Proc. Indian Acad. Sci. (Math. Sci.), Vol. 101, No. 3, December 1991, pp. 169–177.

A multiplier theorem for the sublaplacian on the Heisenberg group

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MS received 24 January 1990

Abstract. A multiplier theorem for the sublaplacian on the Heisenberg group is proved using Littlewood–Paley–Stein theory of g-functions.

Keywords. Multiplier theorem; sublaplacian; Heisenberg group; Littlewood–Paley–Stein theory; g-functions.

1. Introduction

Consider the Heisenberg group H_n and the sublaplacian $\mathscr{L}$ on H_n. $\mathscr{L}$ is a formally non-negative hypoelliptic differential operator which has a unique self-adjoint extension to $L^2(H_n)$. If φ is a function defined on $\mathbb{R}$ then using spectral theorem one can define the operator $\varphi(\mathscr{L})$. If φ is a bounded function, then $\varphi(\mathscr{L})$ will be bounded on $L^2(H_n)$. In the same spirit one likes to find sufficient conditions on φ so that the operator $\varphi(\mathscr{L})$ will be bounded on $L^p(H_n)$.

This problem was studied by Mauceri [4] and the following result was proved.

If the function φ is $n+3$ times differentiable and satisfies the estimate $|\varphi^{(k)}(t)| \leqslant C(1+|t|)^{-k}, k=0,1,\ldots(n+3)$, then $\varphi(\mathscr{L})$ is bounded operator on $L^p(H_n)$ for all $1<p<\infty$.

This result was proved using the theory of singular integrals on homogeneous spaces developed by Coifman and Weiss [1]. Later Mauceri improved the above result replacing the smoothness condition on φ by a fractional order condition of the order $s>n+2$ (see [5]). Here we propose to give a different proof of the multiplier theorem. We prove:

Theorem. *Let φ be ν times differentiable and satisfies $|\varphi^{(k)}(t)| \leqslant C(1+|t|)^{-k}$ for $k=0,1,\ldots \nu$ where $\nu=n+2$ if n is even and $\nu=n+3$ if n is odd. Then $\varphi(\mathscr{L})$ is a bounded operator on $L^p(H_n)$, $1<p<\infty$.*

Our proof of this theorem is based on Littlewood–Paley–Stein theory of g_k and g_k^* functions. We adapt this method which was originally employed by Stein [6] to prove the Hormander–Mihlin multiplier theorem for the Fourier transform, to the present case. The same technique was successfully employed by Strichartz [7] and by the author [9], [10] to prove some multiplier theorems. One good thing about this approach is that the proof is simple and also we get a sharper result when n is even.

2. Preliminaries

The main reference for this section is [3]. See also [4]. The $(2n+1)$-dimensional Heisenberg group H_n is the nil potent Lie group whose underlying manifold is $\mathbb{C}^n \times \mathbb{R}$. The group structure is given by

$$(z,t)(\xi,s) = (z+\xi, t+s+2\,\mathrm{Im}\, z\cdot\bar{\xi}) \tag{1}$$

where $t, s \in \mathbb{R}$ and $z, \xi \in \mathbb{C}^n$. The Haar measure on H_n is simply the Lebesgue measure $\mathrm{d}z\mathrm{d}s$ on $\mathbb{C}^n \times \mathbb{R}$. For $w=(z,s)$ the homogeneous norm $|w|$ is defined by $|w|^4 = |z|^4 + s^2$.

We next recall the definition of the Fourier transform on H_n. The infinite dimensional representations of H_n are parametrized by $\mathbb{R}\backslash\{0\}$. If $\lambda \neq 0$, then all the representations π_λ can be realized on the same Hilbert space $L^2(\mathbb{R}^n)$. For $(z,s)\in H_n$, $\pi_\lambda(z,s)$ is the operator acting on $L^2(\mathbb{R}^n)$ by the prescription

$$\pi_\lambda(z,s)\varphi(\xi) = \exp(i\lambda s)\exp[i2\lambda(2\xi - x)\cdot y]\varphi(\xi - x), \tag{2}$$

where $z = x+iy$ and $\xi \in \mathbb{R}^n$.

