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Multipliers for the Weyl transform and Laguerre expansions

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Abstract. Let P_k denote the projection of $L^2(\mathbb{R}^n)$ onto the kth eigenspace of the operator $(-\Delta+|x|^2)$ and $S_N^\alpha=(1/A_N^\alpha)\sum_{k=0}^N A_{N-k}^\alpha P_k$. We study the multiplier transform T_N^α for the Weyl transform W defined by $W(T_N^\alpha f)=S_N^\alpha W(f)$. Applications to Laguerre expansions are given.

Keywords. Weyl transform; multipliers; twisted convolution; oscillatory singular integrals; Laguerre expansions; summability.

1. Introduction

Let $\hat{f}$ denote the Fourier transform of the function f and let $B=\{\xi:|\xi|\leqslant 1\}$ be the unit ball in $\mathbb{R}^n$. For $\alpha>0$ define the multiplier operator

$$(T_\alpha f)\hat{}(\xi)=(1-|\xi|^2)^\alpha\chi_B(\xi)\hat{f}(\xi). \tag{1}$$

It is well known that these operators are unbounded on $L^p(\mathbb{R}^n)$ for $p\leqslant(2n/(n+1+2\alpha))$ or $p\geqslant(2n/(n-1-2\alpha))$ for $0<\alpha\leqslant((n-1)/2)$. If $\alpha>((n-1)/2)$ then T_α is bounded on $L^p(\mathbb{R}^n)$ for $1\leqslant p\leqslant\infty$. When $\alpha=0$ a celebrated theorem of Fefferman asserts that T_0 is bounded only on $L^2(\mathbb{R}^n)$. But if we consider only radial functions it is known that T_α is bounded for $(2n/(n+1+2\alpha))<p<(2n/(n-1-2\alpha))$ if $0\leqslant\alpha\leqslant((n-1)/2)$.

In this paper we consider a similar problem for the Weyl transform. General multipliers for the Weyl transform have been studied by Mauceri [5]. Here we are interested in multipliers of the form

$$S_N^\alpha=\frac{1}{A_N^\alpha}\sum_{k=0}^N A_{N-k}^\alpha P_k \tag{2}$$

where P_k are the projections onto the kth eigenspace of the Hermite operator $H=-\Delta+|x|^2$. We call such multipliers Cesaro multipliers. Let W denote the Weyl transform which takes functions f on $\mathbb{C}^n$ into bounded operators on $L^2(\mathbb{R}^n)$. We define the multiplier operators T_N^α by setting

$$W(T_N^\alpha f)=S_N^\alpha W(f). \tag{3}$$

We would like to know for what values of α these operators T_N^α will be uniformly bounded on $L^p(\mathbb{C}^n)$. We prove the following theorem.

Theorem 1. i) *When* $\alpha > n-\frac{1}{2}$, T_N^α *is bounded on* $L^p(\mathbb{C}^n)$ *for* $1 \leqslant p \leqslant \infty$. ii) *When* $\alpha = n-\frac{1}{2}$, T_N^α *is bounded on* $L^p(\mathbb{C}^n)$ *for* $1 < p < \infty$. iii) *When* $0 < \alpha < n-\frac{1}{2}$, T_N^α *cannot be bounded unless p lies in the interval* $(4n/(2n+1+2\alpha)) < p < (4n/(2n-1-2\alpha))$. iv) *If we consider only radial functions then* T_N^α *is bounded on* $L^p(\mathbb{C}^n)$ *for* $(4n/(2n+1+2\alpha)) < p < (4n(2n-1-2\alpha))$ *when* $0 \leqslant \alpha \leqslant n-\frac{1}{2}$.

The study of the Weyl multipliers naturally involves the study of expansions in terms of the Laguerre functions $L_N^{n-1}(\frac{1}{2}|z|^2)\exp(-\frac{1}{4}|z|^2)$. Let $v_N^2 = (N!/(N+n-1)!)$ and consider the functions

$$\varphi_N(r) = v_N L_N^{n-1}(\tfrac{1}{2}r^2)\exp(-\tfrac{1}{4}r^2). \tag{4}$$

These functions form an orthonormal system for $L^2(\mathbb{R}_+, r^{2n-1}\,\mathrm{d}r)$. So one would like to study the summability of the series

$$f(r) = \sum_{N=0}^{\infty} R_N(f) L_N^{n-1}(\tfrac{1}{2}r^2)\exp(-\tfrac{1}{4}r^2) \tag{5}$$

where

$$R_N(f) = v_N^2 \int_0^\infty f(r) L_N^{n-1}(\tfrac{1}{2}r^2)\exp(-\tfrac{1}{4}r^2) r^{2n-1}\,\mathrm{d}r.$$

If $C_N^\alpha f$ stand for the Cesaro means of order α of the above series then Theorem 1 can be interpreted as summability results for the Laguerre expansions in the space $L^p(\mathbb{R}_+, r^{2n-1}\,\mathrm{d}r)$. Thus we have the following theorem.

