(\psi))_{t\geq 0}$ satisfies the following equation in S_{-q}: a.s.,*

$$Y_t(\psi) = \psi + \int_0^t A^*(Y_s(\psi)) \cdot dB_s + \int_0^t L^*(Y_s(\psi))\, ds \qquad (3.7)$$

for all $t \geq 0$.

Proof: From Proposition 3.2 and the estimate (3.4), it follows that for $t \geq 0$, a.s.

$$\int_0^t \|A^* Y_s(\psi)\|^2_{HS(-q)}\, ds + \int_0^t \|L^*\psi_s(\psi)\|_{-q} ds < \infty.$$

Hence the right hand side of (3.7) is well defined. Let $\varphi \in C^\infty$. Then by Ito's formula

$$X_t(\varphi) \;=\; \varphi + \int_0^t X_s(A\varphi)\cdot dB_s + \int_0^t X_s(L\varphi) ds$$

where the integrals on the right hand side are understood as $C^\infty(\mathbb{R}^d)$ valued processes given by $(t, x, \omega) \to \int_0^t A\varphi(X(s, x, \omega)\cdot dB_s$ and $(t, s, \omega) \to \int_0^t L\varphi(X(s, x, \omega))ds$. Since ψ has compact support, by localising, we may assume that these processes have their supports contained in the support of ψ, a fixed compact set not depending on t and ω. In particular, they belong to $S \subset S_p$. Using Proposition 3.1, we get, for $\varphi \in S$,

$$\begin{aligned}
\langle Y_t(\psi), \varphi\rangle &= \langle \psi, X_t(\varphi)\rangle \\
&= \langle \psi, \varphi\rangle + \int_0^t \langle \psi, X_s(A\varphi)\rangle \cdot dB_s + \int_0^t \langle \psi, X_s(L\varphi)\rangle\, ds \\
&= \langle \psi, \varphi\rangle + \int_0^t \langle A^* Y_s(\psi), \varphi\rangle \cdot dB_s + \int_0^t \langle L^* Y_s(\psi), \varphi\rangle\, ds \\
&= \langle \psi + \int_0^t A^* Y_s(\psi)\cdot dB_s + \int_0^t L^* Y_s(\psi)\, ds, \quad \varphi\rangle
\end{aligned}$$

and the result follows. In the above calculations we have used the fact that $T\int_0^t A^* Y_s(\psi)\cdot dB_s = \int_0^t TA^*(Y_s(\psi))\cdot dB_s$ for any bounded linear functional $T : S_{-q} \to \mathbb{R}$. □

4 Probabilistic representations

In this section we prove the probabilistic representations of solutions to the initial value problem for the parabolic operator $\partial_t - L^*$. We also show uniqueness for the solutions of the initial value problem under the 'Monotonicity conditions'. We first prove some estimates on $\|\delta_x\|_{-p}$ and $\|\partial^\gamma \delta_x\|_{-p}$ that are required later.

Theorem 4.1 *$\delta_x \in S_{-p}$ iff $p > \frac{d}{4}$. Further if $p > \frac{d}{4}$, then*

$$\lim_{|x|\to\infty} \|\delta_x\|_{-p} = 0.$$

In particular, if for a multi-index $\gamma \in \mathbb{Z}_+^d$, $p > \frac{d}{4} + \frac{|\gamma|}{2}$, then

$$\sup_{x\in\mathbb{R}^d} \|\partial^\gamma \delta_x\|_{-p} < \infty.$$

Proof: Since $\partial^\gamma : S_{-q} \to S_{-q-\frac{|\gamma|}{2}}$ is a continuous operator we have for $p > \frac{d}{4} + \frac{|\gamma|}{2}$,

$$\|\partial^\gamma \delta_x\|_{-p} \le C \|\delta_x\|_{(-p-\frac{|\gamma|}{2})}.$$

Further from Theorem (2.1) of [17], it follows that for any compact set K contained in $\mathbb{R}^d$, $\sup_{x\in K} \|\delta_x\|_{-(p+\frac{|\gamma|}{2})} < \infty$. Since $p + \frac{|\gamma|}{2} > \frac{d}{4}$, the last statement of the theorem now follows from the second statement.

