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On regularity of twisted spherical means and special Hermite expansions

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Abstract. Regularity properties of twisted spherical means are studied in terms of certain Sobolev spaces defined using Laguerre functions. As an application we prove a localisation theorem for special Hermite expansions.

Keywords. Spherical means; twisted convolution; Sobolev spaces; special Hermite series.

1. Introduction

Regularity of the spherical means $f(x,t)$ defined on $\mathbb{R}^n$ for f in $L^1_{\text{loc}}(\mathbb{R}^n)$ by the equation

$$f(x,t) = \int_{S^{n-1}} f(x-ty)\,d\sigma(y) \tag{1.1}$$

has been the subject of several papers. In 1976, Stein [10] proved that when $n \geqslant 3$ and $p > n/(n-1)$ the spherical means $f(x,t)$ of a function f in $L^p(\mathbb{R}^n)$ are bounded and continuous as a function of t for almost every x in $\mathbb{R}^n$. Later more refined results were obtained by Oberlin–Stein [4], Peyriere–Sjolin [7] and Sjolin [8]. To wit, the following theorem has been proved by Sjolin [8]. Let $H^s(\mathbb{R})$ stand for the usual L^2 Sobolev space of order s and let $\varphi \in C_0^\infty(0,\infty)$. Then the following is true.

Theorem (Sjolin). *If $n \geqslant 2$, $2n/2n-1 \leqslant p \leqslant 2$ and $s = n(1-1/p)-(1/2)$ then for any f in $L^p(\mathbb{R}^n)$ one has*

$$\left(\int_{\mathbb{R}^n} \|\varphi(\cdot) f(x,\cdot)\|^2_{H^s}\,dx\right)^{1/2} \leqslant C\left(\int |f(x)|^p\,dx\right)^{1/p}.$$

The above result has been extended from $\mathbb{R}^n$ to any compact symmetric space by Colzani [1]. His study of the spherical means on these spaces is essentially group theoretic and is based on the theory of spherical functions. In this setting the proofs turned out to be simpler and the results quite general. In order to state the main result of Colzani it is necessary to recall the definition of the spherical means on a compact symmetric space.

Let (G,K) be a compact symmetric pair and $X = G/K$ be the associated symmetric space. Functions on X are identified with right K-invariant functions on G. Let $G = KAK$ be a Cartan decomposition of G and let $dg = dk\, w(t)dt\, dk'$ be the corresponding decomposition of the Haar measure dg on G. Here $w(t)$ is an appropriate weight function defined on A. If $f \in L^1(G)$ we define the spherical means $f(g,t)$, $g \in G$ and $t \in A$ by

$$f(g,t) = \int_K \int_K f(gktk')\mathrm{d}k\,\mathrm{d}k'. \tag{1.2}$$

Clearly, $f(g,t)$ is K-invariant as a function of g.

Using the Peter-Weyl theorem we can expand $f(g,t)$ in terms of zonal spherical functions. To see this let us recall the Peter-Weyl theorem for right K-invariant functions of G. Let $\hat{G}_K$ be the subset of the dual of G consisting of equivalence classes of class 1 representations of G and let $\{\varphi_\lambda\}$, $\lambda \in \hat{G}_K$ be the associated system of zonal spherical functions. Then the Peter-Weyl theorem reads

$$f = \sum_{\lambda \in \hat{G}_K} d_\lambda f * \varphi_\lambda, \tag{1.3}$$

where d_λ stands for the dimension of λ. Since the spherical functions φ_λ satisfy the relation

$$\int_K \varphi_\lambda(gkg')\mathrm{d}k = \varphi_\lambda(g)\varphi_\lambda(g') \tag{1.4}$$

the spherical means $f(g,t)$ have the expansion

$$f(g,t) = \sum_{\lambda \in \hat{G}_K} d_\lambda f * \varphi_\lambda(g)\varphi_\lambda(t). \tag{1.5}$$

The above expression for $f(g,t)$ suggests that it is natural to study the regularity properties of $f(g,t)$ in terms of some sort of Sobolev spaces associated to the system $\{\varphi_\lambda\}$. For $t \in A$ we get $\psi_\lambda(t) = d_\lambda^{1/2}\varphi_\lambda(t)$ so that $\{\psi_\lambda\}$ is an orthonormal basis for $L^2(A, w(t)\mathrm{d}t)$. We then introduce the Sobolev space $L_s^2(A, w(t)\mathrm{d}t)$ to be the set of all functions $f = \Sigma \hat{f}(\lambda)\psi_\lambda$ for which the norm

$$\|f\|_{L_s^2}^2 = \sum_{\lambda \in \hat{G}_K} (1 + |\lambda|^2)^s |\hat{f}(\lambda)|^2 \tag{1.6}$$

is finite. Here $|\lambda|$ denotes the norm of the highest weight corresponding to λ.

Measuring the regularity of the spherical means $f(g,t)$ in terms of the above Sobolev spaces Colzani established the following.

Theorem (Colzani). *Let $X = G/K$ be a compact rank one symmetric space of real dimension n. If $f \in L^p(X)$, $1 < p \leqslant 2$, and $s = n(1-(1/p)) - (1/2)$ then*

$$\int_G \|f(g,.)\|_{L_s^2}^2 \mathrm{d}g \leqslant C\|f\|_p^2.$$

Using the above regularity theorem he was able to establish a localisation theorem for spherical harmonic expansions of L^2 functions on compact rank one symmetric spaces.

Our goal in this paper is to establish similar regularity results for the twisted spherical means on $\mathbb{C}^n$ and to prove an almost everywhere convergence result for the twisted spherical means. Before stating the results we need to set the notations up. The twisted convolution $f \times g$ of two functions f and g defined on $\mathbb{C}^n$ is the function

$$f \times g(z) = \int_{\mathbb{C}^n} f(z-w)g(w)\exp\left(\frac{i}{2}\operatorname{Im} z.\bar{w}\right)dw. \tag{1.7}$$

If μ_r stands for the normalized surface measure on the sphere $\{z\in\mathbb{C}^n:|z|=r\}$ the twisted spherical means is defined to be

$$f \times \mu_r(z) = \int_{|w|=r} f(z-w)\exp\left(\frac{i}{2}\operatorname{Im} z.\bar{w}\right)d\mu_r. \tag{1.8}$$

In this paper we are interested in the regularity of these twisted spherical means.