Theorem 2. i) *The partial sums* $C_N f$ *of the series* (1.5) *converges to f in the norm iff* $(4n/(2n+1)) < p < (4n/(2n-1))$. ii) *When* $\alpha > n-\frac{1}{2}$, $C_N^\alpha f$ *converges to f in the norm for all p*, $1 \leqslant p \leqslant \infty$. iii) $\alpha = n-\frac{1}{2}$ *is the critical index in the sense that when* $\alpha < n-\frac{1}{2}$ *there is an* L^1 *function f such that* $C_N^\alpha f$ *will not converge in the norm to f.* iv) *When* $\alpha = n-\frac{1}{2}$, $C_N^\alpha f$ *converges to f whenever* $1 < p < \infty$.

This paper is arranged as follows. In the next section we collect relevant facts about the Weyl transform and express the operators T_N^α as twisted convolution operators. Using that representation one easily proves that they are bounded on $L^p(\mathbb{C}^n)$ when $\alpha > n-\frac{1}{2}$. In §3 we consider the case $\alpha = n-\frac{1}{2}$. In §4, we consider radial functions from which the results about the Laguerre series (Theorem 2) will be deduced.

2. Preliminaries

We first recall the definition of the twisted convolution on $\mathbb{C}^n$. Let $\omega(z,v) = \exp(-(i/2)\operatorname{Im} z\cdot\bar{v})$ and let $\mathrm{d}v\,\mathrm{d}\bar{v}$ stand for the Lebesgue measure on $\mathbb{C}^n$. Then the product

$$fXg(z) = \int_{\mathbb{C}^n} f(z-v)g(v)\omega(z,v)\,\mathrm{d}v\,\mathrm{d}\bar{v} \tag{6}$$

is called the twisted convolution of f and g. Let $B(L^2(\mathbb{R}^n))$ stand for the algebra of bounded linear operators on $L^2(\mathbb{R}^n)$. To define the Weyl transform we consider the operator valued function $W(z)$ taking values in $B(L^2(\mathbb{R}^n))$ defined by

$$W(z)\varphi(\xi) = \exp\{ix\cdot(\tfrac{1}{2}y+\xi)\}\varphi(\xi+y) \tag{7}$$

where $z=x+iy\in\mathbb{C}^n$ and $\xi\in\mathbb{R}^n$. The Weyl transform of a function f is defined by

$$W(f)=\int_{\mathbb{C}^n} f(z)W(z)\,\mathrm{d}z\,\mathrm{d}\bar{z}. \tag{8}$$

Observe that $W(f)$ is a bounded operator on $L^2(\mathbb{R}^n)$.

For results concerning the Weyl transform we refer to the paper of Mauceri [5]. The Weyl transform enjoys many properties of the Fourier transform. For example there is an inversion formula and also a Plancherel theorem. Moreover, Weyl transform of the twisted convolution is the product of the Weyl transforms, i.e.,

$$W(fXg)=W(f)W(g). \tag{9}$$

Given a bounded operator M on $L^2(\mathbb{R}^n)$, one can define a multiplier operator T_M by setting $W(T_Mf)=MW(f)$. Conditions on M have been found in [5] to ensure that T_M is bounded on $L^p(\mathbb{C}^n)$. Here we are interested in multipliers which are special functions of the Hermite operator $H=-\Delta+|x|^2$. The n-dimensional Hermite functions $\{\Phi_\alpha\}$ are the eigenfunctions for the operator H. Let P_k denote the projection of $L^2(\mathbb{R}^n)$ onto the eigenspace spanned by $\{\Phi_\alpha:|\alpha|=k\}$. Our multipliers are then

$$S_N^\alpha=\frac{1}{A_N^\alpha}\sum_{k=0}^{N} A_{N-k}^\alpha P_k \tag{10}$$

where A_N^α are the binomial coefficients $A_N^\alpha=\binom{N}{\alpha}$.

