

**HOLOMORPHIC SOBOLEV SPACES ASSOCIATED TO  
COMPACT SYMMETRIC SPACES**

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Technical Report no. 2007/31  
July 13, 2007

# HOLOMORPHIC SOBOLEV SPACES ASSOCIATED TO COMPACT SYMMETRIC SPACES

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Dedicated to the memory of Mischa Cotlar

ABSTRACT. Using Gutzmer's formula, due to Lassalle, we characterise the images of Sobolev spaces under the Segal-Bargmann transform on compact Riemannian symmetric spaces. We also obtain necessary and sufficient conditions on a holomorphic function to be in the image of smooth functions and distributions under the Segal-Bargmann transform.

## 1. INTRODUCTION

In 1994 Brian Hall [12] studied the Segal-Bargmann transform on a compact Lie group  $G$ . For  $f \in L^2(G)$  let  $f * h_t$  be the convolution of  $f$  with the heat kernel  $h_t$  associated to the Laplacian on  $G$ . The Segal-Bargmann transform of  $f$ , also known as the heat kernel transform, is just the holomorphic extension of  $f * h_t$  to the complexification  $G_{\mathbb{C}}$  of  $G$ . The main result of Hall is a characterisation of the image of  $L^2(G)$  as a weighted Bergman space. This extended the classical results of Segal and Bargmann [4], where the same problem was considered on  $\mathbb{R}^n$ . Later, in [20], Stenzel treated the case of compact symmetric spaces obtaining a similar characterisation. Recently, some surprising results came out on Heisenberg groups (see Krötz-Thangavelu-Xu [16]

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1991 *Mathematics Subject Classification.* Primary: 58 J 35; Secondary: 22 E 30, 43 A 90.

*Key words and phrases.* Gutzmer's formula, Sobolev spaces, Bergman spaces, Segal-Bargmann transform, symmetric spaces.

) and Riemannian symmetric spaces of noncompact type ( see Krötz-Olafsson-Stanton [17]).

In 2004, Hall and Lewkeeratiyutkul [14] considered the Segal-Bargmann transform on Sobolev spaces  $\mathbb{H}^{2m}(G)$  on compact Lie groups. They have shown that the image can be characterised as certain holomorphic Sobolev spaces. The problem of treating the Segal-Bargmann transform on Sobolev spaces defined over compact symmetric spaces remains open. Our aim in this article is to characterise the image of  $\mathbb{H}^m(X)$  under the Segal-Bargmann transform as a holomorphic Sobolev space when  $X$  is a compact symmetric space.

Using an interesting formula due to Lassalle [18], called the Gutzmer's formula, Faraut [7] gave a nice proof of Stenzel's result. In this article we show that his arguments can be extended to treat Sobolev spaces as well. For the proof of our main theorem we need some estimates on derivatives of the heat kernel on a noncompact Riemannian symmetric space. This is achieved by using a result of Flensted-Jensen [8]. We also remark that the image of the Sobolev spaces turn out to be Bergman spaces defined in terms of certain weight functions. These weight functions are not necessarily nonnegative. Nevertheless, they can be used to define weighted Bergman spaces. This is reminiscent of the case of the heat kernel transform on the Heisenberg group. However, if we do not care about the isometry property of the Segal-Bargmann transform, then the images can be characterised as weighted Bergman spaces with nonnegative weight functions. Further, the isometry property of the heat kernel transform can be regained either by changing the original Sobolev norm into a different but equivalent one or by equipping the weighted Bergman space (with the positive weight function) with the previously defined norm (with the oscillating weight function)(see Theorems 3.3 and 3.5). That the weight function can be chosen to be nonnegative follows easily when the complexification of the noncompact dual of the compact symmetric space is of complex type. We use a reduction technique due to Flensted-Jensen to treat the general case.

In Section 4, we characterise the image of  $C^\infty(X)$  under the heat kernel transform. By using good estimates on the heat kernel on non-compact Riemannian symmetric spaces, recently proved by Anker and Ostellari [3], we obtain necessary and sufficient conditions on a holomorphic function to be in the image of  $C^\infty(X)$ . This extends the result of Hall and Lewkeeratiyutkul [14] to all compact symmetric spaces. We also characterise the image of distributions under the heat kernel transform, settling a conjecture stated in [14]. The results in Section 4 depend on the characterisation of holomorphic Sobolev spaces in terms of the holomorphic Fourier coefficients of a function. This in turn depends on the duality between Sobolev spaces  $\mathbb{H}_t^m(X_{\mathbb{C}})$  of positive order and  $\mathbb{H}_t^{-m}(X_{\mathbb{C}})$  of negative order. The latter spaces are easily shown to be Bergman spaces with non-negative weights.

The plan of the paper is as follows. We set up notation and collect relevant results on compact symmetric spaces and their complexifications in Section 2. We also indicate how Gutzmer's formula is used to study the image of  $L^2$  under the Segal-Bargmann transform. In Section 3, we introduce and obtain various characterisations of holomorphic Sobolev spaces  $\mathbb{H}_t^s(X_{\mathbb{C}})$ . Finally, in Section 4, we characterise the images of  $C^\infty$  functions and distributions on  $X$ .

## 2. COMPACT RIEMANNIAN SYMMETRIC SPACES: NOTATIONS AND PRELIMINARIES

The aim of this section is to set up notation and recall the main results from the literature, which are needed in the sequel. The general references for this section are the papers of Lassalle [18], [19] and Faraut [7]. See also Helgason [15] and Flensted-Jensen [8].

**2.1. Compact symmetric spaces and their duals.** We consider a compact Riemannian symmetric space  $X = U/K$ , where  $(U, K)$  is a compact symmetric pair. By this we mean the following:  $U$  is a connected compact Lie group and  $(U^\theta)_0 \subset K \subset U^\theta$ , where  $\theta$  is an involutive automorphism of  $U$  and  $(U^\theta)_0$  is the connected component

of  $U^\theta = \{g \in U : \theta(g) = g\}$  containing the identity. We may assume that  $K$  is connected and  $U$  is semisimple. We denote by  $\mathfrak{u}$  and  $\mathfrak{k}$  the Lie algebras of  $U$  and  $K$  respectively so that  $\mathfrak{k} = \{Y \in \mathfrak{u} : d\theta(Y) = Y\}$ . The base point  $eK \in X$  will be denoted by  $o$ .

Let  $\mathfrak{p} = \{Y \in \mathfrak{u} : d\theta(Y) = -Y\}$  so that  $\mathfrak{u} = \mathfrak{k} \oplus \mathfrak{p}$ . Let  $\mathfrak{a}$  be a Cartan subspace of  $\mathfrak{p}$ . Then  $A = \exp \mathfrak{a}$  is a closed connected abelian subgroup of  $U$ . Every  $g \in U$  has a decomposition  $g = k \exp H, k \in K, H \in \mathfrak{p}$  which in general is not unique. The maximal torus of the symmetric space  $X = U/K$  is defined by  $A_0 = \{\exp H.o : H \in \mathfrak{a}\}$ , which can be identified with the quotient  $\mathfrak{a}/\Gamma$ , where  $\Gamma = \{H \in \mathfrak{a} : \exp H \in K\}$ .

Let  $U_{\mathbb{C}}$  (resp.  $K_{\mathbb{C}}$ ) be the universal complexification of  $U$  (resp.  $K$ ). As  $U$  is compact, we can identify  $U_{\mathbb{C}}$  as a closed subgroup of  $GL(N, \mathbb{C})$  for some  $N$ . The group  $K_{\mathbb{C}}$  sits inside  $U_{\mathbb{C}}$  as a closed subgroup. We may then consider the complex homogeneous space  $X_{\mathbb{C}} = U_{\mathbb{C}}/K_{\mathbb{C}}$ , which is a complex variety and gives the complexification of the symmetric space  $X = U/K$ . The Lie algebra  $\mathfrak{u}_{\mathbb{C}}$  of  $U_{\mathbb{C}}$  is the complexified Lie algebra  $\mathfrak{u}_{\mathbb{C}} = \mathfrak{u} + i\mathfrak{u}$ . For every  $g \in U_{\mathbb{C}}$  there exists  $u \in U$  and  $X \in \mathfrak{u}$  such that  $g = u \exp iX$ .

We let  $G = K \exp i\mathfrak{p}$ , which forms a closed subgroup of  $U_{\mathbb{C}}$ , whose Lie algebra is given by  $\mathfrak{g} = \mathfrak{k} + i\mathfrak{p}$ . It can be shown that  $G$  is a real linear reductive Lie group, which is semisimple whenever  $U$  is and  $(G, K)$  forms a noncompact symmetric pair relative to the restriction of the involution  $\theta$  to  $G$ . The symmetric space  $Y = G/K$  is called the noncompact dual of the compact symmetric space  $X$ . The set  $i\mathfrak{a}$  is a Cartan subspace for the symmetric space  $G/K$ . Let  $\Sigma = \Sigma(\mathfrak{g}, i\mathfrak{a})$  be the system of restricted roots. It is then known that  $\Sigma(\mathfrak{g}, i\mathfrak{a}) = \Sigma(\mathfrak{u}_{\mathbb{C}}, \mathfrak{a}_{\mathbb{C}})$ . Let  $\mathfrak{t}$  be a Cartan subalgebra of  $\mathfrak{u}$  containing  $\mathfrak{a}$  and let  $\Sigma(\mathfrak{u}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$  be the corresponding root system for the complex semisimple Lie algebra  $\mathfrak{u}_{\mathbb{C}}$ . Then the elements of  $\Sigma(\mathfrak{g}, i\mathfrak{a})$  are precisely the roots in  $\Sigma(\mathfrak{u}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ , that have a nontrivial restriction to  $\mathfrak{a}_{\mathbb{C}}$ , which explains the terminology 'restricted roots'.

