A multiplier theorem for the sublaplacian on the Heisenberg group

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Abstract. A multiplier theorem for the sublaplacian on the Heisenberg group is proved using Littlewood–Paley–Stein theory of $g$-functions.

Keywords. Multiplier theorem; sublaplacian; Heisenberg group; Littlewood–Paley–Stein theory; $g$-functions.

1. Introduction

Consider the Heisenberg group $H_n$ and the sublaplacian $\mathcal{L}$ on $H_n$. $\mathcal{L}$ is a formally non-negative hypoelliptic differential operator which has a unique self-adjoint extension to $L^2(H_n)$. If $\varphi$ is a function defined on $\mathbb{R}$ then using spectral theorem one can define the operator $\varphi(\mathcal{L})$. If $\varphi$ is a bounded function, then $\varphi(\mathcal{L})$ will be bounded on $L^2(H_n)$. In the same spirit one likes to find sufficient conditions on $\varphi$ so that the operator $\varphi(\mathcal{L})$ will be bounded on $L^p(H_n)$.

This problem was studied by Mauceri [4] and the following result was proved. If the function $\varphi$ is $n+3$ times differentiable and satisfies the estimate $|\varphi^{(k)}(t)| \leq C(1+|t|)^{-k}$, $k = 0, 1, \ldots, (n+3)$, then $\varphi(\mathcal{L})$ is bounded operator on $L^p(H_n)$ for all $1 < p < \infty$.

This result was proved using the theory of singular integrals on homogeneous spaces developed by Coifman and Weiss [1]. Later Mauceri improved the above result replacing the smoothness condition on $\varphi$ by a fractional order condition of the order $s > n+2$ (see [5]). Here we propose to give a different proof of the multiplier theorem. We prove:

Theorem. Let $\varphi$ be $v$ times differentiable and satisfies $|\varphi^{(k)}(t)| \leq C(1+|t|)^{-k}$ for $k = 0, 1, \ldots, v$ where $v = n+2$ if $n$ is even and $v = n+3$ if $n$ is odd. Then $\varphi(\mathcal{L})$ is a bounded operator on $L^p(H_n)$, $1 < p < \infty$.

Our proof of this theorem is based on Littlewood–Paley–Stein theory of $g_s$ and $g_s^+$ functions. We adapt this method which was originally employed by Stein [6] to prove the Hormander–Mihlin multiplier theorem for the Fourier transform, to the present case. The same technique was successfully employed by Strichartz [7] and by the author [9], [10] to prove some multiplier theorems. One good thing about this approach is that the proof is simple and also we get a sharper result when $n$ is even.
2. Preliminaries

The main reference for this section is [3]. See also [4]. The $(2n+1)$-dimensional Heisenberg group $H_n$ is the nilpotent Lie group whose underlying manifold is $\mathbb{C}^n \times \mathbb{R}$. The group structure is given by

$$(z,t)(\xi,s) = (z + \xi, t + s + 2 \text{Im} z \cdot \bar{\xi})$$

where $t, s \in \mathbb{R}$ and $z, \xi \in \mathbb{C}^n$. The Haar measure on $H_n$ is simply the Lebesgue measure $dz ds$ on $\mathbb{C}^n \times \mathbb{R}$. For $w = (z,s)$ the homogeneous norm $|w|$ is defined by $|w|^2 = |z|^4 + s^2$.

We next recall the definition of the Fourier transform on $H_n$. The infinite dimensional representations of $H_n$ are parametrized by $\mathbb{R} \setminus \{0\}$. If $\lambda \neq 0$, then all the representations $\pi_{\lambda}$ can be realized on the same Hilbert space $L^2(\mathbb{R}^n)$. For $(z,s) \in H_n$, $\pi_{\lambda}(z,s)$ is the operator acting on $L^2(\mathbb{R}^n)$ by the prescription

$$\pi_{\lambda}(z,s)\varphi(\xi) = \exp(i\lambda s) \exp[i2\lambda(2\xi \cdot x - x) \cdot y] \varphi(\xi - x),$$

where $z = x + iy$ and $\xi \in \mathbb{R}^n$.