The study of the twisted spherical means turns out to be much simpler than the study of spherical means; thanks to the existence of an analogue of the Peter-Weyl theorem. This analogue is the so-called special Hermite expansion which is written as

$$f = (2\pi)^{-n}\sum_{k=0}^{\infty} f \times \varphi_k. \tag{1.9}$$

Here φ_k stands for the Laguerre function

$$\varphi_k(z) = L_k^{n-1}(\tfrac{1}{2}|z|^2)\exp(-(1/4)|z|^2) \tag{1.10}$$

where L_k^{n-1} is the kth Laguerre polynomial of type $(n-1)$. For various results concerning the special Hermite expansions we refer to Strichartz [11] and Thangavelu [14] and also to the monographs by Folland [2] and Thangavelu [16].

We also have an analogue of the expansion (1.5) for the twisted spherical means, namely,

$$f \times \mu_r(z) = (2\pi)^{-n}\sum_{k=0}^{\infty}\frac{k!(n-1)!}{(k+n-1)!}\varphi_k(r) f \times \varphi_k(z). \tag{1.11}$$

This has been proved in [14] and has played a crucial role in the study of spherical means on the Heisenberg group (see [14]). The above expansion suggests that we should measure the regularity of $f \times \mu_r$ in terms of some sort of Sobolev spaces defined using the functions $\varphi_k(r)$. If we define

$$\psi_k(r) = \left(\frac{2^{1-n}k!}{(k+n-1)!}\right)^{1/2}\varphi_k(r) \tag{1.12}$$

then the system $\{\psi_k\}$ forms an orthonormal basis for $L^2(\mathbb{R}_+, r^{2n-1}dr)$ where $\mathbb{R}_+ = (0,\infty)$. We introduce the Sobolev spaces $W_R^s(\mathbb{R}_+)$ to be the set of all functions of the form $f = \Sigma_{k=0}^{\infty} a_k\psi_k$ for which the norms

$$|f|_s^2 = \sum_{k=0}^{\infty}|a_k|^2(2k+n)^{2s} \tag{1.13}$$

are finite. For the twisted spherical means we then establish the following theorem.

Theorem A. *If $f \in L^p(\mathbb{C}^n)$, $1 < p \leqslant 2$, $s = n(1-(1/p)) - (1/2)$ then one has the inequality and*

$$\int_{\mathbb{C}^n} |f \times \mu_r(z)|_s^2 \, dz \leqslant C \|f\|_p^2.$$

The Sobolev space $W_R^s(\mathbb{R}_+)$ turns out to be a subspace of the Laguerre (or twisted) Sobolev space $W_L^s(\mathbb{C}^n)$. This space and another Sobolev space $W_H^s(\mathbb{R}^n)$ (Hermite Sobolev space) will be introduced in the next section. It turns out that $f \in W_R^s(\mathbb{R}_+)$ if and only if the function g defined by $g(z) = f(|z|)$ belongs to $W_L^s(\mathbb{C}^n)$. We also prove the embedding theorem for the Sobolev space W_L^s and W_H^s. As a consequence we also establish a pointwise convergence theorem for the twisted spherical means.

We also study the localisation problem for the special Hermite expansion (1.9). Let $S_N f(z)$ be the partial sums defined by

$$S_N f(z) = (2\pi)^{-n} \sum_{k=0}^{N} f \times \varphi_k(z). \tag{1.14}$$

Suppose we know that $f(w)$ vanishes in a neighborhood of z, say in $|z-w| \leqslant a$. We are interested in finding conditions of f which will ensure that $S_N f(z) \to 0$ as $N \to \infty$ for a.e. z in $|z-w| \leqslant a$. Using the regularity of spherical means for L^2 functions we are able to prove the following theorem.

Theorem B. *Assume $n \geqslant 2$ and f is a compactly supported function in $W_L^{1/2}(\mathbb{C}^n)$. If $f(z) = 0$ on an open set Ω, then $S_N f(z) \to 0$ for a.e. z in Ω as $N \to \infty$.*

This theorem is proved in § 4. We introduce the Sobolev spaces in the next section. Regularity theorems for the spherical means are proved in § 3.

2. Invariant Sobolev spaces

We have introduced the Laguerre Sobolev spaces $W_L^s(\mathbb{C}^n)$ in [14] in connection with the spherical means on the Heisenberg group. There we have called them twisted Sobolev spaces. They have been also introduced in Peetre-Sparr [6] in connection with noncommutative interpolation. Let L be the special Hermite operator defined by

$$L = -\Delta + \frac{1}{4}|z|^2 - i \sum_{j=1}^{n} \left(x_j \frac{\partial}{\partial y_j} - y_j \frac{\partial}{\partial x_j} \right). \tag{2.1}$$

Then the special Hermite functions are eigenfunctions of the operator L and the series (1.9) is nothing but the eigenfunction expansion associated to L. In fact one has

$$L(f \times \varphi_k) = (2k+n) f \times \varphi_k \tag{2.2}$$

for any k. In view of this and spectral theorem one can define L^s by

$$L^s f = (2\pi)^{-n} \sum_{k=0}^{\infty} (2k+n)^s f \times \varphi_k. \tag{2.3}$$

The usual Sobolev space $H^s(\mathbb{R}^n)$ is defined to be the image of $L^2(\mathbb{R}^n)$ under the operator $(-\Delta + 1)^{-(s/2)}$. Similarly, we define $W_L^s(\mathbb{C}^n)$ to be the image of $L^2(\mathbb{C}^n)$ under

L^{-s}. In other words, $f \in W_L^s$ if and only if

$$|f|_s^2 = (2\pi)^{-2n} \sum_{k=0}^{\infty} (2k+n)^{2s} \|f \times \varphi_k\|_2^2 \tag{2.4}$$

is finite. Before studying properties of W_L^s we also introduce the Sobolev spaces W_H^s.

Consider the Hermite operator $H = -\Delta + (1/4)|x|^2$ on $\mathbb{R}^n$ for which the eigenfunctions are the ordinary Hermite functions $\Phi_\mu(x)$. If $P_k f$ stand for the projection of f in $L^2(\mathbb{R}^n)$ into the kth eigenspace spanned by $\{\Phi_\mu : |\mu| = k\}$ then the spectral decomposition of H is

$$Hf = \sum_{k=0}^{\infty} (2k+n) P_k f. \tag{2.5}$$

Using this one can define the operators H^s and $W_H^s(\mathbb{R}^n)$ is defined to be the image of $L^2(\mathbb{R}^n)$ under H^{-s}. This means f belongs to $W_H^s(\mathbb{R}^n)$ if and only if

$$|f|_s^2 = \sum_{k=0}^{\infty} (2k+n)^{2s} \|P_k f\|_2^2 \tag{2.6}$$

is finite. We use the same notation for the norms in W_L^s as well as in W_H^s. The situation will make it clear which space we refer to.