Let $L_N^{n-1}(z)$ denote the Laguerre polynomial of type $(n-1)$. Then Peetre [6] has shown that the projection P_N is the Weyl transform of the function $\varphi_N(z)=L_N^{n-1}(\frac{1}{2}|z|^2)\exp(-\frac{1}{4}|z|^2)$, i.e., we have

$$P_N=W(\varphi_N) \tag{11}$$

Therefore, S_N^α is the Weyl transform of the function $s_N^\alpha(z)$ where

$$s_N^\alpha(z)=\frac{1}{A_N^\alpha}\sum_{k=0}^{N} A_{N-k}^\alpha\varphi_k(z). \tag{12}$$

But in view of the formula (see [8])

$$L_N^{\alpha+\beta+1}(x)=\sum_{k=0}^{N} A_{N-k}^\alpha L_k^\beta(x) \tag{13}$$

we obtain

$$s_N^\alpha(z)=\frac{1}{A_N^\alpha}L_N^{\alpha+n}(\tfrac{1}{2}|z|^2)\exp(-\tfrac{1}{4}|z|^2). \tag{14}$$

Thus we have proved that

$$W(T_N^\alpha f)=W(s_N^\alpha)W(f) \tag{15}$$

or $T_N^\alpha f$ is given by

$$T_N^\alpha f=s_N^\alpha\times f. \tag{16}$$

We see that T_N^α is a twisted convolution operator.

Now it is an easy matter to prove T_N^α are uniformly bounded on $L^p(\mathbb{C}^n)$ for $1 \leqslant p \leqslant \infty$ when $\alpha > n - \frac{1}{2}$. For that purpose we make use of the following asymptotic estimates available for the Laguerre functions. If $\mathscr{L}_N^\delta(r)$ are the normalized Laguerre functions then the following estimates are known (see [4]).

$$|\mathscr{L}_N^\delta(r)| \leqslant C \begin{cases} (r\nu)^{\delta/2} & \text{if } 0 \leqslant r \leqslant \dfrac{1}{\nu} \\ (r\nu)^{-1/4} & \text{if } \dfrac{1}{\nu} < r \leqslant \dfrac{\nu}{2} \\ \{\nu(\nu^{1/3} + |\nu - r|)\}^{-1/4} & \text{if } \dfrac{\nu}{2} < r \leqslant \dfrac{3\nu}{2} \\ \exp(-\gamma r) & \text{if } \dfrac{3\nu}{2} < r \end{cases}$$

where we have set $\nu = 4N + 2\delta + 2$. In view of these estimates one can easily check that

$$\int_{\mathbb{C}^n} |s_N^\alpha(z)| \mathrm{d}z\, \mathrm{d}\bar{z} \leqslant C \tag{17}$$

provided $\alpha > n - \frac{1}{2}$. Hence the operators T_N^α are uniformly bounded on L^p, $1 \leqslant p \leqslant \infty$ when $\alpha > n - \frac{1}{2}$.

It is also easy to see that T_N^α are not uniformly bounded on $L^1(\mathbb{C}^n)$ when $\alpha < n - \frac{1}{2}$. To prove this we suppose that T_N^α are bounded on $L^1(\mathbb{C}^n)$ and express S_N as follows (see [1]).

$$S_N = \sum_{k=0}^{N} A_k^\alpha A_{N-k}^{-\alpha-1} S_k^\alpha. \tag{18}$$

From this it follows that if $T_N = T_N^0$,

$$\| T_N f \|_1 \leqslant C N^\alpha \| f \|_1. \tag{19}$$

Since T_N is the twisted convolution operator with the function

$$L_N^n(\tfrac{1}{2}|z|^2) \exp(-\tfrac{1}{4}|z|^2)$$

the above will imply that

$$\int_0^\infty |L_N^n(\tfrac{1}{2}r^2)| \exp(-\tfrac{1}{4}r^2) r^{2n-1}\, \mathrm{d}r \leqslant C N^\alpha. \tag{20}$$

Now we need a lower bound for the left hand side of (20).

The following estimates have been proved in [4].

$$\| \mathscr{L}_N^{\alpha+\beta}(r) r^{-\beta/2} \|_p \sim N^{(1/p)-(1/2)-(\beta/2)} \quad \text{if } \beta < \frac{2}{p} - \frac{1}{2}$$

$$\| \mathscr{L}_N^{\alpha+\beta}(r) r^{-\beta/2} \|_p \sim N^{(\beta/2)-(1/p)} \quad \text{if } \beta > \frac{2}{p} - \frac{1}{2}$$

for $\alpha+\beta>-1, \alpha>-(2/p)$ and $1 \leqslant p \leqslant 4$. If we use these estimates we immediately get

$$C_1 N^{n-(1/2)} \leqslant \int_0^\infty |L_N^n(\tfrac{1}{2}r^2)| \exp(-\tfrac{1}{4}r^2) r^{2n-1}\,\mathrm{d}r$$
$$\leqslant CN^\alpha$$

and this can hold for large N only if $\alpha > n-\frac{1}{2}$. This completes the proof of part (i) of Theorem 1.