The Fourier transform $\hat{f}$ of an L^1 function f on H_n is then the operator valued function

$$\hat{f}(\lambda) = \int_{H_n} f(w)\,\pi_\lambda(w)\,\mathrm{d}w. \tag{3}$$

Then we have the following Plancherel formula:

$$\|f\|_2^2 = \frac{2^{n-1}}{\pi^{n+1}}\int |\lambda|^n \,\|\hat{f}(\lambda)\|_{HS}^2\,\mathrm{d}\lambda, \tag{4}$$

where $\|\cdot\|_{HS}$ is the Hilbert-Schmidt norm. We also have an inversion formula

$$f(w) = \int \mathrm{tr}(\pi_\lambda(w)^*\hat{f}(\lambda))|\lambda|^n\,\mathrm{d}\lambda, \tag{5}$$

where tr is the canonical semifinite trace.

For each $\lambda\neq 0$ we can select an orthonormal basis for $L^2(\mathbb{R}^n)$. Let $\Phi_\alpha^\lambda(x) = (2|\lambda|^{\frac{1}{2}})^{n/2}\,\Phi_\alpha((2|\lambda|^{\frac{1}{2}}x))$ where Φ_α are the Hermite functions on $\mathbb{R}^n$.Then $\{\Phi_\alpha^\lambda\}$ is an orthonormal basis for $L^2\{\mathbb{R}^n)$. Let $P_N(\lambda)$ denote the projection of $L^2(\mathbb{R}^n)$ onto the eigenspace spanned by $\{\Phi_\alpha^\lambda : |\alpha| = N\}$. Using these operators $P_N(\lambda)$ we can write the Fourier transform of a zonal function in a simple way.

Let $f(z,s) = f(|z|,s)$ be a zonal function and $\tilde{f}(z,\lambda)$ be the Fourier transform in the s-variable.

$$\tilde{f}(z,\lambda) = \int \exp(i\lambda s) f(z,s)\,\mathrm{d}s. \tag{6}$$

Define $R_N(\lambda,f)$ by the formula

$$R_N(\lambda,f) = C_N\frac{N!}{(N+n-1)!}\int_0^\infty \tilde{f}(r,\lambda)L_N^{n-1}(2|\lambda|r^2)\exp(-|\lambda|r^2)r^{2n-1}\,\mathrm{d}r, \tag{7}$$

where L_N^{n-1} are the Laguerre polynomials of type $(n-1)$. Then one has

$$\hat{f}(\lambda) = \sum_{N=0}^{\infty} R_N(\lambda, f) P_N(\lambda). \tag{8}$$

And the Plancherel formula takes the form

$$\|f\|_2^2 = \frac{2^{n-1}}{\pi^{n+1}} \int \sum_{N=0}^{\infty} |R_N(\lambda, f)|^2 \frac{(N+n-1)!}{N!} |\lambda|^n \, d\lambda. \tag{9}$$

On H_n consider the following left invariant vector fields.

$$Z_j = \frac{\partial}{\partial z_j} + i\bar{z}_j \frac{\partial}{\partial t}, \quad \bar{Z}_j = \frac{\partial}{\partial \bar{z}_j} - iz_j \frac{\partial}{\partial t}. \tag{10}$$

The sublaplacian $\mathscr{L}$ is then defined by

$$\mathscr{L} = -\frac{1}{2} \sum_{j=1}^{n} (Z_j \bar{Z}_j + \bar{Z}_j Z_j). \tag{11}$$

Each representation π_λ determines a Lie algebra representation $d\pi_\lambda$. It can be shown that $d\pi_\lambda(\mathscr{L})$ is a closable operator. Its closure is denoted by $H(\lambda)$ and it has the following spectral decomposition:

$$H(\lambda) = \sum_{N=0}^{\infty} (2N+n)|\lambda| P_N(\lambda). \tag{12}$$

For any reasonable function φ on $\mathbb{R}$, using spectral theorem, one can define the operator $\varphi(\mathscr{L})$. It can be shown that $\varphi(\mathscr{L})$ is a convolution operator with kernel k i.e. $\varphi(\mathscr{L})f = k * f$. The Fourier transform of k is given by

$$\hat{k}(\lambda) = \sum_{N=0}^{\infty} \varphi((2N+n)|\lambda|) P_N(\lambda). \tag{14}$$

All these things will be made use of in the following sections.