We need the following integration formulas on  $X, X_{\mathbb{C}}$  and  $Y$ . A general reference for these formulas is Helgason [15] (Chap.I, Section 5.2). We choose a positive root system  $\Sigma^+$  and denote by  $(\mathfrak{ia})_+ = \{H \in \mathfrak{ia} : \alpha(H) > 0, \alpha \in \Sigma^+\}$  a positive Weyl chamber. Define  $J_0(H) = \prod_{\alpha \in \Sigma^+} (\sin(\alpha, iH))^{m_\alpha}$ , where  $m_\alpha$  is the dimension of the root space  $\mathfrak{g}_\alpha$ . Let the  $U$ -invariant measure on  $X$  be denoted by  $dm_0$ . Then, integration on  $X$  is given by the formula

$$\int_X f(x) dm_0(x) = c_0 \int_K \int_{\mathfrak{a}/\Gamma} f(k \exp H.o) J_0(H) dk dH.$$

For a proof of this formula, see Faraut [7] (Theorem 1V.1.1). We have a similar formula on the complexification.

Each point  $z \in X_{\mathbb{C}}$  can be written as  $z = g \exp(H).o$ , where  $g \in U$  and  $H \in \mathfrak{ia}$ . If  $g_1 \exp(H_1).o = g_2 \exp(H_2).o$ , then there exists  $w \in W$  such that  $H_2 = w.H_1$ . If we choose  $H \in \overline{\mathfrak{ia}_+}$ , then  $H$  is unique. Let  $dm$  be the  $U_{\mathbb{C}}$  invariant measure on  $X_{\mathbb{C}}$ . Then we have

$$\int_{X_{\mathbb{C}}} f(z) dm(z) = c \int_U \int_{(\mathfrak{ia})_+} f(g \exp H.o) J(H) dg dH,$$

where  $J(H) = \prod_{\alpha \in \Sigma^+} (\sinh 2(\alpha, H))^{m_\alpha}$ . (see Theorem IV.2.4 in Faraut [7]; the powers  $m_\alpha$  are missing in the formula for  $J(H)$ ). Finally, we also need an integration formula on the noncompact dual  $Y = G/K$ . If  $dm_1$  is the  $G$  invariant measure on  $Y$ , then

$$\int_Y f(y) dm_1(y) = c_1 \int_K \int_{\mathfrak{ia}} f(k \exp(H).o) J_1(H) dk dH,$$

where  $J_1(2H) = J(H)$  defined above.

**2.2. Gutzmer's formula.** For results in this section, we refer to the papers of Lassalle [18],[19] and the article by Faraut [7]. We closely follow the notations used in Faraut [7].

Given an irreducible unitary representation  $(\pi, V)$  of  $U$  and a function  $f \in L^1(U)$  we define

$$\hat{f}(\pi) = \int_U f(g) \pi(g) dg$$

where  $dg$  is the Haar measure on  $U$ . When  $f$  is a function on  $X$ , so that it can be considered as a right  $K$  invariant function on  $U$ , it can be shown that  $\hat{f}(\pi) = 0$  unless the representation  $(\pi, V)$  is spherical, which means that  $V$  has a unique  $K$  invariant vector. When  $(\pi, V)$  is spherical and  $u$  is the unit invariant vector, then  $\hat{f}(\pi)v = (v, u)\hat{f}(\pi)u$ . This means that  $\hat{f}(\pi)$  is of rank one. Let  $\hat{U}_K$  be the subset of the unitary dual  $\hat{U}$  containing spherical representations (also called class one representations). Then,  $\hat{U}_K$  is in one to one correspondence with a discrete subset  $\mathcal{P}^+$  of  $\mathfrak{a}^*$ , called the set of restricted dominant weights.

For each  $\lambda \in \mathcal{P}^+$  let  $(\pi_\lambda, V_\lambda)$  be a spherical representation of  $U$  of dimension  $d_\lambda$ . Let  $\{v_j^\lambda, 1 \leq j \leq d_\lambda\}$  be an orthonormal basis for  $V_\lambda$  with  $v_1^\lambda$  being the unique  $K$ -invariant vector. Then the functions

$$\varphi_j^\lambda(g) = (\pi_\lambda(g)v_1^\lambda, v_j^\lambda)$$

form an orthogonal family of right  $K$  invariant analytic functions on  $U$ . Note that each  $\varphi_j^\lambda(g)$  is right  $K$ -invariant and hence they can be considered as functions of the symmetric space. When  $x = g.o \in X$ , we simply denote by  $\varphi_j^\lambda(x)$  the function  $\varphi_j^\lambda(g.o)$ . The function  $\varphi_1^\lambda(g)$  is  $K$  biinvariant, called an elementary spherical function. It is usually denoted by  $\varphi_\lambda$ .

For  $f \in L^2(X)$ , we define its Fourier coefficients  $\hat{f}_j(\lambda), 1 \leq j \leq d_\lambda$  by

$$\hat{f}_j(\lambda) = \int_X f(x) \overline{\varphi_j^\lambda(x)} dm_0(x).$$

The Fourier series of  $f$  is written as

$$f(x) = \sum_{\lambda \in \mathcal{P}} d_\lambda \sum_{j=1}^{d_\lambda} \hat{f}_j(\lambda) \varphi_j^\lambda(x)$$

and the Plancherel theorem reads as

$$\int_X |f(x)|^2 dm_0(x) = \sum_{\lambda \in \mathcal{P}} d_\lambda \sum_{j=1}^{d_\lambda} |\hat{f}_j(\lambda)|^2.$$

Defining  $A_\lambda(f) = d_\lambda^{-\frac{1}{2}} \hat{f}(\pi_\lambda)$ , the Plancherel formula can be put in the form

$$\int_X |f(x)|^2 dm_0(x) = \sum_{\lambda \in \mathcal{P}} d_\lambda \|A_\lambda(f)\|^2.$$

Let  $\Omega$  be an  $U$  invariant domain in  $X_{\mathbb{C}}$  and let  $\mathcal{O}(\Omega)$  stand for the space of holomorphic functions on  $\Omega$ . The group  $U$  acts on  $\mathcal{O}(\Omega)$  by  $T(g)f(z) = f(g^{-1}z)$ . For each  $\lambda \in \mathcal{P}^+$  the matrix coefficients  $\varphi_j^\lambda$  extend to  $X_{\mathbb{C}}$  as holomorphic functions. When  $f \in \mathcal{O}(\Omega)$ , it can be shown that the series

$$\sum_{\lambda \in \mathcal{P}} d_\lambda \sum_{j=1}^{d_\lambda} \hat{f}_j(\lambda) \varphi_j^\lambda(z)$$

converges uniformly over compact subsets of  $\Omega$ . Thus we have the expansion

$$f(z) = \sum_{\lambda \in \mathcal{P}} d_\lambda \sum_{j=1}^{d_\lambda} \hat{f}_j(\lambda) \varphi_j^\lambda(z),$$

called the Laurent expansion of  $f$ . The following formula, known as Gutzmer's formula, is very crucial for our main result.

**Theorem 2.1.** (*Gutzmer's formula*) *For every  $f \in \mathcal{O}(X_{\mathbb{C}})$  and  $H \in \mathfrak{ia}$ , we have*

$$\int_U |f(g \cdot \exp(H) \cdot o)|^2 dg = \sum_{\lambda \in \mathcal{P}^+} d_\lambda \|A_\lambda(f)\|^2 \varphi_\lambda(\exp(2H) \cdot o).$$

This theorem is due to Lasalle; we refer to [18] and [19] for a proof. See also Faraut [7]. Polarisation of the above formula gives

$$\begin{aligned} & \int_U f(g \cdot \exp(H) \cdot o) \overline{h(g \cdot \exp(H) \cdot o)} dg \\ &= \sum_{\lambda \in \mathcal{P}^+} d_\lambda \left( \sum_{j=1}^{d_\lambda} \hat{f}_j(\lambda) \overline{\hat{h}_j(\lambda)} \right) \varphi_\lambda(\exp(2H) \cdot o), \end{aligned}$$

for any two  $f, h \in \mathcal{O}(X_{\mathbb{C}})$ .



**2.3. Segal-Bargmann transform.** We now turn our attention to the Segal-Bargmann or heat kernel transform on  $X$ . Let  $D$  stand for the Laplace operator on the symmetric space, defined in Faraut [7]. The functions  $\varphi_j^\lambda$  turn out to be eigenfunctions of  $D$ , with eigenvalues  $\kappa(\lambda) = -(|\lambda|^2 + 2\rho(\lambda))$ , where  $\rho$  is the half sum of positive roots. We let  $\Delta = D - |\rho|^2$  so that the eigenvalues of  $\Delta$  are given by  $-|\lambda + \rho|^2$ . Note that our  $\delta$  differs from the standard Laplacian  $D$  by a constant. To avoid further notation we have denoted the shifted Laplacian by the symbol  $\Delta$ , which is generally used for the unshifted one.