3. Cesaro multipliers when $\alpha \equiv n-\frac{1}{2}$

In this section we will prove that T_N^α are uniformly bounded on L^p, $1<p<\infty$ when $\alpha=n-\frac{1}{2}$. If we examine the estimates for the normalized Laguerre functions stated in the previous section it is easy to see that

$$\int_{|z|\leqslant N^{-1/2}} |s_N^\alpha(z)|\,\mathrm{d}z\,\mathrm{d}\bar{z} \leqslant C$$
$$\int_{|z|>N^{1/2}} |s_N^\alpha(z)|\,\mathrm{d}z\,\mathrm{d}\bar{z} \leqslant C \tag{21}$$

even when $\alpha=n-\frac{1}{2}$. Here the constant C is independent of N. If $A(N)$ denote the set $\{z: N^{-1/2} \leqslant |z| \leqslant N^{1/2}\}$ and if $\chi_{A(N)}$ stand for the characteristic function of $A(N)$ then it is enough to consider twisted convolution with the kernel

$$s_N^\alpha(z)\chi_{A(N)}(z). \tag{22}$$

Again we split this kernel into two parts. First we consider

$$\tilde{s}_N^\alpha(z) = s_N^\alpha(z)\chi_{B(N)}(z) \tag{23}$$

where $B(N)=\{z: 1 \leqslant |z| \leqslant N^{1/2}\}$. When $\alpha=n-\frac{1}{2}$ we have the estimate

$$|\tilde{s}_N^\alpha(z)| \leqslant C|z|^{-2n}\chi_{B(N)}(z) \tag{24}$$

Now we will prove the following proposition.

PROPOSITION 3.1.

The operator $f \to \tilde{s}_N^\alpha \times f$ is bounded on L^p for $1<p<\infty$.

Proof. Observe that the kernel $\tilde{s}_N^\alpha$ is in $L^p(\mathbb{C}^n)$ for all $p>1$. The proposition is then an immediate consequence of the following proposition whose proof can be found in [5].

PROPOSITION 3.2

Suppose that the kernel k is in $L^{1+\varepsilon}(\mathbb{C}^n)$ for $0<\varepsilon<\varepsilon_0$. Then the twisted convolution operator $Kf=k\times f$ is bounded on $L^p(\mathbb{C}^n)$ for $1<p<\infty$.

Thus we are left with the kernel $s_N^\alpha(z)\chi_{D(N)}(z)$ where $D(N)=\{z: N^{-1/2} \leqslant |z| \leqslant 1\}$.

When $N^{-1/2} \leqslant |z| \leqslant 1$ using the asymptotic formula for the Laguerre functions (see [8], page 199) we see that

$$s_N^\alpha(z) = C|z|^{-2n}\cos\left\{2N^{1/2}|z| - (\alpha+n)\frac{\pi}{2} - \frac{\pi}{4}\right\} + N^{-1/2}|z|^{-2n-1}O(1).$$

Since the error term is uniformly integrable we need to consider the twisted convolution with the kernel $K_N^\alpha(z)$ given by

$$K_N^\alpha(z) = |z|^{-2n}\psi(N^{1/2}|z|)\chi_{D(N)}(z) \tag{25}$$

where we have set

$$\psi(\theta) = \cos\left\{2\theta - (\alpha+n)\frac{\pi}{2} - \frac{\pi}{4}\right\}.$$

PROPOSITION 3.3

The operator $f \to K_N^\alpha \times f$ is bounded on L^p for $1 < p < \infty$.

Proof. The proof follows in several steps. First we reduce the problem to the study of ordinary convolution with the kernel $k_N^\alpha(z)$. This is done as follows.

Consider a kernel $K(x, y)$ which satisfies the estimates

$$|K(x,y)| \leqslant C|x-y|^{-2n}$$

and assume that the operator

$$Tf(x) = \int K(x,y)f(y)\,\mathrm{d}y \tag{26}$$

is bounded on $L^p(\mathbb{R}^{2n})$, $1 \leqslant p \leqslant \infty$. Consider now the truncated operators T_ε where

$$T_\varepsilon f(x) = \int_{|x-y|\leqslant\varepsilon} K(x,y)f(y)\,\mathrm{d}y \tag{27}$$

Then we have the following Lemma due to Ricci and Stein [7].