The Hermite and Laguerre Sobolev spaces enjoy an invariant property which is not shared by the usual Sobolev spaces $H^s(\mathbb{R}^n)$. Namely, W_H^s is invariant under the Fourier transform and W_L^s invariant under the symplectic Fourier transform. The invariance of W_H^s under the Fourier transform follows from the fact that $F(P_k f) = (-i)^k P_k f$ where Ff is the Fourier transform of f. This follows from the well known properties of the Hermite functions. The invariance of W_L^s can be seen as follows.

Let $F_s f$ be the symplectic Fourier transform of f which is defined by

$$F_s f(z) = \int f(w) \exp\left(-\frac{i}{2} \operatorname{Im} z \cdot \bar{w}\right) dw. \tag{2.7}$$

We can write this as $F_s f(z) = f \times 1(z)$. In view of this

$$F_s f \times \varphi_k = f \times (1 \times \varphi_k) \tag{2.8}$$

and if we can show that

$$1 \times \varphi_k = (-1)^k \varphi_k \tag{2.9}$$

the invariance of W_L^s under F_s will follow immediately. But (2.9) is an easy consequence of the generating function identity

$$\sum_{k=0}^{\infty} r^k \varphi_k(z) = (1-r)^{-n} \exp(-(1/4)(1+r/1-r)|z|^2) \tag{2.10}$$

valid for any $0 \leqslant r < 1$.

We now bring out the relation between the Hermite Sobolev space W_H^s and the ordinary Sobolev space H^s. If we let

$$A_j = \left(\frac{\partial}{\partial x_j} + \frac{1}{2} x_j\right), \quad j = 1, 2, \dots n. \tag{2.11}$$

and A_j^* the adjoint of A_j then it is easily seen that

$$H=\left(-\Delta+\frac{1}{4}|x|^2\right)=\frac{1}{2}\sum_{j=1}^{n}(A_jA_j^*+A_j^*A_j).$$

In view of this it is clear that $H^m f$ is a finite linear combination of terms of the form $x^\alpha\partial_x^\beta f$ where $|\alpha|+|\beta|\leqslant 2m$ when m is an integer. Thus, in this case $W_H^m(\mathbb{R}^n)\subset H^{2m}(\mathbb{R}^n)$. This inclusion is true for any $m\geqslant 0$.

Theorem 2.1 (i) *If* $s\geqslant 0$ *then* $W_H^s(\mathbb{R}^n)\subset H^{2s}(\mathbb{R}^n)$. (ii) *Conversely, if* $s\geqslant 0$ *and* $\varphi\in C_0^\infty(\mathbb{R}^n)$ *then there is a constant* $C=C(\varphi)$ *depending on* φ *such that*

$$\int|H^s(\varphi f)|^2\mathrm{d}x\leqslant C\int|(-\Delta+1)^s f|^2\mathrm{d}x.$$

Proof. In order to prove (i) it is enough to show that the operator $(-\Delta+1)^s(-\Delta+|x|^2)^{-s}$ is bounded on $L^2(\mathbb{R}^n)$. In [15] we have proved that $(-\Delta+|x|^2)^{-s}$ is a pseudo differential operator whose symbol $\sigma(x,\xi)$ satisfies the estimates

$$|\partial_x^\alpha\partial_\xi^\beta\sigma(x,\xi)|\leqslant C(1+|x|+|\xi|)^{-2s-|\alpha|-|\beta|},\tag{2.12}$$

where C is independent of x. Therefore, it follows that $(-\Delta+1)^s(-\Delta+|x|)^2)^{-s}$ is a pseudo differential operator with symbol in $S_{1,0}^0$ and hence is bounded on $L^2(\mathbb{R}^n)$.

To establish (ii) we look at the operator $(-\Delta+|x|^2)^s\varphi(x)(-\Delta+1)^{-s}$. Again, because of the compactly supported φ, the above operator is a pseudo differential operator whose symbol b satisfies

$$|\partial_x^\alpha\partial_\xi^\beta b(x,\xi)|\leqslant C_{\alpha\beta}(\varphi)(1+|\xi|)^{-|\beta|}$$

where $C_{\alpha\beta}(\varphi)$ is independent of x. This proves part (ii).

We remark that a similar inclusion is true for the Laguerre Sobolev spaces also. This follows from the relation

$$L=-\frac{1}{2}\sum_{j=1}^{n}(Z_j\bar{Z}_j+\bar{Z}_jZ_j)\tag{2.13}$$

where the vector fields Z_j and $\bar{Z}_j$ are defined by

$$Z_j=\left(\frac{\partial}{\partial z_j}+\frac{1}{2}\bar{z}_j\right),\bar{Z}_j=\left(\frac{\partial}{\partial\bar{z}_j}-\frac{1}{2}z_j\right).\tag{2.14}$$

The proof runs along similar lines and we leave the details to the interested reader.

We now turn our attention to the relation between the Sobolev spaces W_L^s, W_H^s and W_R^s introduced in §1. If for any space V of functions on $\mathbb{C}^n\simeq\mathbb{R}^{2n}$, $(V)_R$ stands for the subspace of all radial functions in V then we have the following relation:

$$(W_L^s(\mathbb{C}^n))_R=(W_H^s(\mathbb{R}^{2n}))_R=W_R^s(\mathbb{R}_+).\tag{2.15}$$

To see the first equality we observe that when f is radial $Lf=Hf$. This follows from the fact that f is annihilated by operator

$$\sum_{j=1}^{n}\left(x_j\frac{\partial}{\partial y_j}-y_j\frac{\partial}{\partial x_j}\right).$$

This observation has been used in the study of Hermite expansions on $\mathbb{R}^{2n}$ for radial functions. Thus as far as radial functions are concerned L and H have the same spectral decomposition and hence the equality $(W_L^s)_R=(W_H^s)_R$.

The equality $W_R^s(\mathbb{R}_+)=(W_L^s(\mathbb{C}^n))_R$ follows from the following fact. When f is a radial function of $\mathbb{C}^n$ then the special Hermite expansion of f reduces to the Laguerre expansion. More precisely, one has

$$f\times\varphi_k(z)=c_n(f,\psi_k)\psi_k(r). \tag{2.16}$$

It is now obvious from this relation that $W_R^s(\mathbb{R}_+)$ is equal to $(W_L^s)_R$. With these observations we end discussion on the various Sobolev spaces and turn our attention to embedding theorems.