The proof of the first part of the theorem uses the generating function for Hermite functions given by Mehler's formula (see [20], page 2). First we note that

$$(2n+d)^{-2p} = \frac{1}{(2p-1)!} \int_0^\infty t^{2p-1}\, e^{-(2n+d)t} dt.$$

Hence,

$$\begin{aligned} \|\delta_x\|_{-p}^2 &= \sum_{n=0}^\infty (2n+d)^{-2p} \sum_{|k|=n} |h_k(x)|^2 \\ &= \frac{1}{(2p-1)!} \int_0^\infty t^{2p-1}\, g(t,x) dt \end{aligned}$$

where

$$g(t,x)\sum_{n=0}^{\infty} e^{-(2n+d)t} \sum_{|k|=n} |h_k(x)|^2.$$

Using Mehler's formula ,

$$\begin{aligned} g(t,x) &= e^{-dt}\pi^{-\frac{d}{2}}(1-e^{-4t})^{-\frac{d}{2}} \\ &\quad \times e^{-\frac{1}{2}(\frac{1+e^{-4t}}{1-e^{-4t}})2|x|^2+(\frac{e^{-2t}}{1-e^{-4t}})2|x|^2} \\ &= e^{-dt}\pi^{-\frac{d}{2}}(1-e^{-4t})^{-\frac{d}{2}} \\ &\quad \times e^{-(tanh\ t)|x|^2}. \end{aligned}$$

It is easy to see that for all x

$$\begin{aligned} & g(t,x) \sim (1-e^{-4t})^{-\frac{d}{2}}, \quad t \to 0 \\ \text{and} \qquad & g(t,x) \sim e^{-dt}e^{-(tanh\ t)|x|^2}, \quad t \to \infty \end{aligned}$$

where $a(t) \sim b(t) \leftrightarrow \frac{a(t)}{b(t)} \to c\ (>0)$. It follows that for all $\epsilon > 0$,

$$\int_0^{\epsilon} t^{2p-1} g(t,x)dt < \infty$$

iff $p > \frac{d}{4}$. Also $g(t,x) \to 0$ as $|x| \to \infty$, for every $t > 0$. Hence by the dominated convergence theorem,

$$\begin{aligned} \lim_{|x|\to\infty} \|\delta_x\|_{-p}^2 &= \lim_{|x|\to\infty} \frac{1}{(2p-1)!} \int_0^{\infty} t^{2p-1} g(t,x)dt \\ &= 0. \end{aligned}$$

□

Proposition 4.2 *Let $\psi \in \mathcal{E}'$ with representation (3.1). Let $p > \frac{d}{4} + \frac{N}{2}$ where $N = order(\psi) + 2d$. Let $(Y_t(\psi))$ be the S_{-p} valued continuous adapted process defined by (3.3). Then for all $T > 0$,*

$$\sup_{t \le T} E\|Y_t(\psi)\|_{-p}^2 < \infty.$$

Proof: Using the representation (3.3) for $Y_t(\psi)$ and the result of Theorem 4.1,

$$E\|Y_t(\psi)\|_{-p}^2 \leq C \cdot \sum_{|\alpha|\leq N} \sum_{\substack{|\gamma|\leq|\alpha| \\ |\beta_i|\leq|\alpha|}} \int_V E\{P_\gamma(\partial^{\beta_1}X_1, \dots \partial^{\beta_d}X_d)(t,x,\omega)\}^2 dx.$$

Hence it suffices to show,

$$\sup_{t\leq T} \sup_{x\in V} E\{P_\gamma(\partial^{\beta_1}X_1, \dots \partial^{\beta_d}X_d)(t,x,\omega)\}^2 < \infty.$$