The Fourier transform $\hat{f}$ of an $L^1$ function $f$ on $H_n$ is then the operator valued function

$$\hat{f}(\lambda) = \int_{H_n} f(w) \pi_{\lambda}(w) dw.$$  

(3)

Then we have the following Plancherel formula:

$$\|f\|^2_2 = \frac{2^{n-1}}{n+1} \int |\lambda|^n \|\hat{f}(\lambda)\|^2_{HS} d\lambda,$$

(4)

where $\|\cdot\|_{HS}$ is the Hilbert-Schmidt norm. We also have an inversion formula

$$f(w) = \int \text{tr}(\pi_{\lambda}(w)^* \hat{f}(\lambda)) |\lambda|^n d\lambda,$$

(5)

where $\text{tr}$ is the canonical semifinite trace.

For each $\lambda \neq 0$ we can select an orthonormal basis for $L^2(\mathbb{R}^n)$. Let $\Phi_{\lambda}(x) = (2|\lambda|)^{n/2} \Phi_{\lambda}(2|\lambda| \cdot x)$ where $\Phi_{\lambda}$ are the Hermite functions on $\mathbb{R}^n$. Then $\{\Phi_{\lambda}\}$ is an orthonormal basis for $L^2(\mathbb{R}^n)$. Let $P_N(\lambda)$ denote the projection of $L^2(\mathbb{R}^n)$ onto the eigenspace spanned by $\{\Phi_{\lambda}; |\lambda| = N\}$. Using these operators $P_N(\lambda)$ we can write the Fourier transform of a zonal function in a simple way.

Let $f(z,s) = f(|z|,s)$ be a zonal function and $\hat{f}(z,\lambda)$ be the Fourier transform in the $s$-variable.

$$\hat{f}(z,\lambda) = \int \exp(i\lambda s) f(z,s) ds.$$  

(6)

Define $R_N(\lambda, f)$ by the formula

$$R_N(\lambda, f) = C_N \frac{N!}{(N + n - 1)!} \int_0^{\infty} \hat{f}(r, \lambda) L_N^{-1}(2|\lambda|r^2) \exp(-|\lambda|r^2) r^{2n-1} dr,$$

(7)
Multiplier theorem

where $L_N^{-1}$ are the Laguerre polynomials of type $(n-1)$. Then one has

$$\hat{f}(\lambda) = \sum_{N=0}^{\infty} R_N(\lambda, f) P_N(\lambda).$$

(8)

And the Plancherel formula takes the form

$$\|f\|_2 = \frac{2^{n-1}}{\pi} \int \sum_{N=0}^{\infty} |R_N(\lambda, f)|^2 \frac{(N+n-1)!}{N!} |\lambda|^n d\lambda.$$

(9)

On $H_n$ consider the following left invariant vector fields.

$$Z_j = \frac{\partial}{\partial \bar{z}_j} + i \bar{z}_j \frac{\partial}{\partial t}, \quad \bar{Z}_j = \frac{\partial}{\partial z_j} - iz_j \frac{\partial}{\partial t}.$$  

(10)

The sublaplacian $\mathcal{L}$ is then defined by

$$\mathcal{L} = -\frac{1}{2} \sum_{j=1}^{n} (Z_j \bar{Z}_j + \bar{Z}_j Z_j).$$

(11)

Each representation $\pi_j$ determines a Lie algebra representation $d\pi_j$. It can be shown that $d\pi_j(\mathcal{L})$ is a closable operator. Its closure is denoted by $H(\lambda)$ and it has the following spectral decomposition:

$$H(\lambda) = \sum_{N=0}^{\infty} (2N+n) |\lambda| P_N(\lambda).$$

(12)

For any reasonable function $\varphi$ on $\mathbb{R}$, using spectral theorem, one can define the operator $\varphi(\mathcal{L})$. It can be shown that $\varphi(\mathcal{L})$ is a convolution operator with kernel $k$ i.e. $\varphi(\mathcal{L})f = kf$. The Fourier transform of $k$ is given by

$$\hat{k}(\lambda) = \sum_{N=0}^{\infty} \varphi((2N+n) |\lambda|) P_N(\lambda).$$

(14)

All these things will be made use of in the following sections.