Given  $f \in L^2(X)$ , the function  $u(g, t)$ , defined by the expansion

$$u(g, t) = \sum_{\lambda \in \mathcal{P}^+} d_\lambda e^{-t|\lambda + \rho|^2} \sum_{j=1}^{d_\lambda} \hat{f}_j(\lambda) \varphi_j^\lambda(g)$$

solves the heat equation

$$\partial_t u(g, t) = \Delta u(g, t), \quad u(g, 0) = f(g).$$

Defining the heat kernel  $\gamma_t(g)$  by

$$\gamma_t(g) = \sum_{\lambda \in \mathcal{P}^+} d_\lambda e^{-t|\lambda + \rho|^2} \varphi_\lambda(g),$$

we can write the solution as

$$u(g, t) = f * \gamma_t(g) = \int_U f(h) \gamma_t(h^{-1}g) dh.$$

The heat kernel  $\gamma_t$  is analytic, strictly positive and satisfies  $\gamma_t * \gamma_s = \gamma_{t+s}$ . Moreover, it extends to  $X_{\mathbb{C}}$  as a holomorphic function. It can be shown that for each  $f \in L^2(X)$ , the function  $u(g, t) = f * \gamma_t(g)$  also extends to  $X_{\mathbb{C}}$  as a holomorphic function. The transformation taking  $f$  into the holomorphic function  $u(z, t) = f * \gamma_t(g.o)$ ,  $z = g.o$ ,  $g \in U_{\mathbb{C}}$  is called the Segal-Bargmann or heat kernel transform.

The image of  $L^2(X)$  under this transform was characterised as a weighted Bergman space by Stenzel in [20], which was an extension of the result of Hall [12] for the case of compact Lie groups. Another proof of Stenzel's theorem was given by Faraut in [7] using Gutzmer's

formula. Since we are going to use similar arguments in our characterisations of holomorphic Sobolev spaces, it is informative to briefly sketch the proof given by Faraut [7].

Let  $\gamma_t^1$  be the heat kernel associated to the Laplace-Beltrami operator  $\Delta_G$  on the noncompact Riemannian symmetric space  $Y = G/K$ . Then,  $\gamma_t^1$  is given by

$$\gamma_t^1(g) = \int_{i\mathbf{a}} e^{-t(|\mu|^2 + |\rho|^2)} \psi_\mu(g) |c(\mu)|^{-2} d\mu,$$

where  $\psi_\mu$  are the spherical functions of the pair  $(G, K)$ . This is the standard representation of the heat kernel on a noncompact symmetric space using Fourier inversion. Here  $c(\mu)$  is the  $c$ -function associated to  $Y = G/K$ . The heat kernel  $\gamma_t^1$  is characterised by the defining property

$$\int_Y \gamma_t^1(g) \psi_{-\mu}(g) dm_1(g) = e^{-t(|\mu|^2 + |\rho|^2)}, \quad \mu \in i\mathbf{a},$$

where  $dm_1$  is the  $G$  invariant measure on  $Y$ . In view of the integration formula mentioned earlier this reads as

$$\int_{i\mathbf{a}} \gamma_t^1(\exp(H).o) \psi_\mu(\exp(H).o) J_1(H) dH = e^{-t(|\mu|^2 + |\rho|^2)}.$$

Note that both sides of the above equation admit holomorphic extensions as a function of  $\mu \in \mathbf{a}_\mathbb{C}$  and hence the above equation is valid for all  $\mu \in \mathbf{a}_\mathbb{C}$ . In particular,

$$\int_Y \gamma_t^1(g) \psi_{-i\mu}(g) dm_1(g) = e^{t(|\mu|^2 - |\rho|^2)}, \quad \mu \in i\mathbf{a}.$$

We can now prove the following result, which characterises the image of  $L^2(X)$  under the Segal-Bargmann transform. Define  $p_t(z)$  on  $X_\mathbb{C}$  by

$$p_t(z) = p_t(g \exp(H).o) = \gamma_{2t}^1(\exp(2H).o), \quad g \in U, H \in i\mathbf{a}.$$

**Theorem 2.2.** (Stenzel) *A holomorphic function  $F \in \mathcal{O}(X_\mathbb{C})$  is of the form  $f * \gamma_t$  for some  $f \in L^2(X)$  if and only if*

$$\int_{X_\mathbb{C}} |F(z)|^2 p_t(z) dm(z) < \infty.$$

*Proof.* The integration formula on  $X_{\mathbb{C}}$  together with Gutzmer's formula leads to

$$\int_{X_{\mathbb{C}}} |F(z)|^2 p_t(z) dm(z) = c_1 \sum_{\lambda \in \mathcal{P}^+} d_{\lambda} \|A_{\lambda}(f)\|^2 \times \\ e^{-2t|\lambda+\rho|^2} \int_{i\mathfrak{a}} \varphi_{\lambda}(\exp(2H).o) \gamma_{2t}^1(\exp(2H).o) J_1(2H) dH.$$

We now make use of the well known relation

$$\varphi_{\lambda}(\exp(H).o) = \psi_{-i(\lambda+\rho)}(\exp(H).o).$$

Using this and recalling the defining relation for  $\gamma_t^1$ , we get

$$\int_{i\mathfrak{a}} \varphi_{\lambda}(\exp(2H).o) \gamma_{2t}^1(\exp(2H).o) J_1(2H) dH = c e^{2t|\lambda+\rho|^2} e^{-2t|\rho|^2}$$

for some constant  $c$ . Hence

$$\int_{X_{\mathbb{C}}} |F(z)|^2 p_t(z) dm(z) = c_t \int_X |f(x)|^2 dm_0(x).$$

This completes the proof of the theorem.  $\square$

**2.4. Some estimates for the heat kernel on  $G/K$ .** The heat kernel  $\gamma_t^1$  on the noncompact dual  $Y = G/K$  of  $X = U/K$  is explicitly known only when  $G$  is a complex Lie group, see Gangolli [9]. This happens precisely when we are dealing with compact Lie groups as symmetric spaces. In this case, we have explicit formulas even for derivatives of the heat kernel and this has been made use of by Hall and Lewkeeratiyutkul [14] in their study of holomorphic Sobolev spaces associated to compact Lie groups. In 2003, Anker and Ostellari [3] has sketched a proof for the following estimate for the heat kernel  $\gamma_t^1$ . For a fixed  $t > 0$ , their main result says that  $\gamma_t^1(\exp H)$  behaves like

$$\Phi(H)^{1/2} e^{-t|\rho|^2} e^{-\frac{1}{4t}|H|^2}, \quad H \in i\mathfrak{a},$$

where  $\Phi$  is the function defined on  $i\mathfrak{a}$  by

$$\Phi(H) = \prod_{\alpha \in \Sigma^+} \left( \frac{(\alpha, H)}{\sinh(\alpha, H)} \right)^{m_{\alpha}}.$$

The following remarks on the  $\Phi$  function are important. Note that the product is taken with respect to all the restricted roots for the pair

$(\mathfrak{g}, i\mathfrak{a})$ . The product remains unaltered even if we take it over all roots in  $\Sigma(\mathfrak{u}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$  since  $(\alpha, H) = 0$  for all elements of  $\Sigma(\mathfrak{u}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$  which are not in  $(\mathfrak{g}, i\mathfrak{a})$ . We note that

$$\Phi(H) = \prod_{\alpha \in \Sigma^+} \left( \frac{(\alpha, H)}{\sinh(\alpha, H)} \right)^{m_\alpha} = J_1(H)^{-1} \prod_{\alpha \in \Sigma^+} (\alpha, H)^{m_\alpha}.$$

We make use of these facts later.

Complete proof of the above estimate for the heat kernel is not yet available but we believe the arguments of Anker and Ostellari are sound. The estimates are known to be true in several particular cases by different methods. In an earlier paper Anker [1] has established slightly weaker estimates, (which are polynomially close to the optimal estimates) whenever  $G$  is a normal real form. These are good enough for some purposes. For example, in the characterisations of the images of smooth functions and distributions the polynomial discrepancies do not really matter. We are thankful to the referee for pointing this out.

For the study of holomorphic Sobolev spaces on  $X_{\mathbb{C}}$ , we also need estimates on the  $t$ -derivatives of  $\gamma_t^1$ . We do not have any result available in the literature, except when  $G$  is complex or  $G/K$  is of rank one. However, there is a powerful method of reduction to the complex case by Flensted-Jensen, using which we can express the heat kernel  $\gamma_t^1$  on  $G/K$  in terms of the heat kernel  $\Gamma_t$  on  $U_{\mathbb{C}}/U$ . As the latter heat kernel is known explicitly, we can get estimates for  $\gamma_t^1$  and its derivatives. We recall this result from Flensted-Jensen [8] and state the result using our notation. (In [8] the group  $G$  stands for a complex Lie group, and  $G_0$  the real group whose Lie algebra  $\mathfrak{g}_0$  is a real form of  $\mathfrak{g}$ . This should not cause any confusion. We refer the reader to [8] ( Theorem 6.1 and Example on page 131) for details.)