This follows from Theorem 2.2 and completes the proof. □

We now consider solutions to the following initial value problem

$$\left.\begin{array}{lcl} \frac{\partial\psi_t}{\partial t} & = & L^*\psi_t \\ \psi_0 & = & \psi \end{array}\right\} \qquad (4.1)$$

for $\psi \in \mathcal{E}'$. By a solution to (4.1) we mean an S_{-p}-valued continuous function $\psi : [0,T] \to S_{-p}$ for some $p > 0$, such that the equation

$$\psi_t = \psi + \int_0^t L^*\psi_s \; ds$$

holds in S_{-q}, where $q > p$ is such that $L^*\psi_s \in S_{-q}, 0 \leq s \leq T$ and is Bochner integrable on $[0,T]$ with respect to Lebesgue measure. Note that, in our definition the initial value ψ belongs to S_{-p}. Note also that we can take $q > p$ given by Proposition 3.2 .

Define $\psi_t = EY_t(\psi), t \geq 0$. If ψ has compact support, then the previous proposition implies that ψ_t is a well defined element of an appropriate Hilbert space S_{-p}. The following theorem gives us the existence of solutions of equation (4.1) and a stochastic representation of its solutions.

Theorem 4.3 *Let $\psi \in \mathcal{E}'$ have representation (3.1) and let p be as in Proposition 4.2. Then $(\psi_t)_{0\leq t\leq T}$ is an S_{-p}-valued solution of (4.1).*

Proof : Let $q > p$ be as in Proposition 3.2. Because of Proposition (4.2) we can take expectations in (3.7):

$$\begin{aligned} E\|\int_0^t A^*Y_s(\psi)\cdot dB_s\|_{-q}^2 &\leq E\int_0^t \|A^*Y_s(\psi)\|_{HS(-q)}^2 ds \\ &\leq C\cdot E\int_0^t \|Y_s(\psi)\|_{-p}^2 ds \\ &< \infty. \end{aligned}$$

Similarly,

$$\begin{aligned} E\left\|\int_0^t L^*Y_s(\psi)ds\right\|_{-q} &\leq E\int_0^t \|L^*Y_s(\psi)\|_{-q}ds \\ &\leq CE\int_0^t \|Y_s(\psi)\|_{-p}ds < \infty. \end{aligned}$$

In particular, $E\int_0^t A^*Y_s(\psi)\cdot dW_s = 0$. Taking expectations in (3.7), we get

$$\psi_t = S_t(\psi) = EY_t(\psi) = \psi + \int_0^t EL^*Y_s(\psi)ds.$$

Since $L^* : S_{-p}\cap\mathcal{E}'(K) \to S_{-q}\cap\mathcal{E}'(K)$ is a bounded operator (Proposition 3.2), $EL^*Y_s(\psi) = L^*EY_s(\psi) = L^*\psi_s$. □

We now consider the uniqueness of solutions to the initial value problem (4.1). For $p > 0$, let $q > p$ be as in Proposition 3.2 ($q > [p] + 2$) so that $A^* : S_{-p}\cap\mathcal{E}' \to L(I\!R^r, S_{-q}\cap\mathcal{E}')$ and $L^* : S_{-p}\cap\mathcal{E}' \to S_{-q}\cap\mathcal{E}'$ are bounded operators. The pair (A^*, L^*) is said to satisfy the monotonicity inequality in $S_{-q}\cap\mathcal{E}'$ if and only if there exists a constant $C = C(p)$, such that,

$$2\langle\varphi, L^*\varphi\rangle_{-q} + \|A^*\varphi\|_{HS(-q)}^2 \leq C\ \|\varphi\|_{-q}^2 \qquad (4.2)$$

holds for all $\varphi\in S_{-p}\cap\mathcal{E}'$.