3. Littlewood–Paley–Stein theory on $H_n$

In [2] Folland has shown that the sublaplacian $\mathcal{L}$ generates a contraction semigroup $T^t$ which satisfies all the conditions required to develop a Littlewood–Paley–Stein theory (see [6]). As in Stein [6] we define, for each positive integer $k$, the following functions

$$\langle g_k(f, w) \rangle^2 = \int_0^\infty t^{2k-1} |\partial_t^k T^t f(w)|^2 dt$$

(15)

$$\langle g_k^* (f, w) \rangle^2 = \int_{H_n} \int_0^\infty t^{-n} (1 + t^{-2} |v|^4)^{-k} |\partial_v^k T^t f(v^{-1} w)|^2 dt dv.$$ 

(16)

For these functions we will prove the following theorem.
Theorem 3.1. (i) For \( k \geq 1 \), \( \|g_k(f)\|_2 = 2^{-k} \|f\|_2 \).
(ii) For \( 1 < p < \infty \), \( C_1 \|f\|_p \leq \|g_k(f)\|_p \leq C_2 \|f\|_p \).
(iii) If \( k > (n+1)/2 \) and \( p > 2 \), then \( \|g_k^+(f)\|_p \leq C \|f\|_p \).

Proof. The inequality \( \|g_k(f)\|_p \leq C_2 \|f\|_p \) follows from the general theory. The reverse inequality can be easily deduced once we have (i). When \( k > (n+1)/2 \), the function \((1 + |\lambda|^4)^{-k}\) is integrable and hence one can prove (iii) using (i). This is routine and well known. So, it remains to prove (i).

We prove (i) when \( k = 1 \). The case \( k > 1 \) is similar. From the definition it follows that

\[
\|g_1(f)\|_2^2 = \int_{0}^{\infty} \int_{\mathbb{R}^n} t|\partial_t T^t f(w)|^2 \, dw \, dt.
\]

In view of the Plancherel formula (4) the integral becomes

\[
\int_{\mathbb{R}^n} |\partial_t T^t f(w)|^2 \, dw = \frac{2^{n+1}}{\pi^{n+1}} \int_{\mathbb{R}^n} |\lambda|^n \|\partial_t (T^t f)(\lambda)\|_{HS}^2 \, d\lambda.
\]

Since \( T^t f = \exp(-t\mathcal{L}) f \), we see that

\[
(\partial_t T^t f)(\lambda) = -H(\lambda) \exp(-tH(\lambda)) \hat{f}(\lambda)
\]

and hence its squared Hilbert-Schmidt norm is given by the expression

\[
\sum_{\alpha} ((2|\alpha|+n)|\lambda|)^2 \exp(-2t(2|\alpha|+n)|\lambda|) \langle \Phi_\alpha^+, \hat{f}(\lambda)^* \hat{f}(\lambda) \Phi_\alpha^+ \rangle.
\]

If we use this in (18) and integrate with respect to \( t \, dt \), we will get

\[
\|g_1(f)\|_2^2 = 2^{-2} \frac{2^{n+1}}{\pi^{n+1}} \int |\lambda|^n \|\hat{f}(\lambda)\|_{HS}^2 \, d\lambda.
\]

And this proves that \( \|g_1(f)\|_2 = 2^{-1} \|f\|_2 \).

4. The multiplier theorem

Let us set \( Mf = \varphi(\mathcal{L}) f \). To prove the multiplier theorem what we need is the following pointwise inequality.

\[
g_{k+1}(Mf) \leq C g_k^+(f)
\]

for some integer \( k > (n+1)/2 \). For then the multiplier theorem for \( p > 2 \) will follow immediately from Theorem 3.1. For \( p < 2 \) one can use duality to conclude that \( M \) is bounded on \( L^p(H^n) \).