Theorem 2.2 (i) *If* $s>\frac{n}{4}$, $W_H^s(\mathbb{R}^n)\subset L^\infty(\mathbb{R}^n)\cap C(\mathbb{R}^n)$. (ii) *If* $s>\frac{n}{2}$, $W_L^s(\mathbb{C}^n)\subset L^\infty(\mathbb{C}^n)\cap C(\mathbb{C}^n)$. *In both cases the inclusion is continuous.*

Proof. We first Prove (i). Writing

$$f(x)=\sum_\mu \hat{f}(\mu)\Phi_\mu(x), \tag{2.17}$$

where $\hat{f}(\mu)=(f,\Phi_\mu)$ and applying Cauchy-Schwarz we have

$$|f(x)|^2\leqslant\left(\sum_\mu(2|\mu|+n)^{2s}|\hat{f}(\mu)|^2\right)B(x) \tag{2.18}$$

where

$$B(x)=\sum_\mu(2|\mu|+n)^{-2s}|\Phi_\mu(x)|^2. \tag{2.19}$$

The sum on the right hand side of (2.18) is nothing but $|f|_s^2$ and consequently

$$\sup_x|f(x)|\leqslant C|f|_s \tag{2.20}$$

would follow once we show that $B(x)$ is bounded.

That $B(x)$ is bounded can be proved in two ways. Writing $\Phi_k(x,x)=\sum_{|\mu|=k}(\Phi_\mu(x))^2$ we see that

$$B(x)=\sum_{k=0}^{\infty}(2k+n)^{-2s}\Phi_k(x,x). \tag{2.21}$$

In [13] we have established the estimate

$$\Phi_k(x,x)\leqslant C(2k+n)^{(n/2)-1} \tag{2.22}$$

with C independent of x. Therefore,

$$B(x) \leqslant C \sum_{k=0}^{\infty} (2k+n) - 2s + (n/2) - 1 \tag{2.23}$$

the sum being convergent as $s > n/4$. We can also prove the boundedness of $B(x)$ without using the estimate (2.22) but using the generating function for the Hermite functions. One has

$$\sum_{k=0}^{\infty} r^k \Phi_k(x,x) = \pi^{-(n/2)}(1-r)^{-(n/2)} \exp\left(-\left(\frac{1}{2}\right)\left(\frac{1-r}{1+r}|x|^2\right)\right). \tag{2.24}$$

Taking $r = e^{-2t}$ we obtain

$$\sum_{k=0}^{\infty} \exp(-(2k+n)t)\Phi_k(x,x) = (2\pi)^{-(n/2)}(\sinh 2t)^{-(n/2)}$$

$$\exp\left(-\left(\frac{1}{2}\right)(\tanh t)|x|^2\right) \tag{2.25}$$

Integrating the above identity against $t^{2s-1}dt$ we get

$$\sum_{k=0}^{\infty} (2k+n)^{-2s}\Phi_k(x,x)$$

$$= \frac{(2\pi)^{-(n/2)}}{\Gamma(2s)} \int_0^{\infty} (\sinh 2t)^{-(n/2)} t^{2s-1} \exp\left(-\left(\frac{1}{2}\right)(\tanh t)|x|^2\right) dt$$

$$\leqslant \frac{(2\pi)^{-(n/2)}}{\Gamma(2s)} \int_0^{\infty} t^{2s-1} \exp(-(nt))(1-\exp(-4t))^{-(n/2)} dt \leqslant C \tag{2.26}$$

since $s > (n/4)$.

We prove (ii) using an interesting identity satisfied by the special Hermite functions. For each pair of multi-indices μ and ν the special Hermite function $\Phi_{\mu\nu}$ is defined by

$$\Phi_{\mu\nu}(z) = (2\pi)^{-(n/2)} \int \exp(ix\cdot\zeta)\Phi_\mu\left(\zeta + \frac{1}{2}y\right)\Phi_\nu\left(\zeta - \frac{1}{2}y\right) d\zeta. \tag{2.27}$$

These functions have very interesting properties. To wit, one has

$$\Phi_{\mu\nu} \times \Phi_{\alpha\beta} = (2\pi)^{n/2} \delta_{\nu\alpha} \Phi_{\mu\beta}. \tag{2.28}$$

They form an orthonormal basis for $L^2(\mathbb{C}^n)$ and one has

$$f \times \varphi_k = (2\pi)^n \sum_{|\nu|=k} \sum_{\mu} (f, \Phi_{\mu\nu})\Phi_{\mu\nu}. \tag{2.29}$$

For all these and other interesting properties we refer to the monograph [16].

The identity which we need is the following.

PROPOSITION 2.1

For any z in $\mathbb{C}^n$ one has

$$\sum_{\beta} |\Phi_{\alpha\beta}(z)|^2 = (2\pi)^{-n}.$$

Proof. The functions $\{\bar{\Phi}_{\nu\beta}(w)\}$ form an orthonormal basis for $L^2(\mathbb{C}^n)$. We expand the function

$$\Phi_{\alpha\mu}(z-w)\exp(i/2)(\operatorname{Im} z.\bar{w}).$$

in terms of $\Phi_{\nu\beta}$. We have

$$\int \Phi_{\alpha\mu}(z-w)\exp(i/2)(\operatorname{Im} z\cdot\bar{w})\Phi_{\nu\beta}(w)dw = \Phi_{\alpha\mu}\times\Phi_{\nu\beta}(z) = \delta_{\mu\nu}(2\pi)^{n/2}\Phi_{\alpha\beta}(z).$$

Thus we have

$$\Phi_{\alpha\mu}(z-w)\exp(i/2)(\operatorname{Im} z.\bar{w}) = (2\pi)^{n/2}\sum_{\beta}\Phi_{\alpha\beta}(z)\bar{\Phi}_{\mu\beta}(w).$$

The orthonormality of the functions $\Phi_{\mu\beta}$ now proves the proposition.