Theorem 4.4 *Let $\psi \in \mathcal{E}'(K)$ have representation 3.1. Let $p > \frac{d}{4} + \frac{N}{2}$ and $N =$ order $\psi + 2d$ (In particular $\psi \in S_{-p}$). Let $q > [p] + 2$. Suppose the pair (A^*, L^*) satisfies (4.2). Then, the initial value problem (4.1) has a unique S_{-p} valued solution given by $\psi_t = EY_t(\psi)$.*

Proof: The existence has been proved. It suffices to show uniqueness. Let ψ'_t be another S_{-p}-valued solution. Let $\varphi_t = \psi_t - \psi'_t$. Then (φ_t) satisfies,

$$\varphi_t = \int_0^t L^* \varphi_s \, ds, \ 0 \le t \le T$$

in S_{-q} for $q > [p] + 2$. Hence

$$\begin{aligned} \|\varphi_t\|_{-q}^2 &= 2 \int_0^t \langle \varphi_s, L^* \varphi_s \rangle_{-q} \, ds \\ &\le \int_0^t \{2\langle \varphi_s, L^* \varphi_s \rangle_{-q} + \|A^* \varphi_s\|_{HS(-q)}^2\} ds \\ &\le C \cdot \int_0^t \|\varphi_s\|_{-q}^2 \, ds. \end{aligned}$$

Now, the Gronwall inequality implies $\varphi_t \equiv 0, 0 \le t \le T$. □

Remark: When $\sigma_j^i(x)$ and $b^i(x)$ are constants (independent of x), the inequality (4.2) for (A^*, L^*) was proved in [8].

We now show that the expected value of the 'stochastic fundamental solution' $(Y_t(\delta_x)) = (\delta_{X(t,x)})$ of equation (3.5) gives us the fundamental solution of (4.1). Let $P(t, x, A) = P(X(t, x) \in A)$ be the transition function of the diffusion $(X(t, x))$.

Theorem 4.5 *Let ψ be an integrable function with compact support . Let $p > \frac{d}{4} + 1$.*

a) Then for $0 \le t \le T$ and $x \in I\!R^d, P(t, x, \cdot) \in S_{-p}$ and

$$P(t, x, \cdot) = E\delta_{X(t,x)} = EY_t(\delta_x)$$

b)Let (ψ_t) be the S_{-p}-valued solution of equation (4.1) given by $\psi_t = EY_t(\psi)$. Then

$$\psi_t = \int \psi(x)\ P(t,x,\cdot)dx$$

where the integral in the right hand side above is an S_{-p} valued Bochner integral.

Proof: a) The equality $E\delta_{X(t,x)} = EY_t(\delta_x)$ follows from the fact that if $\phi \in S$,

$$\begin{aligned}\langle EY_t(\delta_x), \phi\rangle &= E\langle Y_t(\delta_x), \phi\rangle \\ &= E\langle \delta_x, \phi(X(t,\cdot))\rangle \\ &= E\phi(X(t,x)) = E\langle \delta_{X(t,x)}, \phi\rangle \\ &= \langle E\delta_{X(t,x)}, \phi\rangle.\end{aligned}$$

On the other hand, we have $E\phi(X(t,x)) = \langle P(t,x,\cdot), \phi\rangle$ and the equality asserted in a) follows. To show that $P(t,x,\cdot)$ belongs to S_{-p}, we first note that using Theorem (2.1) of [17], there exists a polynomial $P(x)$ such that

$$\begin{aligned}\|EY_t(\delta_x)\|_{-p} &= \|E\delta_{X(t,x)}\|_{-p} \\ &\leq E\|\delta_{X(t,x)}\|_{-p} \\ &\leq (E|P(X(t,x))|)\ \|\delta_0\|_{-p}.\end{aligned}$$

It follows that $P(t,x,\cdot)$ belongs to S_{-p}.