Recall that  $U$  is a maximal compact subgroup of  $U_{\mathbb{C}}$ . We let  $K_c$  stand for the noncompact group whose Lie algebra is  $\mathfrak{k} + i\mathfrak{k}$ , a subalgebra of  $\mathfrak{u}_{\mathbb{C}} = \mathfrak{u} + i\mathfrak{u}$ . In [8], Flensted-Jensen has proved that there is a one to one correspondence between  $K$ -biinvariant functions on  $G$  and certain functions on  $U_{\mathbb{C}}$ , that are right  $U$ -invariant and left  $K_c$  invariant

( see Theorem 5.2 in [8]). This isomorphism is denoted by  $f \rightarrow f^\eta$  and satisfies  $f^\eta(g) = f(g\theta(g)^{-1})$  for all  $g \in G$ . Let  $g_t$  and  $G_t$  be the Gauss kernels on  $G/K$  and  $U_{\mathbb{C}}/U$  respectively as defined by Flensted-Jensen. These are almost the heat kernels  $\gamma_t^1$  and  $\Gamma_t$  differing from them only by multiplicative constants. The formula connecting  $g_t$  and  $G_t$  is given by

$$g_t(x) = \int_{K_c} G_t(hx)dh, \quad x \in G.$$

The above formula has to be interpreted using the isomorphism  $f \rightarrow f^\eta$ .

The above formula connecting  $g_t$  and  $G_t$  leads to a similar formula for  $\gamma_t^1$  and  $\Gamma_t$ . For a reader not familiar with the work of Flensted-Jensen, the above formula might appear a bit mysterious. However, the mystery can be unravelled if we recall that  $f^\eta(\exp H) = f(\exp(2H))$  for  $H \in \mathfrak{p}$ . If we properly take care of the definitions of various inner products and Laplacians, then the final formula connecting the two heat kernels take the form

$$\gamma_t^1(\exp H) = \int_{K_c} \Gamma_{t/4}(h \exp(H/2))dh, \quad H \in \mathfrak{ia}.$$

It can be directly checked that the function defined by the integral on the right hand side solves the heat equation on  $G/K$ , which follows by the invariance of the Laplacian. We are indebted to the referee for this reasoning, leading to the correct scaling of the heat kernels in the above formula.

We have the following explicit formula for the heat kernel  $\Gamma_t$  obtained by Gangolli [9]:

$$\Gamma_t(\exp H) = c(4t)^{-n/2} \prod \frac{(\alpha, H)}{\sinh(\alpha, H)} e^{-t|\rho|^2} e^{-\frac{1}{4t}|H|^2}$$

where the product is taken over all positive roots in  $\Sigma(\mathfrak{u}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ . Using this formula and the connection between  $\gamma_t^1$  and  $\Gamma_t$  we can prove the following estimate.

**Theorem 2.3.** *For every  $s > t, m \in \mathbb{N}$  and  $H \in \mathfrak{ia}$  we have*

$$|\partial_t^m \gamma_t^1(\exp H)| \leq C_{s,t,m} e^{-\frac{1}{4s}|H|^2}.$$

*Proof.* First consider the case  $m = 0$ . Since  $|\exp H| \leq |h \exp H|$  (see [8], eqn. 6.5) the formula for  $\gamma_t^1$  in terms of  $\Gamma_t$ , gives

$$\gamma_t^1(\exp H)e^{\frac{1}{4s}|H|^2} \leq \int_{K_c} \Gamma_{t/4}(h \exp(H/2))e^{\frac{1}{4s}|h \exp H|^2} dh.$$

We only need to show that the right hand side is a bounded function of  $H$ . In view of the formula for  $\Gamma_t$ , we see that  $\Gamma_{t/4}(h \exp(H/2))e^{\frac{1}{4s}|h \exp H|^2}$  is bounded by a constant times  $\Gamma_{r/4}(h \exp(H/2))$ , where  $r = (st)/(s - t)$ . Thus, using the Flensted-Jensen formula once again, we see that  $\gamma_t^1(\exp H)e^{\frac{1}{4s}|H|^2}$  is bounded by a constant times  $\gamma_r^1(\exp H)$ , which is clearly bounded.

In the case of derivatives, we need to show that the function defined by

$$\int_{K_c} P_{t,s}(h \exp(H/2))\Gamma_{r/4}(h \exp(H/2))dh$$

is bounded for any polynomial  $P_{t,s}$ . The spherical Fourier transform of this function on  $G$  can be expressed as the spherical Fourier transform on  $U_{\mathbb{C}}/U$  of the integrand, (evaluated at  $h = \text{identity}$ ), which can be calculated in terms of derivatives of the spherical Fourier transform of  $\Gamma_{r/4}$ , which is a Gaussian. The latter is a Schwartz function, which means that the spherical Fourier transform of the integral is a Schwartz function on  $G$  and hence bounded.

We would like to conclude this proof with a couple of remarks. The above connection between the 'two Fourier transforms' is stated and proved as Theorem 6.1 in [8]. For the case of the Gauss-kernel (alias heat kernel) Flensted-Jensen has explicitly discussed this connection at the end of Section 6 in [8] (see Example on page 131). We also take this opportunity to indicate another proof suggested by the referee: the time derivative of  $\Gamma_t$  pulls down a polynomial factor in  $H$ , with coefficients that depend on  $t$ . Thus,

$$|\partial_t^m \Gamma_t(\exp H)| \leq C_{t,m,\epsilon} e^{\epsilon|H|^2} \Gamma_t(\exp H).$$

In view of the case  $m = 0$ , this gives us the desired estimate. □

### 3. HOLOMORPHIC SOBOLEV SPACES

In this section we introduce and study holomorphic Sobolev spaces  $H^s(X_{\mathbb{C}})$  for any  $s \in \mathbb{R}$ . When  $s = -m$  is a negative integer we show that  $H^s(X_{\mathbb{C}})$  is a weighted Bergman space. But when  $s = m$  is a positive integer  $H^s(X_{\mathbb{C}})$  can be described as the completion of a weighted Bergman space with respect to a smaller norm. Later, using the reduction formula of Flensted-Jensen [8], we show that we can choose a positive weight function, so that  $H^m(X_{\mathbb{C}})$  can be described as a weighted Bergman space in all the cases.

**3.1. Holomorphic Sobolev spaces.** Recall that for each real number  $s$ , the Sobolev space  $\mathbb{H}^s(X)$  of order  $s$  can be defined as the completion of  $C^\infty(X)$  under the norm  $\|f\|_{(s)} = \|(1 - \Delta)^{\frac{s}{2}} f\|_2$ . In view of Plancherel theorem, a distribution  $f$  on  $X$  belongs to  $H^s(X)$  if and only if

$$\sum_{\lambda \in \mathcal{P}^+} d_\lambda (1 + |\lambda + \rho|^2)^s \|A_\lambda(f)\|^2 < \infty.$$

We define  $\mathbb{H}_t^s(X_{\mathbb{C}})$  to be the image of  $\mathbb{H}^s(X)$  under the heat kernel transform. This can be made into a Hilbert space, simply by transferring the Hilbert space structure of  $\mathbb{H}^s(X)$  to  $\mathbb{H}_t^s(X_{\mathbb{C}})$ . This means that if  $F = f * \gamma_t, G = g * \gamma_t$ , where  $f, g \in \mathbb{H}^s(X)$  then  $(F, G)_{\mathbb{H}_t^s(X_{\mathbb{C}})} = (f, g)_{\mathbb{H}^s(X)}$ . Then, it is clear that the heat kernel transform is an isometric isomorphism from  $\mathbb{H}^s(X)$  onto  $\mathbb{H}_t^s(X_{\mathbb{C}})$ . We are interested in realising  $\mathbb{H}_t^s(X_{\mathbb{C}})$  as weighted Bergman spaces.

The spherical functions  $\varphi_j^\lambda, 1 \leq j \leq d_\lambda, \lambda \in \mathcal{P}^+$  form an orthogonal system in  $\mathbb{H}^s(X)$  for every  $s \in \mathbb{R}$ . More precisely,

$$(\varphi_j^\lambda, \varphi_k^\mu)_{\mathbb{H}^s(X)} = \delta_{j,k} \delta_{\lambda,\mu} d_\lambda^{-1} (1 + |\lambda + \rho|^2)^s.$$

From the definition of  $\mathbb{H}_t^s(X_{\mathbb{C}})$  it is clear that the holomorphically extended spherical functions  $\varphi_j^\lambda(g \exp(iH) \cdot o), 1 \leq j \leq d_\lambda, \lambda \in \mathcal{P}^+$  form an orthogonal system in  $\mathbb{H}_t^k(X_{\mathbb{C}})$  :

$$(\varphi_j^\lambda, \varphi_k^\mu)_{\mathbb{H}_t^s(X_{\mathbb{C}})} = \delta_{j,k} \delta_{\lambda,\mu} d_\lambda^{-1} e^{2t|\lambda + \rho|^2} (1 + |\lambda + \rho|^2)^s.$$

Suitably normalised, they form an orthonormal basis for  $\mathbb{H}_t^s(X_{\mathbb{C}})$ . This motivates us to define the holomorphic Fourier coefficients as follows.