Lemma 3.1. *T_ε is bounded on $L^p(\mathbb{R}^{2n})$, $1 \leqslant p \leqslant \infty$ and the norm of T_ε is independent of ε.*

Let $P(x, y)$ be a real polynomial. Consider then the operator

$$\tilde{T}_1 f(x) = \int_{|x-y|\leqslant 1} K(x,y)\exp(iP(x,y))f(y)\,\mathrm{d}y \tag{28}$$

where the kernel K is as above. Then we have the following result.

Lemma 3.2. *The operator $\tilde{T}_1$ is bounded on $L^p(\mathbb{R}^{2n})$, $1 \leqslant p \leqslant \infty$ whenever T is bounded on $L^p(\mathbb{R}^{2n})$.*

A proof of this lemma can be found in [7] and it uses Lemma 3.1 together with an induction argument on the degree of the polynomial $P(x, y)$. Since the twisted convolution with $K_N^\alpha(z)$ is an oscillatory singular integral of the above form, Proposition 3.3 will be proved once we show that the ordinary convolution operator $K_N^\alpha * f$ is bounded on $L^p(\mathbb{R}^{2n})$, for $1<p<\infty$.

In twisted convolution the polynomial $P(x,y)$ is of first degree in x and y. It is easy to see that the roles of T and $\tilde{T}_1$ can be reversed in Lemma 3.2. Since the twisted convolution with K_N^α is bounded on $L^2(\mathbb{C}^n)$ we immediately see that $K_N^\alpha * f$ is bounded on $L^2(\mathbb{R}^{2n})$. Hence in view of the Marcinkiewicz interpolation theorem it is enough to prove that $K_N^\alpha * f$ is weak type $(1,1)$.

Further reduction is affected by rescaling. It is enough to consider convolution with the kernel

$$R_N(x)=|x|^{-2n}\psi(|x|)\chi_{E(N)}(x)$$

where $E(N)=\{x:1\leqslant|x|\leqslant N^{1/2}\}$. If we have a weak type $(1,1)$ analogue of Lemma 3.1 for truncated operators, then the claim that $K_N^\alpha * f$ is of weak type $(1,1)$ will follow from the following theorem due to Christ [3].

Theorem. *Let $K(x)=|x|^{-2n}\psi(|x|)\chi_{\{|x|\geqslant 1\}}(x)$. Then convolution with $K(x)$ is of weak type (1,1).*

So we are left with proving the following.

Lemma 3.3. *Let T and T_ε be as in Lemma* 3.1. *Then if T is of weak type* (1, 1), *so is T_ε.*

Proof. By scale invariance we can assume that $\varepsilon=1$. Let us write $\mathbb{R}^{2n}$ as a union of disjoint cubes Q_j having centres at y_j and each Q_j having side length $\frac{1}{4}$ with sides parallel to the coordinate axes. We will show that

$$|\{x\in Q_j:|T_1 f(x)|>\lambda\}|\leqslant\frac{c}{\lambda}\int_{5Q_j}|f(y)|\,dy \tag{29}$$

where $|E|$ is the Lebesgue measure of E. The Lemma will then follow by summing over j noticing that the doubles of the cubes have bounded overlap property.

Let us write $f=f_1+f_2+f_3$ where $f_1=f\chi_{2Q_j}$ and $f_2=f\chi_{5Q_j\setminus 2Q_j}$. When $x\in Q_j$ and $y\in 2Q_j$ we have $|x-y|\leqslant 1$ and hence for x in Q_j

$$T_1 f_1(x)=\int K(x,y)f_1(y)\,dy=Tf_1(x).$$

Since T is of weak type $(1,1)$

$$\left|\left\{x\in Q_j:|T_1 f_1(x)|>\frac{\lambda}{3}\right\}\right|\leqslant\frac{c}{\lambda}\int_{2Q_j}|f(y)|\,dy. \tag{30}$$

When $x\in Q_j$ and $y\in 5Q_j/2Q_j$ we have $|x-y|\geqslant\frac{1}{4}$. Since the kernel satisfies the estimate $K(x,y)|\leqslant C|x-y|^{-2n}$ it is integrable when $\frac{1}{4}\leqslant|x-y|\leqslant 1$. Hence

$$T_1 f_2(x)=\int_{\frac{1}{4}\leqslant|x-y|\leqslant 1}K(x,y)f_2(y)\,dy \tag{31}$$

is bounded on L^1 and in particular

$$\left|\left\{x\in Q_j:|T_1f_2(x)|>\frac{\lambda}{3}\right\}\right|\leq\frac{c}{\lambda}\int_{5Q_j\setminus 2Q_j}|f(y)|\,\mathrm{d}y$$

Finally, when $x\in Q_j$, $y\in\mathbb{R}^{2n}\setminus 5Q_j$ we have $|x-y|>1$ and hence $T_1f_3(x)=0$. Thus we have proved that

$$|\{x\in Q_j:|T_1f(x)|>\lambda\}|\leq\frac{c}{\lambda}\int_{5Q_j}|f(y)|\,\mathrm{d}y \tag{32}$$

This completes the proof of Lemma 3.3.