b) To prove b) we first show the Bochner integrability of $P(t,x,\cdot)$ in S_{-p}. This follows from the calculations above for part a) and the fact that by Theorem 2.2,

$$\begin{aligned}\sup_{x\in\ \mathrm{supp}\ \psi} \|P(t,x,\cdot)\|_{-p} &\leq \sup_{x\in\ \mathrm{supp}\ \psi} (E|P(X(t,x))|)\ \|\delta_0\|_{-p} \\ &< \infty.\end{aligned}$$

Hence the integral $\int \psi(x)\ P(t,x,\cdot)\ dx$ is well defined and belongs to S_{-p}. Further since ψ is an integrable function with compact support, the representation of $Y_t(\psi)$ given by equation (3.3) reduces to

$$Y_t(\psi) = \int_V \psi(x)\ \delta_{X(t,x)}\ dx$$

where $\text{supp}(\psi) \subset V$, V an open set with compact closure. Hence

$$\begin{aligned} \psi_t &= E\ Y_t(\psi) = \int_V \psi(x)\ E\delta_{X(t,x)}\ dx \\ &= \int \psi(x)\ P(t,x,\cdot)\ dx. \end{aligned}$$

□

Suppose now that $(X(t,x))$ has a density $p(t,x,y)$, i.e.

$$P(t,x,A) = \int_A p(t,x,y)\ dy.$$

We shall assume for the rest of the section the following integrability condition on $p(t,x,y)$: For every compact set $K \subset \mathbb{R}^d$,

$$\iint_{K\times K} p(t,x,y)\ dx\ dy < \infty.$$

Corollary: Let $K \subseteq \mathbb{R}^d$ be a compact set and ψ be as in Theorem 4.5. If $\text{supp}(\psi) \subseteq K$, then $\psi_t = EY_t(\psi)$ is given by a locally integrable function $f(y)$ where $f(y) = \int \psi(x)\ p(t,x,y)\ dx$.

Proof: Let $\phi \in C_c^\infty(\mathbb{R}^d)$. Then

$$\begin{aligned} \langle f,\phi\rangle &= \int f(y)\ \phi(y)\ dy = \int\ \phi(y)\ (\int \psi(x)\ p(t,x,y)\ dx)dy \\ &= \int\ \psi(x)(\int\ p(t,x,y)\ \phi(y)\ dy)dx \\ &= \int\ \psi(x)\ \langle P(t,x,\cdot),\phi\rangle dx \\ &= \left\langle \int \psi(x)\ P(t,x,\cdot)\ dx, \phi \right\rangle \\ &= \langle \psi_t,\phi\rangle \end{aligned}$$

where the third equality follows from our assumptions on $p(t,x,y)$ and Fubini's theorem. □

We now consider the self adjoint case $L^* = L$. We deduce, under some mild integrability conditions, the well known result that the transition density is symmetric.

Theorem 4.6 *Suppose σ^i_j, b^i are C^∞, bounded with bounded derivatives of all orders. Suppose (A^*, L^*) satisfy the monotonicity condition (4.2). Suppose further that $L^* = L$. Then for $0 < t \le T, p(t,x,y) = p(t,y,x)$ for every (x,y) outside a set of zero Lebesgue measure in $I\!R^d \times I\!R^d$.*

Proof: Let $\psi \in C^\infty_c(I\!R^d)$. Let $\psi(t,x) = E\psi(X(t,x))$. Under our assumptions on σ^i_j, b^i and ψ, and the assumption $L = L^*$, it is well known (see [1], p.47) that $\psi(t,x)$ is a classical solution of the initial value problem (4.1). On the other hand by the monotonicity condition (4.2), we have uniqueness of the initial value problem (Theorem 4.4) and hence by Corollary to Theorem 4.5, we have

$$\begin{aligned} \int \psi(y)\ p(t,x,y)\ dy &= E\psi(X^x_t) = \psi(t,x) \\ &= \int \psi(y)\ p(t,y,x)\ dy \end{aligned}$$

for a.e. $x \in I\!R^d$. Since ψ is arbitrary, this implies $p(t,x,y) = p(t,y,x)$ for every (x,y) outside a set of zero Lebesgue measure and completes the proof. □

In the constant coefficient case we can deduce the following well known result.