We can now complete the proof of theorem 2.2. Writing

$$f(z) = \sum_{\mu}\sum_{\nu}(f,\Phi_{\mu\nu})\Phi_{\mu\nu}(z) \tag{2.30}$$

we obtain the inequality

$$|f(z)|^2 \leqslant AB(z) \tag{2.31}$$

where

$$\begin{aligned} A &= \sum\sum(2|\nu|+n)^{2s}|(f,\Phi_{\mu\nu})|^2,\\ B &= \sum\sum(2|\nu|+n)^{-2s}|\Phi_{\mu\nu}(z)|^2. \end{aligned} \tag{2.32}$$

In view of the relation (2.29) we get

$$A \leqslant (2\pi)^{-2n}|f|_s^2. \tag{2.33}$$

From the definition of $\Phi_{\mu\nu}$ it is clear that $\bar{\Phi}_{\mu\nu}(-z) = \Phi_{\nu\mu}(z)$ and hence

$$B(z) = \sum_{\nu}(2|\nu|+n)^{-2s}\sum_{\mu}|\Phi_{\mu\nu}(z)|^2 \leqslant (2\pi)^{-n}\sum_{k=0}^{\infty}(2k+n)^{-2s+n-1}$$

in view of the proposition. The last sum is finite precisely when $s > n/2$. Hence we have proved

$$|f(z)| \leqslant C|f|_s$$

for all z in $\mathbb{C}^n$. Hence the theorem.

3. Regularity of twisted spherical means

In this section we study the regularity properties of the spherical means $f\times\mu_r$ as a function of r in terms of the Sobolev spaces $W_R^s(\mathbb{R}_+)$. We first show that the spherical means are regularising in the following sense.

Theorem 3.1 *If $f \in W_L^s(\mathbb{C}^n)$ then for almost every $z \in \mathbb{C}^n$, $f \times \mu_r(z)$ belongs to $W_R^{s+(n-1)/2}(\mathbb{R}_+)$ and one has*

$$\int |f \times \mu_r(z)|_{s+(n-1)/2}^2 \mathrm{d}z \leqslant C |f|_s^2.$$

Proof. The proof is based on the formula (1.11). From the definition of $W_R^s(\mathbb{R}_+)$ it follows that

$$|f \times \mu_r(z)|_{s+(n-1)/2}^2 \leqslant C \sum_{k=0}^{\infty} (2k+n)^{2s} |f \times \varphi_k(z)|^2, \tag{3.1}$$

where we have used that fact $k!(n-1)!/(k+n-1)! = 0(k^{-n+1})$. From (3.1) we immediately obtain the theorem.

Using the above theorem together with the embedding theorem of the previous section one can prove the following almost everywhere convergence result for the spherical means.

Theorem 3.2 *If $f \in W_L^s(\mathbb{C}^n)$ with $s > 1/2$ then $w_{2n} f \times \mu_r(z)$ converges to $f(z)$ for almost every z in $\mathbb{C}^n$ as $r \to 0$, where w_{2n} is the measure of the unit sphere in $\mathbb{C}^n$.*

Proof. When $s > 1/2$ the embedding theorem shows that

$$\sup_{r>0} |f \times \mu_r(z)| \leqslant C |f \times \mu_r(z)|_{s+(n-1)/2}$$

and hence one has the inequality

$$\int \sup_{r>0} |f \times \mu_r(z)|^2 \mathrm{d}z \leqslant C |f|_s^2 \tag{3.2}$$

It is easy to see that $f \times \mu_r(z) \to f(z)$ as $r \to 0$ whenever f is a Schwartz class function. Hence the theorem follows from the above maximal inequality in a routine fashion.

We now turn our attention to the spherical means of L^p functions.

Theorem 3.3 (i) *If $f \in L^1(\mathbb{C}^n)$ and $s < -\frac{1}{2}$ one has*

$$\int |f \times \mu_r(z)|_s^2 \mathrm{d}z \leqslant C \|f\|_1^2.$$

(ii) *If $f \in L^p(\mathbb{C}^n)$, $1 < p \leqslant 2$ and $s = n(1-1/p) - \frac{1}{2}$,*

$$\int |f \times \mu_r(z)|_s^2 \mathrm{d}z \leqslant C \|f\|_p^2.$$

Proof. The proof of (i) is rather easy. In fact,

$$|f \times \mu_r(z)|_s^2 \leqslant C \sum_{k=0}^{\infty} (2k+n)^{2s-(n-1)} |f \times \varphi_k(z)|^2$$

and this gives, upon using the estimate

$$\int |f \times \varphi_k(z)|^2 \mathrm{d}z \leqslant C(2k+n)^{n-1} \|f\|_1^2,$$

the inequality

$$\int |f \times \mu_r(z)|_s^2 \mathrm{d}z \leqslant C \|f\|_1^2 \sum_{k=0}^{\infty} (2k+n)^{2s}.$$

The last sum is finite when $s < -\frac{1}{2}$.

In order to prove (ii) we use the mapping properties of the fractional powers L^{-s} of the operator L. Proceeding as above we have

$$\int |f \times \mu_r(z)|_s^2 \mathrm{d}z \leqslant C \sum_{k=0}^{\infty} (2k+n)^{2s-(n-1)} \|f \times \varphi_k\|_2^2 \tag{3.3}$$

and therefore, we only need to show that where $s = n\left(1 - \frac{1}{p}\right) - \frac{1}{2}$ we have

$$\sum_{k=0}^{\infty} (2k+n)^{-2n(1/p-1/2)} \|f \times \varphi_k\|_2^2 \leqslant C \|f\|_p^2. \tag{3.4}$$

This inequality follows from the more general result given in the next proposition.

PROPOSITION 3.1

If $0 < s < n$ and $\frac{1}{q} = \frac{1}{p} - \frac{s}{n}$ where $1 < p < q < \infty$ then $\| L^{-s} f\|_q \leqslant C \|f\|_p$ for f in $L^p(\mathbb{C}^n)$.

Proof. Let e^{-tL} be the semigroup defined by

$$e^{-tL} f = (2\pi)^{-n} \sum_{k=0}^{\infty} \exp(-(2k+n)t) f \times \varphi_k. \tag{3.5}$$

It is clear that $e^{-tL} f = f \times K_t$ where the kernel K_t is given by

$$K_t(z) = (2\pi)^{-n} \sum_{k=0}^{\infty} \exp(-(2k+n)t) \varphi_k(z). \tag{3.6}$$

Using the generating function identity (2.10) we see that

$$K_t(z) = (4\pi)^{-n} (\sinh t)^{-n} \exp(-(1/4) \coth t |z|^2. \tag{3.7}$$

In view of the formula

$$L^{-s} f = \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1} e^{-tL} f \,\mathrm{d}t \tag{3.8}$$

the kernel of L^{-s} is given by

$$K(z) = \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1} K_t(z) \mathrm{d}t. \tag{3.9}$$

Using (3.7) in (3.9) it is now an easy matter to prove the estimate

$$|K(z)| \leqslant C|z|^{-2n+2s}.$$

The proposition now follows from the Hardy-Littlewood-Sobolev theorem for fractional integration (see Stein [9]).