For a holomorphic function  $F$  on  $X_{\mathbb{C}}$  we define its holomorphic Fourier coefficients by

$$\tilde{F}_j(\lambda) = \int_{X_{\mathbb{C}}} F(z) \overline{\varphi_j^\lambda(z)} p_t(z) dm(z).$$

Note that the holomorphic Fourier coefficients depend on  $t$ , which we have suppressed for the sake of simplicity. (For us  $t$  is fixed throughout). The integration formula on  $X_{\mathbb{C}}$  shows that

$$\tilde{F}_j(\lambda) = \int_{\mathfrak{ia}} \int_U F(g \exp H.o) \overline{\varphi_j^\lambda(g \exp H.o)} \gamma_{2t}^1(\exp 2H) J_1(2H) dg dH.$$

When  $F = f * \gamma_t$ , it follows from the polarised form of the Gutzmer's formula that  $\tilde{F}_j(\lambda) = e^{t|\lambda+\rho|^2} \hat{f}_j(\lambda)$ . This leads to the following characterisation.

**Theorem 3.1.** *A holomorphic function  $F$  on  $X_{\mathbb{C}}$  belongs to  $\mathbb{H}_t^s(X_{\mathbb{C}})$  if and only if*

$$\sum_{\lambda \in \mathcal{P}^+} d_\lambda \left( \sum_{j=1}^{d_\lambda} |\tilde{F}_j(\lambda)|^2 \right) (1 + |\lambda + \rho|^2)^s e^{-2t|\lambda+\rho|^2} < \infty.$$

**Corollary 3.2.** *The spaces  $\mathbb{H}_t^s(X_{\mathbb{C}})$  and  $\mathbb{H}_t^{-s}(X_{\mathbb{C}})$  are dual to each other and the duality bracket is given by*

$$(F, G) = \int_{X_{\mathbb{C}}} F(z) \overline{G(z)} p_t(z) dm(z).$$

*Proof.* From the (polarised) Gutzmer's formula we see that

$$\begin{aligned} & \int_{\mathfrak{ia}} \int_U F(g \exp H.o) \overline{G(g \exp H.o)} \gamma_{2t}^1(\exp H) J_1(2H) dg dH \\ &= \sum_{\lambda \in \mathcal{P}^+} d_\lambda \left( \sum_{j=1}^{d_\lambda} \hat{f}_j(\lambda) \overline{\hat{g}_j(\lambda)} \right) = \sum_{\lambda \in \mathcal{P}^+} d_\lambda \left( \sum_{j=1}^{d_\lambda} \tilde{F}_j(\lambda) \overline{\tilde{G}_j(\lambda)} \right) e^{-2t|\lambda+\rho|^2}, \end{aligned}$$

where  $F = f * \gamma_t$  and  $G = g * \gamma_t$ . Since  $\mathbb{H}^s(X)$  and  $\mathbb{H}^{-s}(X)$  are dual to each other under the duality bracket

$$(f, g) = \sum_{\lambda \in \mathcal{P}^+} d_\lambda \left( \sum_{j=1}^{d_\lambda} \hat{f}_j(\lambda) \overline{\hat{g}_j(\lambda)} \right),$$



it follows that the series

$$\sum_{\lambda \in \mathcal{P}^+} d_\lambda \left( \sum_{j=1}^{d_\lambda} \tilde{F}_j(\lambda) \overline{\tilde{G}_j(\lambda)} \right) e^{-2t|\lambda+\rho|^2}$$

converges whenever  $F \in \mathbb{H}_t^s(X_{\mathbb{C}})$  and  $G \in \mathbb{H}_t^{-s}(X_{\mathbb{C}})$ . This proves the corollary.  $\square$

Note that the duality bracket between  $\mathbb{H}_t^s(X_{\mathbb{C}})$  and  $\mathbb{H}_t^{-s}(X_{\mathbb{C}})$ , which can be put in the form

$$(F, G) = \int_{\mathfrak{ia}} \int_U F(g \exp H.o) \overline{G(g \exp H.o)} \gamma_{2t}^1(\exp(2H)) J_1(2H) dg dH$$

involves only the heat kernel  $\gamma_{2t}^1$ , but not its derivatives. This fact is crucial for some purposes.

**3.2.  $\mathbb{H}_t^m(X_{\mathbb{C}})$  as weighted Bergman spaces.** In proving Stenzel's theorem, we have made use of the crucial fact

$$\int_{\mathfrak{ia}} \gamma_{2t}^1(\exp(2H).o) \varphi_\lambda(\exp(2H)) J_1(2H) dH = c e^{2t|\lambda+\rho|^2}$$

for some positive constant  $c$ . Differentiating the above identity  $m$  times with respect to  $t$  we get

$$\int_{\mathfrak{ia}} \partial_t^m \gamma_{2t}^1(\exp(2H).o) \varphi_\lambda(\exp(2H)) J_1(2H) dH = c 2^m |\lambda + \rho|^{2m} e^{2t|\lambda+\rho|^2}.$$

In view of Gutzmer's formula, the natural weight function for  $\mathbb{H}_t^m(X_{\mathbb{C}})$  should be

$$w_t^m(z) = (1 + \partial_t)^m p_t(z).$$

But unfortunately, this weight function need not be positive and hence in defining a Bergman space with respect to  $w_t^m(z)$  we have to be careful.

Let  $\mathcal{F}_t^m(X_{\mathbb{C}})$  be the space of all  $F \in \mathcal{O}(X_{\mathbb{C}})$  such that

$$\int_{X_{\mathbb{C}}} |F(z)|^2 |w_t^m(z)| dm(z) < \infty.$$

We equip  $\mathcal{F}_t^m(X_{\mathbb{C}})$  with the sesquilinear form

$$(F, G)_m = \int_{X_{\mathbb{C}}} F(z) \overline{G(z)} w_t^m(z) dm(z).$$

We show below that this defines a pre-Hilbert structure on  $\mathcal{F}_t^m(X_{\mathbb{C}})$ . Let  $\mathcal{B}_t^m(X_{\mathbb{C}})$  be the completion of  $\mathcal{F}_t^m(X_{\mathbb{C}})$  with respect to the above inner product. We have the following characterisation of  $\mathbb{H}_t^m(X_{\mathbb{C}})$ .

**Theorem 3.3.** *For every nonnegative integer  $m$  we have  $\mathbb{H}_t^m(X_{\mathbb{C}}) = \mathcal{B}_t^m(X_{\mathbb{C}})$  and the heat kernel transform is an isometric isomorphism from  $\mathbb{H}^m(X)$  onto  $\mathcal{B}_t^m(X_{\mathbb{C}})$  upto a multiplicative constant.*

*Proof.* We first check that the sesquilinear form defined above is indeed an inner product. Let  $F, G \in \mathcal{F}_t^m(X_{\mathbb{C}})$ . In view of the integration formula on  $X_{\mathbb{C}}$ , the sesquilinear form is given by

$$(F, G)_m = \int_{\mathbf{ia}} \int_U F(u \exp(H).o) \overline{G(u \exp(H).o)} J_1(2H) du dH.$$

Then, by Gutzmer's formula, we have

$$\int_U |F(u \exp(H).o)|^2 du = \sum_{\lambda \in \mathcal{P}^+} d_{\lambda} \|A_{\lambda}(F)\|^2 \varphi_{\lambda}(\exp(2H))$$

for all  $H \in \mathbf{ia}$ . Since the left hand side is integrable with respect to  $|w_t^m(\exp(H).o)| J_1(2H)$ , so is the right hand side. By Fubini, we get

$$\begin{aligned} & \int_{\mathbf{ia}} \int_U |F(u \exp(H).o)|^2 w_t^m(\exp(H).o) J_1(2H) du dH \\ &= \sum_{\lambda \in \mathcal{P}^+} d_{\lambda} \|A_{\lambda}(F)\|^2 \int_{\mathbf{ia}} \varphi_{\lambda}(\exp(2H)) w_t^m(\exp(H).o) J_1(H) du dH. \end{aligned}$$

If we use the relation  $\varphi_{\lambda}(\exp H) = \psi_{-i(\lambda+\rho)}(\exp H)$ , the integral on the right hand side becomes a constant multiple of

$$\int_{\mathbf{ia}} (1 + \partial_t)^m \gamma_{2t}^1(\exp H) \psi_{-i(\lambda+\rho)}(\exp H) J_1(H) dH,$$

which is just  $e^{2t|\lambda+\rho|^2} (1 + |\lambda + \rho|^2)^m$ . This proves that

$$\begin{aligned} & \int_{X_{\mathbb{C}}} |F(z)|^2 w_t^m(z) dm(z) dz \\ &= \sum_{\lambda \in \mathcal{P}^+} d_{\lambda} e^{2t(|\lambda+\rho|^2)} (1 + |\lambda + \rho|^2)^m \|A_{\lambda}(F)\|^2 \end{aligned}$$

and hence the sesquilinear form is indeed positive definite.