Proposition 4.7 *Suppose σ^i_j, b^i are constants and that $(A^*, L^*) = (A, L)$ satisfy the monotonicity inequality (4.2). Then $p(t,x,y) = p(t,0,y-x)$ for almost every (x,y) with respect to the Lebesgue measure on $I\!R^d \times I\!R^d$.*

Proof: Let ψ be a continuous function with compact support. By Corollary to Theorem 4.5, the distribution $EY_t(\psi)$ is given by the locally integrable function $\psi(t,y) = \int \psi(x)\ p(t,x,y)\ dx$. On the other hand, by the uniqueness of solutions to the SDE (3.7) (see [7],[8]) we have a.s. $Y_t(\psi) = \tau_{X_t}(\psi)$ for all $t \geq 0$, where (X_t) is the diffusion $(X(t,x))$ with $x = 0$. In particular $EY_t(\psi) = E\tau_{X_t}(\psi)$. The latter distribution is given by a locally integrable function of y. We then have for a.e. y,

$$\begin{aligned}\int \psi(x)\ p(t,x,y)\ dx &= \psi(t,y) = EY_t(\psi) = E\tau_{X_t}(\psi) \\ &= \int \tau_x\psi(y)\ p(t,0,x)\ dx \\ &= \int \psi(x)\ p(t,0,y-x)\ dx.\end{aligned}$$

Since ψ is arbitrary, it follows that $p(t,x,y) = p(t,0,y-x)$ for almost every (x,y) with respect to Lebesgue measure in $\mathbb{R}^d \times \mathbb{R}^d$. □

Define $S_t(\psi) = EY_t(\psi), t \geq 0$ for $\psi \in \mathcal{E}'$. Then $S_t : \mathcal{E}' \to S'$. Let $(T_t)_{t\geq 0}$ be the semigroup corresponding to the diffusion (X_t) solving (2.1) i.e. for $f \in \mathcal{S}, T_tf(x) = Ef(X(t,x))$. We have the following result:

Theorem 4.8 *a) $T_t : \mathcal{S} \to C^\infty$ and we have $S_t = T_t^*$ in the sense that*

$$\langle S_t(\psi), \phi\rangle = \langle \psi, T_t\phi\rangle$$

for all $\psi \in \mathcal{E}'$ and $\phi \in \mathcal{S}$.

b) Let $K \subset \mathbb{R}^d$ be a compact set and $p > 0$. Then for $q > \frac{5}{4}d + [p] + 1$, $S_t : S_{-p} \cap \mathcal{E}'(K) \to S_{-q}$ is a bounded linear operator. Further, for any $T > 0$,

$$\sup_{t\leq T} \|S_t\|_H < C(T)$$

where $\|\cdot\|_H$ is the operator norm on the Banach space H of bounded linear operators from $S_{-p} \cap \mathcal{E}'(K)$ to S_{-q}.

Proof: a) Clearly for $f \in S$, that $T_tf(x) = Ef(X(t,x))$ is $\mathbb{C}^\infty$ follows from Theorem 2.2 and the dominated convergence theorem. In other words, $T_t : S \to C^\infty$. Also if $\psi \in \mathcal{E}'$ and $p > \frac{d}{4} + \frac{N}{2}, N = order(\psi) + 2d$, then Proposition 4.2 implies $EY_t(\psi) = S_t(\psi) \in S_{-p} \subset S'$. Hence $S_t : \mathcal{E}' \to S'$. We then have for $\phi \in S, \psi \in \mathcal{E}'$,