There is an analogue of Theorem 3.3 for the twisted Hardy space $\mathscr{H}^1$. This space was introduced by Mauceri-Picardello-Ricci in [3] and is defined using twisted convolution instead of ordinary convolution. It can also be defined to the subspace of $L^1(\mathbb{C}^n)$ containing all functions f for which the maximal function

$$f^*(z) = \sup_{t>0} |e^{-tL} f(z)| \tag{3.10)'}$$

is also in $L^1(\mathbb{C}^n)$. There is also an atomic decomposition. Any f in $\mathscr{H}^1$ can be written as

$$f(z) = \sum_{k=1}^{\infty} \lambda_k f_k(z) \tag{3.11}$$

with $C_1 \|f\|_{\mathscr{H}^1} \leqslant \sum_{k=1}^{\infty} |\lambda_k| \leqslant C_2 \|f\|_{\mathscr{H}^1}$ were the atoms f_k satisfy the properties

(i) f_k is supported in a cube $Q(z_k, r_k)$ centred at z_k and half side r_k

(ii) $\|f_k\|_\infty \leqslant (2r_k)^{-2n}$

(iii) $\int f_k(w) \exp\left(-\frac{i}{2} \operatorname{Im} z_k \cdot \bar{w}\right) \mathrm{d}w = 0.$

Using this atomic decomposition we can establish the following result.

Theorem 3.4 *If $f \in \mathscr{H}^1$, $f \times \mu_r(z)$ belongs to $W_R^{-(1/2)}(\mathbb{R}_+)$ for almost every z and one has*

$$\int |f \times \mu_r(z)|^2_{-(1/2)} \mathrm{d}z \leqslant C \|f\|^2_{\mathscr{H}^1}.$$

Proof. We need to prove the inequality

$$\sum_{k=0}^{\infty} (2k+n)^{-n} \|f \times \varphi_k\|_2^2 \leqslant C \|f\|^2_{\mathscr{H}^1}. \tag{3.12}$$

It is enough to prove this when f is an atom. So, let f be supported in $Q(z_0, r)$ and satisfying the above conditions (i), (ii) and (iii).

Using the mean zero condition

$$\int f(w) \exp(-(i/2) \operatorname{Im} z.\bar{w}) \, \mathrm{d}w = 0 \tag{3.13}$$

we rewrite the twisted convolution $f \times \varphi_k(z)$ as

$$\begin{aligned} f \times \varphi_k(z) = \int f(w) \Big(& \varphi_k(z-w) \exp(-(i/2) \operatorname{Im} z.\bar{w}) - \varphi_k(z-z_0) \\ & \exp(-(i/2) \operatorname{Im} z_0(\bar{w} - \bar{z})) \Big) \mathrm{d}w \end{aligned} \tag{3.14}$$

An application of Minkowski's integral inequality gives the estimate

$$\|f\times\varphi_k\|_2\leqslant\int|f(w)|\,\|g_k(.,w)\|_2\,\mathrm{d}w, \tag{3.15}$$

where $g_k(z,w)$ is the function inside the bracket on the right hand side of (3.14).

Now a simple calculation shows that

$$\begin{aligned}\|g_k(.,w)\|_2^2 &= \int(\varphi_k(z-w))^2\,\mathrm{d}z+\int(\varphi_k(z-z_0))^2\,\mathrm{d}z\\ &\quad-\int\varphi_k(z-w)\varphi_k(z-z_0)\exp(-(i/2)\mathrm{Im}(z\cdot\bar{w}-z_0\cdot\bar{w}-z\cdot\bar{z}_0))\,\mathrm{d}z\\ &\quad-\int\varphi_k(z-w)\varphi_k(z-z_0)\exp(i/2\,\mathrm{Im}(z\cdot\bar{w}-z_0\cdot\bar{w}-z\cdot\bar{z}_0))\,\mathrm{d}z.\end{aligned} \tag{3.16}$$

The first two terms are equal to $(2\pi)^n\varphi_k(0)$ each and the third term is

$$\begin{aligned}&\int\varphi_k(z+z_0-w)\varphi_k(z)\exp(i/2\,\mathrm{Im}(w-z_0))\cdot\bar{z}\,\mathrm{d}z\\ &=\int\varphi_k(w-z_0-z)\varphi_k(z)\exp\left(\frac{i}{2}\mathrm{Im}(w-z_0)\cdot\bar{z}\right)\mathrm{d}z\\ &=\varphi_k\times\varphi_k(w-z_0)=(2\pi)^n\varphi_k(w-z_0).\end{aligned}$$

Similarly, the fourth term is also $(2\pi)^n\varphi_k(w-z_0)$. Therefore,

$$\int|g_k(z,w)|^2\,\mathrm{d}z=(2\pi)^n(\varphi_k(0)-\varphi_k(w-z_0)). \tag{3.17}$$

Writing $\varphi_k(0)=\varphi_k(z_0-z_0)$ the mean value theorem gives

$$|\varphi_k(0)-\varphi_k(w-z_0)|\leqslant\sum_{j=1}^{n}\sup\left|\frac{\partial}{\partial w_j}\varphi_k\right||z_0-w|^2. \tag{3.18}$$

Recall that the Laguerre polynomials $L_k^{n-1}(t)$ satisfy the relation

$$\frac{\mathrm{d}}{\mathrm{d}t}L_k^{n-1}(t)=-\,L_{k-1}^{n}(t) \tag{3.19}$$

and $L_k^n(t)\exp(-(1/2)t)$ satisfy the estimate

$$\sup_{t>0}|\,L_k^n(t)\exp(-(1/2)t)|\leqslant Ck^n. \tag{3.20}$$

Using (3.19) and (3.20) in (3.18) we obtain

$$|\varphi_k(0)-\varphi_k(w-z_0)|\leqslant Ck^n|z_0-w|^2. \tag{3.21}$$

As f is supported in $|z_0 - w| \leqslant 2r$ we have from (3.15) and (3.21)

$$\|f \times \varphi_k\|_2 \leqslant Ck^{n/2} \int_{|z_0 - w| \leqslant 2r} |f(w)|\,|z_0 - w|\,\mathrm{d}w \leqslant C(2k+n)^{n/2} r. \tag{3.22}$$

Now we are in a position to estimate the sum (3.12). Using (3.22) the sum taken for $k \leqslant r^{-2}$ gives

$$\sum_{k \leqslant r^{-2}} (2k+n)^{-n} \|f \times \varphi_k\|_2^2 \leqslant C. \tag{3.23}$$

On the other hand using the estimate

$$\|f\|_2^2 = \int_{|z_0 - w| \leqslant 2r} |f(w)|\,\mathrm{d}w \leqslant Cr^{-2n} \tag{3.24}$$

we also get

$$\sum_{k > r^{-2}} (2k+n)^{-n} \|f \times \varphi_k\|_2^2 \leqslant Cr^{2n} \|f\|_2^2 \leqslant C. \tag{3.25}$$

Hence (3.12) is established and consequently the theorem follows.