3. Littlewood–Paley–Stein theory on H_n

In [2] Folland has shown that the sublaplacian $\mathscr{L}$ generates a contraction semigroup T^t which satisfies all the conditions required to develop a Littlewood–Paley–Stein theory (see [6]). As in Stein [6] we define, for each positive integer k, the following functions

$$(g_k(f, w))^2 = \int_0^\infty t^{2k-1} |\partial_t^k T^t f(w)|^2 \, dt \tag{15}$$

$$(g_k^*(f, w))^2 = \int_{H_n} \int_0^\infty t^{-n} (1 + t^{-2}|v|^4)^{-k} |\partial_t^k T^t f(v^{-1}w)|^2 \, dt \, dv. \tag{16}$$

For these functions we will prove the following theorem.

Theorem 3.1. (i) *For* $k \geqslant 1$, $\|g_k(f)\|_2 = 2^{-k}\|f\|_2$.
(ii) *For* $1 < p < \infty$, $C_1\|f\|_p \leqslant \|g_k(f)\|_p \leqslant C_2\|f\|_p$.
(iii) *If* $k > (n+1)/2$ *and* $p > 2$, *then* $\|g_k^*(f)\|_p \leqslant C\|f\|_p$.

Proof. The inequality $\|g_k(f)\|_p \leqslant C_2\|f\|_p$ follows from the general theory. The reverse inequality can be easily deduced once we have (i). When $k > (n+1)/2$, the function $(1+|v|^4)^{-k}$ is integrable and hence one can prove (iii) using (i). This is routine and well known. So, it remains to prove (i).

We prove (i) when $k = 1$. The case $k > 1$ is similar. From the definition it follows that

$$\|g_1(f)\|_2^2 = \int_0^\infty \int_{H_n} t|\partial_t T^t f(w)|^2 \mathrm{d}w\mathrm{d}t. \tag{17}$$

In view of the Plancherel formula (4) the integral becomes

$$\int_{H_n} |\partial_t T^t f(w)|^2 \mathrm{d}w = \frac{2^{n-1}}{\pi^{n-1}} \int |\lambda|^n \|(\partial_t T^t f)\hat{}(\lambda)\|_{HS}^2 \mathrm{d}\lambda. \tag{18}$$

Since $T^t f = \exp(-t\mathscr{L})f$, we see that

$$(\partial_t T^t f)\hat{}(\lambda) = -H(\lambda)\exp(-tH(\lambda))\hat{f}(\lambda) \tag{19}$$

and hence its squared Hilbert-Schmidt norm is given by the expression

$$\sum_\alpha ((2|\alpha|+n)|\lambda|)^2 \exp(-2t(2|\alpha|+n)|\lambda|)(\Phi_\alpha^\lambda, \hat{f}(\lambda)^*\hat{f}(\lambda)\Phi_\alpha^\lambda). \tag{20}$$

If we use this in (18) and integrate with respect to $t\mathrm{d}t$, we will get

$$\|g_1(f)\|_2^2 = 2^{-2}\frac{2^{n-1}}{\pi^{n+1}} \int |\lambda|^n \|\hat{f}(\lambda)\|_{HS}^2 \mathrm{d}\lambda.$$

And this proves that $\|g_1(f)\|_2 = 2^{-1}\|f\|_2$.

4. The multiplier theorem

Let us set $Mf = \varphi(\mathscr{L})f$. To prove the multiplier theorem what we need is the following pointwise inequality.

$$g_{k+1}(Mf) \leqslant C g_k^*(f) \tag{21}$$

for some integer $k > (n+1)/2$. For then the multiplier theorem for $p > 2$ will follow immediately from Theorem 3.1. For $p < 2$ one can use duality to conclude that M is bounded on $L^p(H_n)$.