$$\begin{aligned}\langle S_t(\psi), \phi\rangle &= E\langle Y_t(\psi), \phi\rangle = E\langle \psi, X_t(\phi)\rangle \\ &= E\langle \psi, \phi'(\phi(X(t,\cdot)))\rangle = \langle \psi, \phi' E(\phi(X(t,\cdot)))\rangle \\ &= \langle \psi, T_t(\phi)\rangle\end{aligned}$$

where $\phi' \in C_c^\infty, \phi' = 1$ on support of ψ. Here the last but one equality follows from the fact that $E\phi'(\phi(X(t,\cdot))) = \phi' E\phi(X(t,\cdot))$, and the fact proved below that $E\|\phi'\phi(X(t,\cdot))\|_p \leq E\|\phi'(\phi(X(t,\cdot))\|_{[p]+1} < \infty$ for every $p > 0$ and hence $E\phi'(\phi(X(t,\cdot))$ belongs to S_p for every $p > 0$ and in particular belongs to S. This completes the proof of part a).

b) Without loss of generality, we may assume p is an integer (since $S_{-p} \subseteq S_{-([p]+1)}$, where $[p]$ = greatest integer $\leq p$). Note that $\psi \in S_{-p} \cap \mathcal{E}'(K)$ implies order $\psi \leq 2p$ - this follows from the fact that if p is an integer and if support of ϕ is contained in K then (see [17])

$$\|\phi\|_p^2 \leq C_1 \sum_{|\alpha|+|\beta|\leq 2p} \|x^\alpha \partial^\beta(\phi)\|_0^2 \leq C_2 \|\phi\|_p^2.$$

Hence if q is as in the statement of the theorem and N = order of $\psi + 2d$, then $q > \frac{d}{4} + \frac{N}{2}$, which in turn implies, by Proposition 4.2, that $E\|Y_t(\psi)\|_{-q}^2 < \infty$. In particular, $S_t(\psi) \in S_{-q}$ and hence it suffices to show that there exists $C = C(T,p,K) > 0$ such that for $\psi \in S_{-p} \cap \mathcal{E}'(K), \phi \in S$,

$$|\langle S_t(\psi), \phi\rangle| \leq C\|\psi\|_{-p}\|\phi\|_p.$$

Note that the left hand side above is given by

$$\begin{aligned} |\langle S_t(\psi), \phi\rangle| &= |E\langle Y_t(\psi), \phi\rangle| = |E\langle \psi, X_t(\phi)\rangle| \\ &= |E\langle \psi, \phi' X_t(\phi)\rangle| \end{aligned}$$

where $\phi' \in C_c^\infty, \phi' = 1$ on K. Hence from the above equality we get

$$|\langle S_t(\psi), \phi\rangle| \leq \|\psi\|_{-p}(E\|\phi' X_t(\phi)\|_p^2)^{\frac{1}{2}}.$$

Since (see [17])

$$\|\phi' X_t(\phi)\|_p^2 \leq C \sum_{|\alpha|+|\beta|\leq 2p} \|x^\alpha \partial^\beta(\phi' X_t(\phi))\|_0^2$$

it suffices to show that for $|\alpha| + |\beta| \leq 2p$, there exists a constant $C' > 0$ depending only on T, p and K such that

$$E\|x^\alpha \partial^\beta(\phi' X_t(\phi))\|_0^2 \leq C'\|\phi\|_p^2.$$

Let the support of $\phi' = K'$. We first compute the expression inside the expectation sign in the right hand side above for $\omega \notin \tilde{N}$ where $\tilde{N}$ is as in equation (3.3) :