4. A localisation theorem for special Hermite expansions

In this section we prove Theorem B stated in the introduction. The proof is based on the following fact. If a function $g \in L^2(\mathbb{R}_+, r^{2n-1}\mathrm{d}r)$ then the Fourier-Laguerre coefficients $(g, \psi_k) \to 0$ as $k \to \infty$. Recalling the definition of ψ_k, this means that

$$\int_0^\infty g(r)\varphi_k(r) r^{2n-1}\,\mathrm{d}r = o((2k+n)(n-1)/2) \tag{4.1}$$

as $k \to \infty$. If in addition $g \in W_R^s(\mathbb{R}_+)$ then it follows that

$$\int_0^\infty g(r)\varphi_k(r) r^{2n-1}\,\mathrm{d}r = o((2k+n)(n-1)/2). \tag{4.2}$$

In view of Theorem 3.1, Theorem B is an immediate corollary of the following result.

Theorem 4.1. *Let $n \geqslant 2$ and f be a compactly supported function vanishing in a neighbourhood of a point $z \in \mathbb{C}^n$. Further assume that $f \times \mu_r(z)$ belongs to $W_R^{n/2}(\mathbb{R}_+)$ as a function of r. Then $S_N f(z) \to 0$ as $N \to \infty$.*

Proof. From the relation (1.11) and the orthogonality of φ_k we can write the partial sums $S_N f(z)$ as

$$S_N f(z) = \int_0^\infty f \times \mu_r(z) \left(\sum_{k=0}^{N} \varphi_k(r) \right) r^{2n-1}\,\mathrm{d}r. \tag{4.3}$$

Using the formulas (see Szego [12])

$$\sum_{k=0}^{N} L_k^{n-1}\left(\frac{1}{2}r^2\right)=L_N^n\left(\frac{1}{2}r^2\right), \tag{4.4}$$

$$\frac{1}{2}r^2 L_N^n\left(\frac{1}{2}r^2\right)=(N+n)\,L_N^{n-1}\left(\frac{1}{2}r^2\right)-(N+1)\,L_{N+1}^{n-1}\left(\frac{1}{2}r^2\right) \tag{4.5}$$

we can write (4.3) in the form

$$S_N f(z)=\int_0^\infty g_z(r)((N+n)\varphi_N(r)-(N+1)\varphi_{N+1}(r))r^{2n-1}\,dr \tag{4.6}$$

where we have written $g_z(r)=2r^{-2}f\times\mu_r(z)$. Rewriting

$$\begin{aligned}(N+n)\,L_N^{n-1}\left(\frac{1}{2}r^2\right)&-(N+1)\,L_{N+1}^{n-1}\left(\frac{1}{2}r^2\right)\\ &=(N+n)\left(L_N^{n-1}\left(\frac{1}{2}r^2\right)-L_{N+1}^{n-1}\left(\frac{1}{2}r^2\right)\right)+(n-1)\,L_{N+1}^{n-1}\left(\frac{1}{2}r^2\right)\end{aligned} \tag{4.7}$$

and using the formula

$$L_{N+1}^{n-1}\left(\frac{1}{2}r^2\right)-L_N^{n-1}\left(\frac{1}{2}r^2\right)=L_{N+1}^{n-2}\left(\frac{1}{2}r^2\right) \tag{4.8}$$

we finally express (4.6) as the sum of the following two terms:

$$(n-1)\int_0^\infty g_z(r)\varphi_{N+1}(r)r^{2n-1}\,dr, \tag{4.9}$$

and

$$(N+n)\int_0^\infty g_z(r)\,L_{N+1}^{n-2}\left(\frac{1}{2}r^2\right)\exp(-(1/4)r^2)r^{2n-1}\,dr. \tag{4.10}$$

Now if f is supported in $|w|\leqslant b$ and vanishes in $|z-w|\leqslant a$ then $f\times\mu_r(z)$ as a function of r is supported in $a\leqslant r\leqslant b+|z|$. Consider the integral (4.9) first. As we are assuming that $f\times\mu_r(z)$ belongs to $W_R^{n/2}(\mathbb{R}_+)$ it also belongs to $H^{n/2}(\mathbb{R}_+)$. As it vanishes near $0, g_z(r)$ belongs to $H^{n/2}(\mathbb{R}_+)$ as well. The compactness of the support of g_z implies that $g_z\in W_R^{n/2}(\mathbb{R}_+)$ and hence by the remark (4.2) it follows that

$$\int_0^\infty g_z(r)\varphi_{N+1}(r)r^{2n-1}\,dr=o((2N+n)^{-(1/2)}). \tag{4.11}$$

Thus the term (4.9) tends to 0 as $N\to\infty$.

We write the term (4.10) as

$$(N+n)\int_0^\infty f\times\mu_r(z)\,L_{N+1}^{n-2}\left(\frac{1}{2}r^2\right)\exp(-(1/4)r^2)r^{2n-3}\,dr. \tag{4.12}$$

The function $f \times \mu_r(z)$ belongs to $W_R^{n/2}(\mathbb{R}_+)$ and as it is compactly supported it also belongs to the same space defined using $L_k^{n-2}(1/2r^2)\exp(-(1/4)r^2)$ instead of $\varphi_k(r)$. Therefore,

$$\int_0^\infty f \times \mu_r(z)\, L_{N+1}^{n-2}\left(\frac{1}{2}r^2\right)\exp(-(1/4)r^2)r^{2n-3}\,dr = o(N^{-1}) \tag{4.13}$$

and hence the term (4.10) also converges to 0 as $N \to \infty$. This completes the proof of the theorem.

In the case of compact symmetric spaces of rank one a localisation theorem for spherical harmonic expansions was proved under the only assumption that f is in L^2. As we are dealing with the noncompact situation in order to prove theorem B we have had to assume a further regularity assumption on f. We don't know if the condition $f \in W_L^{1/2}(\mathbb{C}^n)$ is optimal; nevertheless some regularity assumption is essential as the following counter-example shows.