So, we proceed to prove the inequality (21). Let us set $u_t = T^t f$, $U_t = T^t(Mf)$. Then it is easy to see that

$$U_{t+s}(w) = (G_t * u_s)(w) \tag{22}$$

where the Fourier transform of G_t is given by

$$\hat{G}_t(\lambda) = \sum_{N=0}^{\infty} \exp(-(2N+n)|\lambda|t)\varphi((2N+n)|\lambda|)P_N(\lambda). \tag{23}$$

Differentiating (22) k times with respect to t and once with respect to s and putting $t=s$ we obtain

$$\partial_t^{k+1} T^{2t}(Mf) = F_t * \partial_t T^t f, \tag{24}$$

where the Fourier transform of F_t is given by

$$\hat{F}_t(\lambda) = (-1)^k \sum_{N=0}^{\infty} \exp(-(2N+n)|\lambda|t)(2N+n)^k |\lambda|^k \varphi((2N+n)|\lambda|) P_N(\lambda). \tag{25}$$

Therefore, we have

$$|\partial_t^{k+1} T^{2t}(Mf)(w)| \leqslant \int |F_t(v)| |\partial_t T^t f(v^{-1}w)| \, dv.$$

Applying Cauchy–Schwartz inequality

$$|\partial_t^{k+1} T^{2t}(Mf)(w)|^2 \leqslant A_t \cdot B_t(w), \tag{26}$$

where we have written

$$A_t = \int |F_t(v)|^2 (1 + t^{-2}|v|^4)^k \, dv$$

$$B_t(w) = \int (1 + t^{-2}|v|^4)^{-k} |\partial_t T^t(v^{-1}w)|^2 \, dv. \tag{27}$$

Now to complete the proof we need the estimate of the following Lemma.

Lemma. Under the hypothesis of the theorem the estimate $A_t \leqslant C\, t^{-n-2k-1}$ *is valid when* k *is the smallest integer greater than* $(n+1)/2$.

Assuming the lemma for a moment it is easy to establish inequality (21). Indeed, from (26) we have

$$|\partial_t^{k+1} T^{2t}(Mf)(w)|^2 \leqslant C t^{-n-2k-1} B_t(w).$$

Integrating this against t^{2k+1} we get

$$g_{k+1}(Mf, w) \leqslant C g_k^*(f, w).$$

This completes the proof of the multiplier theorem modulo the above lemma.

5. Proof of the Lemma

To prove the Lemma let us write

$$I = \int_{|w| \leqslant \sqrt{t}} |F_t(w)|^2 (1 + t^{-2}|w|^4)^k \, dw \tag{28}$$

$$J = \int_{|w| > \sqrt{t}} |F_t(w)|^2 (1 + t^{-2}|w|^4)^k \, dw. \tag{29}$$

Estimating the integral I is easy. We note that since $|w| \leqslant \sqrt{t}$

$$I \leqslant C \int |F_t(w)|^2 \mathrm{d}w$$

and hence in view of Plancherel formula

$$I \leqslant C \int |\lambda|^n \left(\sum_{N=0}^{\infty} (2N+n)^{2k} |\lambda|^{2k} \exp[-2|\lambda|(2N+n)t] \frac{(N+n-1)!}{N!} \right) \mathrm{d}\lambda$$
$$\leqslant Ct^{-n-2k-1}(\Sigma(2N+n)^{-2}) \leqslant Ct^{-n-2k-1}.$$

This proves the estimate for the integral I.

Next consider J. Let us write $w = (z, s)$. We observe that

$$J \leqslant Ct^{-2k} \int\int (s^2 + |z|^4)^k |F_t(z,s)|^2 \mathrm{d}z\,\mathrm{d}s$$
$$= Ct^{-2k} \int\int |(is - |z|^2)^k F_t(z,s)|^2 \mathrm{d}z\,\mathrm{d}s. \tag{30}$$

If we can show that the integral in (30) is bounded by t^{-n-1} then we are done. If we write the Fourier transform of $G = (is - r^2)^k F_t(z,s)$ in the form

$$\hat{G}(\lambda) = \sum_{N=0}^{\infty} R_N(\lambda, (is - |z|^2)^k F_t) P_N(\lambda)$$

then we need to show that

$$\int \sum_{N=0}^{\infty} |R_N(\lambda, (is - r^2)^k F_t)|^2 \frac{(N+n-1)!}{N!} |\lambda|^n \mathrm{d}\lambda \leqslant Ct^{-n-1} \tag{31}$$

where we have set $|z|^2 = r^2$.