4. Radial functions and Laguerre expansions

When f is a radial function the Weyl transform $W(f)$ reduces to the Laguerre transform and we have the formula

$$W(f)=\sum_{N=0}^{\infty}R_N(f)P_N \tag{33}$$

where $R_N(f)$ are defined by

$$R_N(f)=\frac{N!}{(N+n-1)!}\int_0^{\infty}f(r)L_N^{n-1}(\tfrac{1}{2}r^2)\exp(-\tfrac{1}{4}r^2)r^{2n-1}\,\mathrm{d}r. \tag{34}$$

Therefore, for radial functions

$$T_N^{\alpha}f=\frac{1}{A_N^{\alpha}}\sum_{k=0}^{N}A_{N-k}^{\alpha}R_k(f)L_k^{n-1}(\tfrac{1}{2}|z|^2)\exp(-\tfrac{1}{4}|z|^2). \tag{35}$$

Thus the study of the operators T_N^{α} reduces to the study of the Cesaro means of order α of the series

$$f(r)=\sum_{k=0}^{\infty}R_k(f)L_k^{n-1}(\tfrac{1}{2}r^2)\exp(-\tfrac{1}{4}r^2). \tag{36}$$

Let us set $v_N^2=(N!/(N+n-1)!)$ and define

$$\varphi_N^{n-1}(r)=v_NL_N^{n-1}(\tfrac{1}{2}r^2)\exp(-\tfrac{1}{4}r^2)r^{n-1}.$$

Let $s_N^{\alpha}(r,s)$ be the kernel

$$s_N^{\alpha}(r,s)=\frac{1}{A_N^{\alpha}}\sum_{k=0}^{N}A_{N-k}^{\alpha}\varphi_k^{n-1}(r)\varphi_k^{n-1}(s). \tag{37}$$

Then it is clear that $T_N^{\alpha}f(s)$ is given by

$$T_N^{\alpha}f(s)=s^{-(n-1)}\int_0^{\infty}s_N^{\alpha}(r,s)f(r)r^n\,\mathrm{d}r \tag{38}$$

We will prove the following proposition.

PROPOSITION 4.1

Assume that $0 \leqslant \alpha \leqslant n-\frac{1}{2}$, $f \in L^p(\mathbb{R}_+, r^{2n-1}\,\mathrm{d}r)$ *and p satisfies* $(4n/(2n+1+2\alpha)) < p < (4n/(2n-1-2\alpha))$. *Then*

$$\int_0^\infty |T_N^\alpha f(s)|^p s^{2n-1}\,\mathrm{d}s \leqslant C \int_0^\infty |f(r)|^p r^{2n-1}\,\mathrm{d}r$$

with a constant C independent of N.

Proof. In view of Stein's interpolation theorem for analytic families of operators and the L^1 result for $\alpha > n-\frac{1}{2}$ it is enough to consider the case $\alpha = 0$. So let T_N be the operator defined by

$$T_N f(s) = s^{-(n-1)} \int_0^\infty s_N(r,s) f(r) r^n\,\mathrm{d}r \tag{39}$$

where

$$s_N(r,s) = \sum_{k=0}^N \varphi_k^{n-1}(r)\varphi_k^{n-1}(s). \tag{40}$$

The case $n=1$ was settled by Askey and Wainger in [2] because then we are considering the convergence of the usual Laguerre series. The higher dimensional case is not very different except that certain calculations become more tedious. We will not give a complete detailed proof. Instead we will bring out the analogy between the one dimensional and the higher dimensional case and then refer to the paper of Askey–Wainger.