The above calculation also shows that any  $F \in \mathcal{F}_t^m(X_{\mathbb{C}})$  is the holomorphic extension of  $f * \gamma_t$  for some  $f \in \mathbb{H}^m(X)$ . Indeed, we only have to define  $f$  by the expansion

$$f(g.o) = \sum_{\lambda \in \mathcal{P}^+} d_{\lambda} e^{t|\lambda + \rho|^2} \sum_{j=1}^{d_{\lambda}} \hat{F}_j(\lambda) \varphi_j^{\lambda}(g.o).$$

Here,  $\hat{F}_j(\lambda)$  are the Fourier coefficients of  $F$  defined by

$$\hat{F}_j(\lambda) = \int_X F(x) \overline{\varphi_j^{\lambda}(x)} dm_0(x).$$

Thus, we have proved that  $\mathcal{F}_t^m(X_{\mathbb{C}})$  is contained in  $\mathbb{H}_t^m(X_{\mathbb{C}})$ . And also, the norms are equivalent. To complete the proof of the theorem, it is enough to show that  $\mathcal{F}_t^m(X_{\mathbb{C}})$  is dense in  $\mathbb{H}_t^m(X_{\mathbb{C}})$ .

As we have already observed, the functions  $\varphi_j^{\lambda}$  initially defined on  $X$  have holomorphic extensions to  $X_{\mathbb{C}}$ . From the manner we have defined the holomorphic Sobolev spaces  $\mathbb{H}_t^m(X_{\mathbb{C}})$ , it follows that the functions  $\varphi_j^{\lambda}$ , after suitable normalisation, form an orthonormal basis for  $\mathbb{H}_t^m(X_{\mathbb{C}})$ . The proof will be complete if we can show that all  $\varphi_j^{\lambda}$  belong to  $\mathcal{F}_t^m(X_{\mathbb{C}})$ , since the finite linear combinations of them forms a dense subspace of  $\mathbb{H}_t^m(X_{\mathbb{C}})$ . As

$$\varphi_j^{\lambda} * \gamma_t(g \exp(H).o) = e^{-t|\lambda + \rho|^2} \varphi_j^{\lambda}(g \exp(H).o),$$

by applying Gutzmer's formula to the functions  $\varphi_j^{\lambda}(g \exp(H).o)$ , we only need to check if

$$\int_{i\mathfrak{a}} \varphi_{\lambda}(\exp(2H).o) |w_t^m(\exp(H).o)| J_1(2H) dH < \infty.$$

The functions  $\varphi_{\lambda}$  are known to satisfy the estimate

$$\varphi_{\lambda}(\exp H.o) \leq e^{\lambda(H)}$$

for all  $H \in i\mathfrak{a}$  (see Proposition 2 in Lassalle [18]). The weight function  $w_t^m$  involves derivatives of the heat kernel  $\gamma_t^1$ , for which we have the estimates stated in Theorem 2.3. Using them we can easily see that the above integrals are finite.  $\square$

**3.3. A positive weight function for  $\mathbb{H}_t^m(X_{\mathbb{C}})$ .** In this section, we show that the holomorphic Sobolev spaces  $\mathbb{H}_t^m(X_{\mathbb{C}})$  can be characterised as weighted Bergman spaces with non-negative weight functions. Note that if  $w_t^m$  happens to be positive then  $\mathcal{F}_t^m(X_{\mathbb{C}}) = \mathcal{B}_t^m(X_{\mathbb{C}}) = \mathbb{H}_t^m(X_{\mathbb{C}})$ . We show that it is possible to define a new weight function  $w_{t,\delta}^m$ , which will be positive and  $\mathbb{H}_t^m(X_{\mathbb{C}})$  is precisely the weighted Bergman space defined in terms of  $w_{t,\delta}^m$ . But we lose the isometry property of the heat kernel transform. If we are ready to change the norm on  $\mathbb{H}_t^m(X_{\mathbb{C}})$  into another equivalent norm, the isometry property can also be regained.

The case of compact Lie groups  $H$ , studied by Hall [12], corresponds to the symmetric space  $U/K$ , where  $U = H \times H$  and  $K$  is the diagonal subgroup of  $U$ . This is precisely the case for which the subgroup  $G$  of  $U_{\mathbb{C}}$  is a complex Lie group. Therefore, we do not have to use the result of Flensted-Jensen in getting estimates for the heat kernel on  $G/K$ . In this case, the weight function  $w_t^m$  can be modified to be positive. In [14], Hall and Lewkeeratiyutkul have shown that by a proper choice of  $\delta > 0$ , the kernel  $w_{t,\delta}^m(z) = (\delta + \partial_t^m)p_t(z)$  can be made positive. That this is indeed the case can be easily seen from the explicit formula for  $\gamma_t^1$  in the complex case. The kernel  $w_{t,\delta}^m(\exp H.o)$  turns out to be  $(P_t(H) + \delta)\gamma_{2t}^1(\exp(2H))$ , where  $P_t(H)$  is a polynomial. It is then clear that  $\delta$  can be chosen large enough to make  $(P_t(H) + \delta)$  positive. The same is true in the general case also.

**Theorem 3.4.** *Let  $m$  be a non-negative integer. Then  $F \in \mathbb{H}_t^m(X_{\mathbb{C}})$  if and only if*

$$\int_{X_{\mathbb{C}}} |F(z)|^2 w_{t,\delta}^m(z) dm(z) < \infty.$$

*Moreover, the norm on  $\mathbb{H}_t^m(X_{\mathbb{C}})$  is equivalent to the above weighted  $L^2$  norm.*

*Proof.* To check the positivity of the weight function, we only need to recall that

$$w_{t,\delta}^m(\exp H) = \int_{K_c} (\delta + \partial_t^m)\Gamma_{2t}(h \exp(2H)) dh$$

and the integrand can be made positive by a proper choice of  $\delta$ . By Gutzmer's formula the integral in the theorem reduces to

$$C \sum_{\lambda \in \mathcal{P}^+} d_\lambda \|A_\lambda(f)\|^2 (\delta + |\lambda + \rho|^{2m})$$

if  $F = f * \gamma_t$ . The above is clearly equivalent to the Sobolev norm on  $\mathbb{H}_t^m(X_{\mathbb{C}})$ . If we equip  $\mathbb{H}_t^m(X_{\mathbb{C}})$  with this norm instead of the original norm, then it follows that the heat kernel transform is an isometric isomorphism.  $\square$

Perhaps, it is better to state the characterisation of  $\mathbb{H}_t^m(X_{\mathbb{C}})$  in the following form. Let us set  $W_t^m(z) = p_t(z) + w_{t,\delta}^m(z) = (1 + \delta + \partial_t^m)p_t(z)$  so that  $W_t^m(z) \geq p_t(z)$ . Let  $\mathcal{B}_t^m(X_{\mathbb{C}})$  be the set of all holomorphic functions, which are square integrable with respect to  $W_t^m$ . Equip  $\mathcal{B}_t^m(X_{\mathbb{C}})$  with the sesquilinear form

$$(F, G)_m = \int_{X_{\mathbb{C}}} F(z) \overline{G(z)} w_t^m(z) dm(z).$$

This turns out to be a genuine inner product on  $\mathcal{B}_t^m(X_{\mathbb{C}})$ , turning it into a Hilbert space, which is the same as  $\mathbb{H}_t^m(X_{\mathbb{C}})$ .

**Theorem 3.5.** *The Segal-Bargmann transform is an isometric isomorphism from  $\mathbb{H}^m(X)$  onto  $\mathcal{B}_t^m(X_{\mathbb{C}}) = \mathbb{H}_t^m(X_{\mathbb{C}})$ .*

**3.4. Holomorphic Sobolev spaces of negative order.** The problem of characterising  $\mathbb{H}_t^{-s}(X_{\mathbb{C}})$ ,  $s > 0$  as a weighted Bergman space has a simple solution. In this case, the weight functions are given by the Riemann-Liouville fractional integrals

$$w_t^{-s}(\exp H) = \frac{1}{\Gamma(s)} \int_0^{2t} (2t - r)^{s-1} e^r \gamma_r^1(\exp 2H) dr.$$

Note that unlike  $w_t^m$ , the weight function  $w_t^{-s}$  are always positive.

**Theorem 3.6.** *Let  $s$  be positive. A holomorphic function  $F$  on  $X_{\mathbb{C}}$  belongs to  $\mathbb{H}_t^{-s}(X_{\mathbb{C}})$  if and only if*

$$\int_{X_{\mathbb{C}}} |F(z)|^2 w_t^{-s}(z) dm(z) < \infty.$$

Thus, we can identify  $\mathbb{H}_t^{-s}(X_{\mathbb{C}})$  with  $\mathcal{F}_t^{-s}(X_{\mathbb{C}})$  defined using the weight function  $w_t^{-s}$ .