So, we proceed to prove the inequality (21). Let us set \( u_t = T^t f \), \( U_i = T^i(Mf) \). Then it is easy to see that

\[
U_{t+s}(w) = (G_t * u_s)(w)
\]

where the Fourier transform of \( G_t \) is given by

\[
\hat{G}_t(\lambda) = \sum_{N=0}^{\infty} \exp(-2N+n|\lambda|t) \varphi((2N+n)|\lambda|) F_n(\lambda).
\]

(23)
Multiplier theorem

Differentiating (22) k times with respect to t and once with respect to s and putting s = t we obtain

\[ \partial_t^{k+1} T^{2t}(Mf) = F_t \ast \partial_t T^t f, \]  

(24)

where the Fourier transform of \( F_t \) is given by

\[ \hat{F}_t(\lambda) = (-1)^k \sum_{N=0}^{\infty} \exp(-(2N + n)|\lambda|t)(2N + n)^k |\lambda|^k \varphi((2N + n)|\lambda|)P_N(\lambda). \]

(25)

Therefore, we have

\[ |\partial_t^{k+1} T^{2t}(Mf)(w)| \leq \int |F_t(v)||\partial_t T^tf(v^{-1}w)| dv. \]

Applying Cauchy–Schwarz inequality

\[ |\partial_t^{k+1} T^{2t}(Mf)(w)|^2 \leq A_t B_t(w), \]

(26)

where we have written

\[ A_t = \int |F_t(v)|^2 (1 + t^{-2}|v|^4)^k dv \]

\[ B_t(w) = \int (1 + t^{-2}|v|^4)^{-k}|\partial_t T^t(v^{-1}w)|^2 dv. \]

(27)

Now to complete the proof we need the estimate of the following Lemma.

Lemma. Under the hypothesis of the theorem the estimate \( A_t \leq C t^{-n-2k-1} \) is valid when \( k \) is the smallest integer greater than \((n + 1)/2\).

Assuming the lemma for a moment it is easy to establish inequality (21). Indeed, from (26) we have

\[ |\partial_t^{k+1} T^{2t}(Mf)(w)|^2 \leq C t^{-n-2k-1} B_t(w). \]

Integrating this against \( t^{2k+1} \) we get

\[ g_{k+1}(Mf,w) \leq C g^2(f,w). \]

This completes the proof of the multiplier theorem modulo the above lemma.

5. Proof of the Lemma

To prove the Lemma let us write

\[ I = \int_{|w|<\sqrt{t}} |F_t(w)|^2 (1 + t^{-2}|w|^4)^k dw \]

\[ J = \int_{|w|>\sqrt{t}} |F_t(w)|^2 (1 + t^{-2}|w|^4)^k dw. \]

(28)

(29)
Estimating the integral $I$ is easy. We note that since $|w| \leq \sqrt{t}$

$$I \leq C \int |F_t(w)|^2 \, dw$$

and hence in view of Plancherel formula

$$I \leq C \int |\lambda|^n \left( \sum_{N=0}^{\infty} \frac{(2N + n)^2 |\lambda|^{2k} \exp \left[-2|\lambda|/(2N + n)\right]}{N!} \right)^{(N + n - 1)!} \, d\lambda$$

$$\leq Ct^{-n-2k-1} \left( \sum_{N=0}^{\infty} (2N + n)^{-2} \right) \leq Ct^{-n-2k-1}.$$

This proves the estimate for the integral $I$.

Next consider $J$. Let us write $w = (z, s)$. We observe that

$$J \leq Ct^{-2k} \int \int (s^2 + |z|^4)^k |F_t(z, s)|^2 \, dz \, ds$$

$$= Ct^{-2k} \int \int (is - |s|^2)^k |F_t(z, s)|^2 \, dz \, ds. \quad (30)$$

If we can show that the integral in (30) is bounded by $t^{-n-1}$ then we are done. If we write the Fourier transform of $G = (is - r^2)^k F_t(z, s)$ in the form

$$\hat{G}(\lambda) = \sum_{N=0}^{\infty} R_N(\lambda, (is - |z|^2)^k F_t) P_N(\lambda)$$

then we need to show that

$$\int \sum_{N=0}^{\infty} |R_N(\lambda, (is - r^2)^k F_t)|^2 \frac{(N + n - 1)!}{N!} |\lambda|^n \, d\lambda \leq Ct^{-n-1} \quad (31)$$

where we have set $|z|^2 = r^2$.