Let us write

$$\psi(N, \lambda) = (-1)^k (2N+n)^k |\lambda|^k \exp[-(2N+n)|\lambda| t] \varphi((2N+n)|\lambda|)$$

so that $R_N(\lambda, F_t) = \psi(N, \lambda)$. We define $\psi_k(N, \lambda)$ to be $R_N(\lambda, (is - r^2)^k F_t)$. Then the following estimate is valid.

Lemma 5.1. *Under the hypothesis of the theorem there is an* $\varepsilon > 0$ *such that*

$$|\psi_k(N, \lambda)| \leqslant C \exp[-\varepsilon(2N+n)|\lambda| t. \tag{32}$$

If we use (32) in (29) then the estimate $J \leqslant t^{-n-2k-1}$ is immediate. So we proceed to prove Lemma 5.1.

Recall the definition of $R_N(\lambda, f)$ for a zonal function f.

$$R_N(\lambda, f) = C_n \frac{N!}{(N+n-1)!} \int_0^{\infty} \tilde{f}(r, \lambda)\, L_N^{n-1}(2|\lambda| r^2) \exp(-|\lambda| r^2) r^{2n-1} \mathrm{d}r, \tag{33}$$

where $\tilde{f}(r, \lambda)$ is the Euclidean Fourier transform of f in the s variable. We will prove

(32) when $\lambda > 0$. The case $\lambda < 0$ is completely similar.
Since $(isf)\tilde{}(r, \lambda) = (d/d\lambda)\tilde{f}(r, \lambda)$ we obtain

$$R_N(\lambda, isf) = \frac{d}{d\lambda} R_N(\lambda, f) - C_n \frac{N!}{(N+n-1)!} \int_0^\infty \tilde{f}(r, \lambda) \times \frac{d}{d\lambda}\{L_N^{n-1}(2\lambda r^2)\exp(-\lambda r^2)\} r^{2n-1}\, dr.$$

Now

$$\frac{d}{d\lambda}(L_N^{n-1}(2\lambda r^2)\exp(-\lambda r^2)) = 2r^2 \frac{d}{dr} L_N^{n-1}(2\lambda r^2)\exp(-\lambda r^2) - r^2 L_N^{n-1}(2\lambda r^2)\exp(-\lambda r^2).$$

Using the recursion formula (see [8])

$$r\frac{d}{dr} L_N^{n-1}(r) = N L_N^{n-1}(r) - (N+n-1) L_{N-1}^{n-1}(r) \tag{34}$$

a simple calculation shows that

$$R_N(\lambda, isf) = \frac{d}{d\lambda} R_N(\lambda, f) - \frac{N}{\lambda}(R_N(\lambda, f) - R_{N-1}(\lambda, f)) + R_N(\lambda, r^2 f).$$

Thus we have obtained the formula

$$\psi_1(N, \lambda) \doteq \frac{\partial \psi}{\partial \lambda} - \frac{N}{\lambda}(\psi(N, \lambda) - \psi(N-1, \lambda)). \tag{35}$$

Since $\psi(N, \lambda) = \psi((2N+n)\lambda)$ we can write (35) in the form

$$\psi_1(N, \lambda) = \frac{1}{2}\frac{n}{\lambda}\frac{\partial \psi}{\partial N} + \frac{N}{\lambda}\left(\frac{\partial \psi}{\partial N} - \Delta\psi\right), \tag{36}$$

where $\Delta\psi(N, \lambda) = \psi(N, \lambda) - \psi(N-1, \lambda)$. Define the operators S, D and T by

$$S\psi = \frac{\partial \psi}{\partial N}, \quad D\psi = \frac{\partial \psi}{\partial N} - \Delta\psi, \quad T\psi = ND\psi.$$

So, we have

$$\psi_1(N, \lambda) = \lambda^{-1}\left(\frac{n}{2}S + T\right)\psi(N, \lambda). \tag{37}$$

From this formula we can conclude that

$$\psi_k(N, \lambda) = \lambda^{-k} \sum_{i+j+m=k} a_{ijm} S^i T^j S^m \psi(N, \lambda). \tag{38}$$

Now we observe that $S^m\psi(N, \lambda) = \psi^{(m)}((2N+n)\lambda)(2\lambda)^m$ and by hypothesis of the theorem S^m in essence brings a factor $(2N+n)^{-m}$. We will show that T^j also does the same thing. Then each term in the sum (38) will behave like $\lambda^{-k}(2N+n)^{-k}\psi(N,\lambda)$. Recalling the definition of $\psi(N,\lambda)$ we see that

$$|\psi_k(N,\lambda)| \leqslant C\exp[-\varepsilon(2N+n)\lambda t]$$

as desired.