$$\begin{aligned}
\|x^\alpha \partial^\beta (\phi' X_t(\omega)(\phi))\|_0^2 &= \int_{K'} |x^\alpha \partial^\beta (\phi'(x)\varphi(X(t,x,\omega))|^2 dx \\
&= \int_{K'} |x^\alpha \sum_{|\gamma|+|\gamma'|=|\beta|} \partial^\gamma \phi'(x)\ \partial^{\gamma'} \varphi(X(t,x,\omega))|^2 dx \\
&= \int_{K'} |x^\alpha \sum_{|\gamma|+|\gamma'|=|\beta|} \partial^\gamma \phi'(x) \sum_{|\gamma_1'|,|\gamma_2'|\le|\gamma'|} \\
&\quad P_{\gamma_1'}(\partial^{(\gamma_2')}(X))(t,x,\omega)\partial^{\gamma_1'}\varphi(X(t,x,\omega))|^2 dx
\end{aligned}$$

where $P_{\gamma_1'}(x_1,\dots x_d)$ is a polynomial as in (3.3)and for a multi index $\alpha = (\alpha_1,\dots\alpha_d)$, we use the notation $\partial^{(\alpha)}(X)(t,x,\omega) = (\partial^{\alpha_1}X_1(t,x,\omega),\dots \partial^{\alpha_d}X_d(t,x,\omega))$. Using the change of variable $y = X(t,x,\omega)$, the integral in the last equality above is

$$\begin{aligned}
&\le \sum_{|\gamma|+|\gamma'|=|\beta|} \sum_{|\gamma_1'|,|\gamma_2'|\le|\gamma'|} \int_{K_t'} |(X^{-1}(t,x,\omega))^\alpha \partial^\gamma \phi'(X^{-1}(t,x,\omega)) \\
&P_{\gamma'}(\partial^{(\gamma_2')}(X)(t,X^{-1}(t,x,\omega),\omega))\partial^{\gamma_1'}\phi(x)|^2\ |det(\partial X(t,\omega)^{-1})(X^{-1}(t,x,\omega))|\ dx.
\end{aligned}$$

where $\partial X(t,\omega)$ is the Jacobian of $x \to X(t,x,\omega)$, $\partial X(t,\omega)^{-1}$ is the inverse of $\partial X(t,\omega)$ and $K_t'(\omega) = X(t,K',\omega)$ is the image of K' under the map $X(t,.,\omega)$. Hence from the above, we get

$$\|x^\alpha \partial^\beta (\phi' X_t(\omega)(\phi))\|_0^2 \ \le\ C\ \alpha_1(t)\alpha_2(t) \sum_{|\gamma'|\le|\beta|} \int |\partial^{\gamma'}\phi(x)|^2\ dx$$

for some constant C depending only on p and K' , where

$$\begin{aligned}
\alpha_1(t,\omega) &= \max_{|\beta|\le 2p} \max_{|\gamma_1'|,|\gamma_2'|\le|\beta|} \sup_{x\in K'} |P_{\gamma_1'}(\partial^{(\gamma_2')}(X))(t,x,\omega)|^2 \\
\alpha_2(t,\omega) &= \sup_{x\in K'} |(det(\partial X)^{-1}(t,x,\omega))|.
\end{aligned}$$

Summing over α and β with $|\alpha|+|\beta| \le 2p$, we get

$$\|\phi' X_t(\omega)(\phi))\|_p^2 \ \le\ C\ \alpha_1(t)\alpha_2(t)\|\phi\|_p^2$$

Hence

$$
\begin{aligned}
E\|\phi' X_t(\phi))\|_p^2 &\leq C\|\phi\|_p^2 E(\alpha_1(t)\alpha_2(t)) \\
&\leq C\|\phi\|_p^2 \sup_{t\leq T} \; (E(\alpha_1(t))^2)^{1/2}(E(\alpha_2(t))^2)^{1/2} \\
&\leq C\|\phi\|_p^2
\end{aligned}
$$

for some constant C that changes from line to line, but depends only on p, T, and K. Note that we have used Theorem 2.2 in the last inequality. It now follows that

$$|\langle S_t(\psi), \phi\rangle| \leq C\|\psi\|_{-p}\|\phi\|_p$$

and this completes the proof .

□

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