Let us write

$$\psi(N, \lambda) = (-1)^k (2N + n)^k |\lambda|^k \exp \left[-(2N + n)|\lambda|\right] \varphi((2N + n)|\lambda|)$$

so that $R_N(\lambda, F_t) = \psi(N, \lambda)$. We define $\psi_k(N, \lambda)$ to be $R_N(\lambda, (is - r^2)^k F_t)$. Then the following estimate is valid.

**Lemma 5.1.** Under the hypothesis of the theorem there is an $\varepsilon > 0$ such that

$$|\psi_k(N, \lambda)| \leq C \exp \left[-\varepsilon(2N + n)|\lambda|\right]. \quad (32)$$

If we use (32) in (29) then the estimate $J \leq t^{-n-2k-1}$ is immediate. So we proceed to prove Lemma 5.1.

Recall the definition of $R_N(\lambda, f)$ for a zonal function $f$.

$$R_N(\lambda, f) = C_\ast \frac{N!}{(N + n - 1)!} \int_0^{\infty} \tilde{f}(r, \lambda) L_{N-1}^{-1} (2|\lambda|r^2) \exp(-|\lambda|r^2) r^{2n-1} \, dr, \quad (33)$$

where $\tilde{f}(r, \lambda)$ is the Euclidean Fourier transform of $f$ in the $s$ variable. We will prove
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(32) when \( \lambda > 0 \). The case \( \lambda < 0 \) is completely similar.

Since \((\mathrm{isf}) (r, \lambda) = (d/d\lambda) \tilde{f}(r, \lambda)\) we obtain

\[
\begin{align*}
R_N(\lambda, \text{isf}) = & \frac{d}{d\lambda} R_N(\lambda, f) - C_n \frac{N!}{(N + n - 1)!} \int_{0}^{\infty} \tilde{f}(r, \lambda) \\
& \times \frac{d}{d\lambda} \left\{ L_n^{n-1}(2\lambda r^2) \exp(-\lambda r^2) \right\} r^{2n-1} dr.
\end{align*}
\]

Now

\[
\frac{d}{d\lambda} L_n^{n-1}(2\lambda r^2) \exp(-\lambda r^2)
= 2r^2 \frac{d}{dr} L_n^{n-1}(2\lambda r^2) \exp(-\lambda r^2) - r^2 L_n^{n-1}(2\lambda r^2) \exp(-\lambda r^2).
\]

Using the recursion formula (see [8])

\[
r \frac{d}{dr} L_n^{n-1}(r) = NL_n^{n-1}(r) - (N + n - 1) L_{n-1}^{n-1}(r)
\]

a simple calculation shows that

\[
R_N(\lambda, \text{isf}) = \frac{d}{d\lambda} R_N(\lambda, f) - \frac{N}{\lambda} (R_N(\lambda, f) - R_{N-1}(\lambda, f)) + R_N(\lambda, r^2 f).
\]

Thus we have obtained the formula

\[
\psi_1(N, \lambda) = \frac{\partial \psi}{\partial \lambda} - \frac{N}{\lambda} (\psi(N, \lambda) - \psi(N - 1, \lambda)).
\]

(35)

Since \( \psi(N, \lambda) = \psi((2N + n)\lambda) \) we can write (35) in the form

\[
\psi_1(N, \lambda) = \frac{1}{2} \frac{\partial \psi}{\partial N} + \frac{N}{\lambda} \left( \frac{\partial \psi}{\partial N} - \Delta \psi \right).
\]

(36)

where \( \Delta \psi(N, \lambda) = \psi(N, \lambda) - \psi(N - 1, \lambda) \). Define the operators \( S, D \) and \( T \) by

\[
S\psi = \frac{\partial \psi}{\partial N}, \quad D\psi = \frac{\partial \psi}{\partial N} - \Delta \psi, \quad T\psi = ND\psi.
\]

So, we have

\[
\psi_1(N, \lambda) = \lambda^{-1} \left( \frac{n}{2} S + T \right) \psi(N, \lambda).
\]

(37)

From this formula we can conclude that

\[
\psi_k(N, \lambda) = \lambda^{-k} \sum_{i+j+m=k} a_{ijm} S^i T^j S^m \psi(N, \lambda).
\]

(38)

Now we observe that \( S^m \psi(N, \lambda) = \psi^{(m)}((2N + n)\lambda)(2\lambda)^m \) and by hypothesis of the theorem \( S^m \) in essence brings a factor \( (2N + n)^{-m} \). We will show that \( T^j \) also does
the same thing. Then each term in the sum (38) will behave like \( \lambda^{-k}(2N + n)^{-k}\psi(N, \lambda) \). Recalling the definition of \( \psi(N, \lambda) \) we see that
\[
|\psi_k(N, \lambda)| \leq C \exp[-\epsilon(2N + n)\lambda t]
\]
as desired.