For the operators T^j the following formula is valid.

Lemma 5.2.

$$T^j\psi = \sum C_{pqm} N^p D^q(\Delta^m \psi)$$

where the sum is extended over all p,q,m *satisfying the relation* $j+p \leqslant 2q+m \leqslant 2j$.

Proof. We prove this lemma by induction. We first observe that from the definition of T, the lemma is trivially valid for $j=1$. Now assume the lemma true for some j and consider $T^{j+1}\psi$

$$T^{j+1}\psi = \sum C_{pqm} ND(N^p D^q(\Delta^m\psi)) \tag{39}$$

where $j+p \leqslant 2q+m \leqslant 2j$. We need a formula for $D(N^p D\psi)$.

We claim that

$$D(N^p D\psi) = N^p D^2\psi + \sum_{i=0}^{p-1} a_i N^i D(\Delta\psi) + \sum_{i=0}^{p-2} b_i N^i D\psi. \tag{40}$$

Assuming the claim for a moment we have

$$T^{j+1}\psi = \sum_{p,q,m} C_{pqm} N^{p+1} D^{q+1}(\Delta^m\psi) + \sum_{p,q,m} C_{pqm} \sum_{i=0}^{p-1} a_i N^{i+1} D^q(\Delta^{m+1}\psi)$$
$$+ \sum_{p,q,m} C_{pqm} \sum_{i=0}^{p-2} b_i N^{i+1} D^q(\Delta^m\psi).$$

From this formula it is clear that $T^{j+1}\psi$ is of the desired form.

To prove the claim we first observe that

$$\Delta(\varphi\psi)(N) = \Delta\varphi(N)\psi(N) + \varphi(N-1)\Delta\psi(N). \tag{41}$$

In view of this formula

$$\Delta(N^p D\psi) = \Delta(N^p)D\psi + (N-1)^p D(\Delta\psi). \tag{42}$$

We also have

$$\Delta(N^p) = N^p - (N-1)^p = pN^{p-1} - \sum_{i=0}^{p-2} b_i N^i \tag{43}$$

$$(N-1)^p D(\Delta\psi) = N^p D(\Delta\psi) - \sum_{i=0}^{p-1} a_i N^i D(\Delta\psi) \tag{44}$$

$$\frac{\partial}{\partial N}(N^p D\psi) = pN^{p-1} D\psi + N^p D\left(\frac{\partial\psi}{\partial N}\right). \tag{45}$$

From (42)–(45) it follows that

$$D(N^p D\psi) = N^p D^2\psi + \sum_{i=0}^{p-1} a_i N^i D(\Delta\psi) + \sum_{i=0}^{p-2} b_i N^i D\psi. \tag{46}$$

This proves the claim.

Finally we will show that the action of T^j has the desired properties. We have

$$T^j\psi = \sum C_{pqm} N^p D^q(\Delta^m \psi), \tag{47}$$

where $p+j \leqslant 2q+m \leqslant 2j$. Now using Taylor's formula with integral form of remainder we can write

$$D\psi(N) = \int_0^1 t\psi''(N-1+t,\lambda)\,\mathrm{d}t, \tag{48}$$

where the primes stand for the derivatives with respect to N. From (48) it is clear that the action of D is to bring down the factor N^{-2}. An iteration will show that D^q will bring down the factor of N^{-2q} when applied to ψ. Since $\Delta^m\psi$ brings down N^{-m} the formula (47) shows that T^j acting on ψ brings down the factor

$$\sum C_{pqm} N^p N^{-2q-m}.$$

Since $p+j \leqslant 2q+m$, essentially T^j brings down a factor of N^{-j} as required.

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