Consider the function f which is the characteristic function of the annulus $1 \leqslant |z| \leqslant 2$. As f is radial, the special Hermite expansion takes the form

$$f(z) = c_n \sum_{k=0}^{\infty} \frac{k!(n-1)!}{(k+n-1)!}\left(\int_1^2 \varphi_k(s)s^{2n-1}\,ds\right)\varphi_k(z), \tag{4.14}$$

where c_n is a constant. Hence the spherical means $f \times \mu_r(z)$ has the expansion

$$f \times \mu_r(z) = c_n' \sum_{k=0}^{\infty}\left(\frac{k!(n-1)!}{(k+n-1)!}\right)^2\left(\int_1^2 \varphi_k(s)s^{2n-1}\,ds\right)\varphi_k(r)\varphi_k(z). \tag{4.15}$$

Setting $z=0$ and remembering $\varphi_k(0) = (k+n-1)!/k!(n-1)!$ we get

$$f \times \mu_r(0) = c_n' \sum_{k=0}^{\infty}\left(\frac{k!(n-1)!}{(k+n-1)!}\right)^{1/2}\left(\int_1^2 \varphi_k(s)s^{2n-1}\,ds\right)\psi_k(r). \tag{4.16}$$

It is clear that $f \times \mu_r(0)$ belongs to $L^2(\mathbb{R}_+, r^{2n-1}\,ds)$. We claim that it does not belong to $W_R^{n/2}(\mathbb{R}_+)$ and the partial sums $S_N f(0)$ does not converge to 0.

In order to prove these two claims we need the following asymptotic property of the Laguerre functions (see Szego [12]): for $x > 0$

$$L_k^\alpha(x^2)\exp(-(1/2)x^2)x^\alpha = k^{-\alpha/2}\left(\frac{\Gamma(k+\alpha+1)}{\Gamma(k+1)}\right)J_\alpha(2\sqrt{K}x) + 0(x^{\alpha/2-3/4}) \tag{4.17}$$

where $K = k + (\alpha+1)/2$ and the bound holds uniformly in $0 < x \leqslant x_0$. As $J_\alpha(t)$ behaves like $t^{-1/2}$ as $t \to \infty$ it follows from (4.17) that

$$\varphi_k(s) \sim (2k+n)^{(n-1)/2-1/4}, 1 \leqslant s \leqslant 2 \tag{4.18}$$

as $k \to \infty$. This shows that $f \times \mu_r(0)$ is certainly not in $W_R^{n/2}(\mathbb{R}_+)$.

To see that $S_N f(0)$ does not converge we have

$$S_N f(0) = \int_0^\infty f \times \mu_r(0)\, L_N^n\left(\frac{1}{2}r^2\right)\exp(-(1/4)r^2)r^{2n-1}\,dr. \tag{4.19}$$

A calculation shows that $f \times \mu_r(0) = 1$ for $1 \leqslant r \leqslant 2$ and is zero elsewhere. Thus

$$S_N f(0) = \int_1^2 L_N^n\left(\frac{1}{2}r^2\right) \exp(-(1/4)r^2) r^{2n-1} \mathrm{d}r. \tag{4.20}$$

Again (4.17) shows that $S_N f(0)$ behaves like $N^{n/2-1/4}$ as $N \to \infty$ and hence $S_N f(0)$ does not converge to 0 as $N \to \infty$.

In a recent article [5] Mark Pinsky has related the smoothness of the spherical means with pointwise convergence of the partial seems associated to Fourier integrals and Fourier series. In a similar vein we can also prove pointwise convergence of special Hermite expansions under some smoothness properties of the twisted spherical means. For example, let $n = 1$ and consider the partial sums $S_N f(z)$ which is given in terms of the spherical means as

$$S_N f(z) = \int_0^\infty f \times \mu_r(z)\, L_N^1\left(\frac{1}{2}r^2\right) \exp(-(1/4)r^2) r \,\mathrm{d}r. \tag{4.21}$$

Assume that $f \times \mu_r(z)$ is supported in $0 < r \leqslant a$ and that $\mathrm{d}/\mathrm{d}r f \times \mu_r(z)$ is piecewise continuous, the discontinuities being at $a_1 < a_2 < \ldots < a_m$ where $a_1 > 0$ and $a_m < a$.

Using the relation $\mathrm{d}/\mathrm{d}r\, L_k^\alpha(t) = -L_{k-1}^{\alpha+1}(t)$ we may write (4.21) as

$$S_N f(z) = \int_0^a f \times \mu_r(z) \exp(-(1/4)r^2) \frac{\mathrm{d}}{\mathrm{d}r}\left\{L_{N+1}^0\left(\frac{1}{2}r^2\right)\right\} \mathrm{d}r. \tag{4.22}$$

Integrating by parts we get

$$S_N f(z) = f(z+0) + \int_0^a \frac{\mathrm{d}}{\mathrm{d}r}\{f \times \mu_r(z) \exp(-(1/4)r^2)\}\, L_{N+1}^0\left(\frac{1}{2}r^2\right) \mathrm{d}r, \tag{4.23}$$

where $f(z+0) = \lim_{r \to 0} f \times \mu_r(z)$. By the asymptotic properties of L_N^0 it is easily seen that the integral on the right hand side of (4.23) tends to 0 as $N \to \infty$. Thus $S_N f(z) \to f(z+0)$ as $N \to \infty$. More generally, we can prove the following theorem.

Theorem 4.2. *Assume that $f \times \mu_r(z)$ is compactly supported in r and $\mathrm{d}^n/\mathrm{d}r^n(f \times \mu_r(z))$ is piecewise continuous. Then $S_N f(z) \to f(z+0)$ as $N \to \infty$.*

Proof. The proof follows by iteration. We have

$$S_N f(z) = \int_0^a f \times \mu_r(z) \exp(-(1/4)r^2)\, r^{2n-1} L_N^n\left(\frac{1}{2}r^2\right) \mathrm{d}r. \tag{4.24}$$

If D stands for the operator $1/r\ \mathrm{d}/\mathrm{d}r$ then we have the relation

$$(-1)^n D^n\left(L_{N+n}^0\left(\frac{1}{2}r^2\right)\right) = L_N^n\left(\frac{1}{2}r^2\right). \tag{4.25}$$

Using this to integrate by parts in (4.24) and remembering that $D^{n-1}(f \times \mu_r(z))$ is a continuous function of r we can prove that $S_N f(z) \to f(z+0)$ as $N \to \infty$ as in the case of $n = 1$. The details are left to the reader.

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