*Proof.* Using Gutzmer's formula we have

$$\begin{aligned} & \int_{i\mathbf{a}} \int_U |F(g \exp(H).o)|^2 w_t^{-s}(\exp H.o) J(H) dg dH \\ &= \sum_{\lambda \in \mathcal{P}^+} d_\lambda \|A_\lambda(F)\|^2 \int_{i\mathbf{a}} w_t^{-s}(\exp H.o) \varphi_\lambda(\exp(2H).o) J_1(2H) dH. \end{aligned}$$

Since

$$\int_{i\mathbf{a}} \gamma_r^1(\exp 2H) \varphi_\lambda(\exp(2H).o) J_1(2H) dH = c e^{r|\lambda+\rho|^2},$$

we see that

$$\begin{aligned} & \int_{i\mathbf{a}} \int_U |F(g \exp(H).o)|^2 w_t^{-s}(\exp H.o) J(H) dg dH \\ &= \sum_{\lambda \in \mathcal{P}^+} d_\lambda \|A_\lambda(F)\|^2 \frac{1}{\Gamma(s)} \int_0^{2t} (2t-r)^{s-1} e^{r(1+|\lambda+\rho|^2)} dr. \end{aligned}$$

We show below that

$$\begin{aligned} c_1 (1 + |\lambda + \rho|^2)^{-s} e^{2t(1+|\lambda+\rho|^2)} &\leq \frac{1}{\Gamma(s)} \int_0^{2t} (2t-r)^{s-1} e^{r(1+|\lambda+\rho|^2)} dr \\ &\leq c_2 (1 + |\lambda + \rho|^2)^{-s} e^{2t(1+|\lambda+\rho|^2)}. \end{aligned}$$

The theorem follows immediately from these estimates. To verify our claim, we look at the integral

$$\frac{1}{\Gamma(s)} \int_0^t (t-r)^{s-1} e^{ar} dr = e^{at} \frac{1}{\Gamma(s)} \int_0^t r^{s-1} e^{-ar} dr.$$

The last integral is nothing but

$$e^{at} a^{-s} \left( 1 - \frac{1}{\Gamma(s)} \int_{at}^\infty r^{s-1} e^{-r} dr \right).$$

Since  $\int_{at}^\infty r^{s-1} e^{-r} dr$  goes to 0 as  $a$  tends to infinity, our claim is verified.  $\square$

4. THE IMAGE OF  $C^\infty(X)$  AND  $\mathcal{D}'(X)$  UNDER HEAT KERNEL  
TRANSFORM

In this section, we characterise the images of  $C^\infty(X)$  and  $\mathcal{D}'(X)$ , the spaces of smooth functions and distributions, under the heat kernel transform. We are looking for pointwise estimates on a holomorphic function  $F$  on  $X_{\mathbb{C}}$  that will guarantee that  $F = f * \gamma_t$  for a function  $f \in C^\infty(X)$ . We begin with a necessary condition for functions in the Sobolev space  $\mathbb{H}_t^m(X_{\mathbb{C}})$ . Define the function  $\Phi_0$  on  $\mathfrak{t}_{\mathbb{C}}$  by  $\Phi_0(H) = \prod_{\alpha \in R^+} \frac{(\alpha, H)}{\sinh(\alpha, H)}$ , where the product is taken over all  $R^+$  which is the set of all positive roots in  $\Sigma(\mathfrak{u}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ . Recall that elements of  $\Sigma$  are the elements of  $\Sigma(\mathfrak{u}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$  having a nontrivial restriction to  $i\mathfrak{a}$ . The roots in  $R^+$  give rise to elements of  $\Sigma^+$  and a single  $\alpha \in \Sigma^+$  may be given by several elements of  $R^+$ . (This number is denoted by  $m_\alpha$ .) If we recall the definition of  $\Phi$ , which occurred in the estimates for  $\gamma_t^1$ , we see that  $\Phi(H) = \Phi_0(H)$  as long as  $H \in i\mathfrak{a}$ . We make use of this in what follows.

**Theorem 4.1.** *Let  $m$  be a non-negative integer. Every  $F \in \mathbb{H}_t^m(X_{\mathbb{C}})$  satisfies the estimate*

$$|F(u \exp H)|^2 \leq C(1 + |H|^2)^{-m} \Phi(H) e^{\frac{1}{2i}|H|^2}$$

for all  $u \in U, H \in i\mathfrak{a}$ .

*Proof.* : By standard arguments we can show that the reproducing kernel for the Hilbert space  $\mathbb{H}_t^m(X_{\mathbb{C}})$  is given by

$$K_t^m(g, h) = \frac{1}{(m-1)!} \int_0^\infty s^{m-1} e^{-s} \gamma_{2(t+s)}(gh^*) ds,$$

where  $h \rightarrow h^*$  is the anti-holomorphic anti-involution of  $U_{\mathbb{C}}$  which satisfies  $h^* = h^{-1}$  for  $h \in U$  (see e.g. [13]). Therefore, every  $F \in \mathbb{H}_t^m(X_{\mathbb{C}})$  satisfies the estimate

$$|F(g)|^2 \leq K_t^m(g, g) \|F\|_m.$$

When  $g = u \exp H$  it follows that  $gg^* = u \exp(2H)u^{-1}$  and hence we need to estimate

$$\frac{1}{(m-1)!} \int_0^\infty s^{m-1} e^{-s} \gamma_{2(t+s)}(\exp(2H)) ds.$$

In order to estimate the above integral, we proceed as follows.

Recall that  $\gamma_t$  is the heat kernel associated to the operator  $\Delta = D - |\rho|^2$ , where  $D$  is the Laplace operator on  $X = U/K$ . Let  $D_U$  be the Laplacian on the group  $U$  and let  $\Delta_U = D_U - |\rho|^2$ . Let  $\rho_t(g)$  be the heat kernel associated to  $\Delta_U$ , which is given by

$$\rho_t(g) = \sum_{\pi \in \hat{U}} d_\pi e^{-t\lambda(\pi)^2} \chi_\pi(g),$$

where  $\chi_\pi$  is the character of  $\pi$  and  $\lambda(\pi)^2$  are the eigenvalues of  $\pi$ . When  $\pi = \pi_\lambda$ ,  $\lambda \in \mathcal{P}^+$ , we have  $\lambda(\pi)^2 = |\lambda + \rho|^2$ . We also have

$$\gamma_t(g) = \sum_{\lambda \in \mathcal{P}^+} d_\lambda e^{-t(|\lambda + \rho|^2)} \varphi_\lambda(g).$$

Moreover, we have the relation

$$\int_K \chi_\pi(gk) dk = c_\pi \varphi_\lambda(g),$$

where  $c_\pi = 1$  if  $\pi = \pi_\lambda$  and  $c_\pi = 0$  otherwise. Therefore, we have

$$\gamma_t(g) = \int_K \rho_t(gk) dk$$

and consequently we need to estimate the integral

$$\frac{1}{(m-1)!} \int_0^\infty \left( \int_K \rho_{2(t+s)}(\exp(2H)k) dk \right) s^{m-1} e^{-s} ds.$$

Written explicitly, the above integral is given by the sum

$$\sum_{\pi \in \hat{U}} d_\pi (1 + \lambda(\pi)^2)^{-m} e^{-2t\lambda(\pi)^2} \int_K \chi_\pi(\exp(2H)k) dk.$$

Since  $\pi(\exp(2H))$  is positive definite  $\text{tr} \pi(\exp(2H)) = \|\pi(\exp(2H))\|_1$ , the trace norm of  $\pi(\exp(2H))$ . Using the fact that

$$\|\pi(\exp(2H))\|_1 = \sup\{|\text{tr} \pi(\exp(2H))V| : V^*V = VV^* = I\},$$

we have the estimate

$$\begin{aligned} |\chi_\pi(\exp(2H)k)| &= |\text{tr}(\pi(\exp(2H))\pi(k))| \\ &\leq \text{tr} \pi(\exp(2H)) = \chi_\pi(\exp(2H)). \end{aligned}$$



Therefore, the sum is bounded by

$$C \sum_{\pi \in \hat{U}} d_{\pi} (1 + \lambda(\pi)^2)^{-m} e^{-2t\lambda(\pi)^2} \chi_{\pi}(\exp(2H)).$$

The above sum is related to the reproducing kernel for holomorphic Sobolev spaces on the compact Lie group  $U$  studied by Hall and Lewkeeratiyutkul in [14]. In that paper, using estimates for the heat kernel  $\rho_t$ , they have proved that

$$\begin{aligned} & \sum_{\pi \in \hat{U}} d_{\pi} (1 + \lambda(\pi)^2)^{-m} e^{-2t\lambda(\pi)^2} \chi_{\pi}(\exp(2H)) \\ & \leq C(1 + |H|^2)^{-m} \Phi_0(H) e^{\frac{1}{2t}|H|^2}. \end{aligned}$$

( In [14] the authors have defined the heat kernel for the operator  $\frac{1}{2}\Delta_U$  rather than  $\Delta_U$ .) This estimate immediately gives the required estimate for our kernel since  $\Phi_0(H) = \Phi(H)$ ,  $H \in i\mathfrak{a}$ . This completes the proof of the theorem.  $\square$

Finding suitable pointwise estimates on a holomorphic function sufficient for the membership of the Holomorphic Sobolev spaces is a difficult problem as the proof requires good estimates on the derivatives of the heat kernel  $\gamma_t^1$  on the noncompact dual. Such estimates are not available in the literature. Only recently good estimates on  $\gamma_t^1$  have been obtained by Anker and Ostellari [3] and it is not clear if the same techniques will give us estimates on the derivatives of  $\gamma_t^1$ . So we proceed indirectly to get a sufficient condition. The method avoids estimates on the derivatives but uses only the estimate on  $\gamma_t^1$ . This is done by using Holomorphic Sobolev spaces of negative order.