For the operators \( T^j \) the following formula is valid.

Lemma 5.2.
\[
T^j\psi = \sum C_{pqm} N^p D^q(\Delta^m \psi)
\]
where the sum is extended over all \( p, q, m \) satisfying the relation \( j + p \leq 2q + m \leq 2j \).

Proof. We prove this lemma by induction. We first observe that from the definition of \( T \), the lemma is trivially valid for \( j = 1 \). Now assume the lemma true for some \( j \) and consider \( T^{j+1} \psi \)
\[
T^{j+1} \psi = \sum C_{pqm} N D(N^p D^q(\Delta^m \psi))
\]
where \( j + p \leq 2q + m \leq 2j \). We need a formula for \( D(N^p D\psi) \).

We claim that
\[
D(N^p D\psi) = N^p D^2 \psi + \sum_{i=0}^{p-1} a_i N^i D(\Delta \psi) + \sum_{i=0}^{p-2} b_i N^i D\psi.
\]
(40)

Assuming the claim for a moment we have
\[
T^{j+1} \psi = \sum_{p, q, m} C_{pqm} N^{p+1} D^{q+1}(\Delta^m \psi) + \sum_{p, q, m} C_{pqm} \sum_{i=0}^{p-1} a_i N^{i+1} D^q(\Delta^{m+1} \psi)
\]
\[
+ \sum_{p, q, m} C_{pqm} \sum_{i=0}^{p-2} b_i N^{i+1} D^q(\Delta^m \psi).
\]

From this formula it is clear that \( T^{j+1} \psi \) is of the desired form.

To prove the claim we first observe that
\[
\Delta(\psi(N)) = \Delta \psi(N) \psi(N) + \varphi(N) \Delta \psi(N).
\]
(41)

In view of this formula
\[
\Delta(N^p D\psi) = \Delta(N^p D\psi) + (N - 1) D(\Delta \psi).
\]
(42)

We also have
\[
\Delta(N^p) = N^p - (N - 1)^p = p N^{p-1} - \sum_{i=0}^{p-2} b_i N^i
\]
(43)
\[
(N - 1)^p D(\Delta \psi) = N^p D(\Delta \psi) - \sum_{i=0}^{p-1} a_i N^i D(\Delta \psi)
\]
(44)
\[
\frac{\partial}{\partial N} (N^p D\psi) = p N^{p-1} D\psi + N^p D\left( \frac{\partial \psi}{\partial N} \right).
\]
(45)
From (42)–(45) it follows that

\[ D(N^p D\psi) = N^p D^2 \psi + \sum_{i=0}^{p-1} a_i N^i D(\Delta \psi) + \sum_{i=0}^{p-2} b_i N^i D \psi. \]  

(46)

This proves the claim.

Finally we will show that the action of $T^j$ has the desired properties. We have

\[ T^j \psi = \sum C_{p,q,m} N^p D^q (\Delta^m \psi), \]  

(47)

where $p + j \leq 2q + m \leq 2j$. Now using Taylor's formula with integral form of remainder we can write

\[ D\psi(N) = \int_0^1 t \psi''(N - 1 + t, \lambda) dt, \]  

(48)

where the primes stand for the derivatives with respect to $N$. From (48) it is clear that the action of $D$ is to bring down the factor $N^{-2}$. An iteration will show that $D^q$ will bring down the factor of $N^{-2q}$ when applied to $\psi$. Since $\Delta^m \psi$ brings down $N^{-m}$ the formula (47) shows that $T^j$ acting on $\psi$ brings down the factor

\[ \sum C_{p,q,m} N^p N^{-2q-m}. \]

Since $p + j \leq 2q + m$, essentially $T^j$ brings down a factor of $N^{-j}$ as required.

References