We are interested in establishing the inequality

$$\int_0^\infty |T_N f(s)|^p s^{2n-1}\,\mathrm{d}s \leqslant C \int_0^\infty |f(r)|^p r^{2n-1}\,\mathrm{d}r \tag{41}$$

for $(4n/(2n+1)) < p < (4n/(2n-1))$. Let us make the following reductions. Define $g(r) = f(r^{1/2n})$ and

$$K_N(r,s) = s^{-[(n-1)/2n]} s_N(r^{1/2n}, s^{1/2n}) r^{-[(n-1)/2n]}$$

and let K_N be the operator with kernel $K_N(r,s)$. Then we need to establish the inequality

$$\int_0^\infty |K_N g(s)|^p\,\mathrm{d}s \leqslant C \int_0^\infty |g(r)|^p\,\mathrm{d}r. \tag{42}$$

As in Askey–Wainger we use Pollard's device of considering

$$s_N + s_{N+1} = 2s_N + \varphi_{N+1}^{n-1}(r)\varphi_{N+1}^{n-1}(s). \tag{43}$$

This leads to the formula

$$s_N(r,s) = \frac{N\varphi_{N+1}^{n-1}(s)}{r^2 - s^2}\{\varphi_{N+2}^{n-1}(r) - \varphi_N^{n-1}(r)\} + \text{more terms}.$$

Consequently the kernel $K_N(r,s)$ is given by

$$K_N(r,s)=\frac{N(rs)^{-[(n-1)/2n]}\varphi_{N+1}^{n-1}(s^{1/2n})}{r^{1/n}-s^{1/n}}\{\varphi_{N+2}^{n-1}(r^{1/2n})-\varphi_N^{n-1}(r^{1/2n})\}+\text{more terms.}$$

We just consider the first term which defines the operator

$$A_Ng(s)=s^{-[(n-1)/2n]}N\varphi_{N+1}^{n-1}(s^{1/2n})\int_0^\infty\frac{\varphi_{N+1}^{n-1}(r^{1/2n})-\varphi_N^{n-1}(r^{1/2n})}{r^{1/n}-s^{1/n}}$$
$$g(r)r^{-[(n-1)/2n]}\,\mathrm{d}r. \tag{44}$$

Using the identity

$$(sr)^{-[(n-1)/2n]}(r^{1/n}-s^{1/n})^{-1}=(r-s)^{-1}\sum_{k=1}^{n}(r^{1/n})^{((n+1)/2)-k}(s^{1/n})^{k-((n+1)/2)}$$

we are led to consider operators of the form

$$A_N^kg(s)=(s^{1/n})^{k-((n+1)/2)}N\varphi_{N+1}^{n-1}(s^{1/2n})$$
$$\times\int_0^\infty\frac{\varphi_{N+2}^{n-1}(r^{1/2n})-\varphi_N^{n-1}(r^{1/2n})}{r-s}(r^{1/n})^{((n+1)/2)-k}g(r)\,\mathrm{d}r. \tag{45}$$

Let us now briefly recall how Askey and Wainger established the inequality (42) in the one dimensional case. Dividing the r-range of integration and the s-range each into 6 parts one is led to consider 36 integrals. Using asymptotic estimates for the Laguerre functions some of the integrals can be estimated directly. The remaining integrals can be expressed in one of the following three forms:

$$\varphi(s,N)\int\frac{f(r)}{r-s}\psi(r,N)\,\mathrm{d}r$$

$$|s|^{-1/4}\varphi(s,N)\int\frac{f(r)}{r-s}\psi(r,N)|r|^{1/4}\,\mathrm{d}r$$

$$|s-v|^{-1/4}\varphi(s,N)\int\frac{f(r)}{r-s}\psi(r,N)|r-v|^{1/4}\,\mathrm{d}r.$$

To estimate integrals of the above forms they used a weighted inequality for the Hilbert transform and a weighted Hardy's inequality. Let us recall what these things are. If we define

$$F_{1/4}(x)=x^{-5/4}\int_0^x f(t)t^{1/4}\,\mathrm{d}t$$

then for $\frac{4}{3}<p<4$ one has

$$\|F_{1/4}\|_p\leqslant C\|f\|_p.$$

This is the weighted Hardy's inequality.

The weighted Hilbert transform is defined by

$$\tilde{f}_{1/4}(x) = \lim_{\varepsilon \to 0} |x|^{-1/4} \int_{|x-y|>\varepsilon} \frac{f(y)}{x-y} |y|^{1/4}\, dy.$$

For this Hilbert transform one has

$$\| \tilde{f}_{1/4} \|_p \leqslant C \| f \|_p$$

provided $\frac{4}{3} < p < 4$. The condition $\frac{4}{3} < p < 4$ is imposed by the fact that the weight $|x|^{p/4}$ is in the A_p class of Muckenhoupt iff $p > \frac{4}{3}$.

One can do the same thing in the case when $n > 1$. Instead of the weights $|r|^{\pm(p/4)}$, this time we have to consider the weights $|r|^{\pm\alpha p}$ where $0 < \alpha < 1$. These weights will be in A_p class iff p lies between $1/\alpha$ and $1/(1-\alpha)$ (For facts about the A_p class we refer to the book [9] of Torchinsky).

Using asymptotic estimates for the functions $\varphi_N^{n-1}(r)$ as in [2] one can establish the inequality (42) by appealing to the weighted Hilbert transform and Hardy's inequality. Because of various weights of the form $|r|^{\pm\alpha p}$ the condition $(4n/(2n+1)) < p < (4n/(2n-1))$ for the validity of (42) is naturally imposed. We omit the details.

Finally it remains to prove the necessity of the condition $(4n/(2n+1+2\alpha)) < p < (4n/(2n-1-2\alpha))$ for the boundedness of T_N^α for radial functions. That is the content of the next proposition.

PROPOSITION 4.2

The inequality (41) *holds only if p satisfies the condition* $(4n/(2n+1+2\alpha)) < p < (4n/(2n-1-2\alpha))$.

Proof. As we know that T_N^α are bounded on L^p radial functions when $(4n/(2n+1)) < p < (4n/(2n-1))$ we consider a p satisfying $(4n/(2n-1)) < p \leqslant 4$. Then the conjugate exponent q will satisfy $q < (4n/(2n+1))$. Suppose that we have the inequality

$$\int_0^\infty |T_N^\alpha f(s)|^p s^{2n-1}\, ds \leqslant C \int_0^\infty |f(s)|^p s^{2n-1}\, ds.$$

Expressing the partial sums in terms of the Cesaro means of order α as in §2 we get

$$|R_N(f)| \left(\int_0^\infty |L_N^{n-1}(\tfrac{1}{2}r^2) \exp(-\tfrac{1}{4}r^2)|^p r^{2n-1}\, dr \right)^{1/p} \leqslant C N^\alpha \| f \|_p.$$

By a proper choice of f the above will lead to the inequality

$$\left(\int_0^\infty |L_N^{n-1}(\tfrac{1}{2}r^2) \exp(-\tfrac{1}{4}r^2)|^p r^{2n-1}\, dr \right)^{1/p} \cdot \left(\int_0^\infty |L_N^{n-1}(\tfrac{1}{2}r^2) \exp(-\tfrac{1}{4}r^2)|^q r^{2n-1}\, dr \right)^{1/q} \leqslant C N^{\alpha+n-1}.$$

Again we make use of the norm estimates stated in §2 .

$$\left(\int_0^\infty |L_N^{n-1}(\tfrac{1}{2}r^2)\exp(-\tfrac{1}{4}r^2)|^p r^{2n-1}\,\mathrm{d}r\right)^{1/p}$$
$$= CN^{(n-1)/2}\left(\int_0^\infty |\mathscr{L}_N^{n-1}(r) r^{(n-1)((1/p)-(1/2))}|^p\,\mathrm{d}r\right)^{1/p}.$$

In the notation of §2 we have $\alpha+\beta=n-2$, $\beta=2(n-1)((1/2)-(1/p))$. Since $p>(4n/(2n-1))$ it is easy to see that $\beta>((2/p)-(1/2))$. Therefore,

$$\left(\int_0^\infty |L_N^{n-1}(\tfrac{1}{2}r^2)\exp(-\tfrac{1}{4}r^2)|^p r^{2n-1}\,\mathrm{d}r\right)^{1/p} \geqslant CN^{(n-1)-(n/p)}.$$

For the other term as $q<(4n/(2n+1))$ we will have $\beta=2(n-1)((1/2)-(1/p))<((2/q)-(1/2))$ and hence

$$\left(\int_0^\infty |L_N^{n-1}(\tfrac{1}{2}r^2)\exp(-\tfrac{1}{4}r^2)|^q r^{2n-1}\,\mathrm{d}r\right)^{1/q} \geqslant CN^{((n-1)/2)+n((1/q)-(1/2))}.$$

Hence we have the inequality

$$N^{n-1}N^{((n-1)/2)}N^{((n/2)-(n/p)+(n/q))} \leqslant CN^{\alpha+n-1}$$

This can hold for large N only if

$$n\left(\frac{1}{q}-\frac{1}{p}\right)-\tfrac{1}{2}\leqslant\alpha$$

which after simplification becomes $p\leqslant(4n/(2n-1-2\alpha))$. Similarly one can show that the condition $p\geqslant(4n/(2n+1+2\alpha))$ is also necessary.

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