Let  $n$  be the dimension of the Cartan subspace  $i\mathfrak{a}$  and let  $r$  be the least positive integer for which  $\Pi_{\alpha \in \Sigma^+} |(\alpha, H)|^{m_{\alpha}} \leq C(1 + |H|)^r$ . Determine  $d$  by the condition that the series  $\sum_{\lambda \in \mathcal{P}^+} d_{\lambda}^2 (1 + |\lambda + \rho|^2)^{-d+r+n+1}$  converges. ( Such a  $d$  exists since  $d_{\lambda}$  has a polynomial growth in  $|\lambda|$ .)

**Theorem 4.2.** *Let  $F$  be a holomorphic function on  $X_{\mathbb{C}}$  which satisfies the estimate*

$$|F(u \exp(H))|^2 \leq C(1 + |H|^2)^{-m-d} \Phi(H) e^{\frac{1}{2t}|H|^2}$$

for all  $u \in U$  and  $H \in i\mathbf{a}$ . Then  $F \in \mathbb{H}_t^m(X_{\mathbb{C}})$ .

*Proof.* : In view of Theorem 3.1, which characterises holomorphic Sobolev spaces in terms of the holomorphic Fourier series, we have to show that

$$\sum_{\lambda \in \mathcal{P}^+} d_{\lambda} \left( \sum_{j=1}^{d_{\lambda}} |\tilde{F}_j(\lambda)|^2 \right) (1 + |\lambda + \rho|^2)^m e^{-2t(|\lambda + \rho|^2)} < \infty.$$

In order to estimate the holomorphic Fourier coefficients  $\tilde{F}_j(\lambda)$ , we make use of the estimates on  $\gamma_t^1$ , proved by Anker and Ostellari [3]. They have shown that

$$\gamma_t^1(\exp H) \leq C_t P_t(H) e^{-(\rho, H) - \frac{1}{4t}|H|^2},$$

where  $P_t(H)$  is an explicit polynomial (see the equation 3.1 in [3] for the exact expression for  $P_t$ ). Since  $t$  is fixed, we actually have the estimate

$$\gamma_t^1(\exp H) \leq C_t (\Phi(H))^{\frac{1}{2}} e^{-\frac{1}{4t}|H|^2}.$$

We also know that the holomorphically extended spherical functions  $\varphi_j^{\lambda}$  satisfy the estimates

$$|\varphi_j^{\lambda}(u \exp(H))| \leq \varphi_{\lambda}(\exp(H)).$$

Moreover,  $\varphi_{\lambda}(\exp(H)) = \psi_{-i(\lambda + \rho)}(\exp(H))$  for all  $H \in i\mathbf{a}$  and hence well known estimates on  $\psi_{\lambda}$  leads to

$$|\varphi_j^{\lambda}(u \exp(H))| \leq C e^{|\lambda + \rho||H|} e^{-(\rho, H)}.$$

We refer to Gangolli-Varadarajan [10] (Section 4.6) for these estimates on the spherical functions  $\psi_{\lambda}$ . We also note that  $\Phi(H)(\Phi(2H))^{-1} \leq C e^{2(\rho, H)}$ .

Therefore, making use of the above two estimates, under the hypothesis on  $F$ , we see that  $|\tilde{F}_j(\lambda)|$  is bounded by a constant multiple of the integral

$$\int_{i\mathbf{a}} \Phi(2H) e^{\frac{1}{4t}|H|^2} (1 + |H|^2)^{-m-d} e^{|\lambda + \rho||H|} e^{-\frac{1}{2t}|H|^2} J_1(2H) dH.$$

Recalling the definition of  $J_1(2H)$ , we see that  $\Phi(2H)J_1(2H)$  is bounded by a constant multiple of  $(1 + |H|)^r$ . Thus, the above integral is bounded by

$$\int_{\mathbf{ia}} (1 + |H|^2)^{-m-d+r} e^{|\lambda+\rho||H|} e^{-\frac{1}{4t}|H|^2} dH.$$

The above integral can be easily estimated to give

$$|\tilde{F}_j(\lambda)| \leq C_m (1 + |\lambda + \rho|^2)^{-m-d+r+n+1} e^{t|\lambda+\rho|^2}.$$

This proves our claim and completes the proof of sufficiency.  $\square$

Combining Theorems 4.1 and 4.2 and we obtain the following characterisation of the image of  $C^\infty(X)$  under the Segal-Bargmann transform.

**Theorem 4.3.** *A holomorphic function  $F$  on  $X_{\mathbb{C}}$  is of the form  $F = f * \gamma_t$  with  $f \in C^\infty(X)$  if and only if it satisfies*

$$|F(u \exp(H))| \leq C_m (1 + |H|^2)^{-m/2} (\Phi(H))^{\frac{1}{2}} e^{\frac{1}{4t}|H|^2}$$

for all  $u \in U$ ,  $H \in \mathbf{ia}$  and for all positive integers  $m$ .

This theorem follows from the fact that  $C^\infty(X)$  is the intersection of all the Sobolev spaces  $\mathbb{H}^m(X)$ .

We conclude this section by giving a characterisation of the image of distributions on  $X$  under the heat kernel transform. If  $f$  is a distribution,  $f * \gamma_t$  still makes sense and extends to  $X_{\mathbb{C}}$  as a holomorphic function. We now prove the following theorem which was stated as a conjecture in [14].

**Theorem 4.4.** *A holomorphic function  $F$  on  $X_{\mathbb{C}}$  is of the form  $F = f * \gamma_t$  for a distribution  $f$  on  $X$  if and only if it satisfies the estimate*

$$|F(u \exp(H))| \leq C (1 + |H|^2)^{m/2} (\Phi(H))^{\frac{1}{2}} e^{\frac{1}{4t}|H|^2}$$

for some positive integer  $m$  for all  $u \in U$  and  $H \in \mathbf{ia}$ .

*Proof.* First we prove the sufficiency of the above condition. If we could show that the holomorphic Fourier coefficients of  $F$  satisfy

$$|\tilde{F}_j(\lambda)| \leq A (1 + |\lambda + \rho|^2)^N e^{t(|\lambda+\rho|^2)}$$

for some  $N$ , then by Theorem 3.1 it would follow that  $F = f * \gamma_t$  for some  $f \in \mathbb{H}^{-d}(X)$  for a suitable  $d$ . Since the union of all the Sobolev spaces is precisely the space of distributions, we get the result. In order to prove the above estimate, we can proceed as in the previous theorem. We end up with the integral

$$\int_{i\mathbf{a}} \Phi(2H) e^{\frac{1}{4t}|H|^2} (1 + |H|^2)^{m/2} e^{|\lambda+\rho||H|} e^{-\frac{1}{2t}|H|^2} J(H) dH.$$

As before, this leads to the estimate  $A(1 + |\lambda + \rho|^2)^{m+r+n+1} e^{t(|\lambda+\rho|^2)}$  proving the sufficiency.

For the necessity: since every distribution belongs to some Sobolev space, let us assume  $f \in \mathbb{H}^{-m}(X)$  for a positive integer. Then,  $F = f * \gamma_t$  belongs to  $\mathbb{H}_t^{-m}(X_{\mathbb{C}})$ , whose reproducing kernel is given by

$$K_t^{-m}(g, h) = \sum_{\lambda \in \mathcal{P}} d_{\lambda} (1 + |\lambda + \rho|^2)^m e^{-2t|\lambda+\rho|^2} \sum_{j=1}^{d_{\lambda}} \varphi_j^{\lambda}(g) \overline{\varphi_j^{\lambda}(h^*)}.$$

Proceeding as in Theorem 3.1, we need to estimate

$$\sum_{\pi \in \hat{U}} d_{\pi} (1 + \lambda(\pi)^2)^m e^{-2t\lambda(\pi)^2} \chi_{\pi}(\exp(2H)).$$

To this end, we make use of the Poisson summation formula proved by Eskin [5] and Urakawa [22] as in Hall [13]. According to this formula,

$$\sum_{\pi \in \hat{U}} d_{\pi} e^{-2t\lambda(\pi)^2} \chi_{\pi}(\exp(2H)) = e^{2t|\rho|^2} (8\pi t)^{-\frac{n}{2}} e^{\frac{1}{2t}|H|^2} \Phi(H) k(t, H),$$

where  $k(t, H)$  is known explicitly (see equation 8 in [13]). We need to estimate the  $m$ -th derivative of  $k(t, H)$  with respect to  $t$ .

The above function  $k(t, H)$  has been estimated in [13]. Here, we modify the arguments in [13] to estimate its derivatives. Observe that any derivative falling on  $e^{\frac{1}{2t}|H|^2}$  brings down a factor of  $|H|^2$ . The function  $k(t, H)$  is given by the sum

$$k(t, H) = \sum_{\gamma_0 \in \Gamma \cap \overline{\mathbf{a}^+}} \epsilon(\gamma_0) e^{-\frac{1}{8t}|\gamma_0|^2} p_{\gamma_0}(t, H)$$

with  $p_{\gamma_0}(t, H)$  given by the expression

$$p_{\gamma_0}(t, H) = \pi(H)^{-1} \sum_{\gamma \in W \cdot \gamma_0} \pi(H - \frac{1}{2i}\gamma) e^{\frac{i}{t}(H, \gamma)}.$$

In the above,  $\pi(H) = \prod_{\alpha \in \Delta^+} (\alpha, H)$ ,  $W$  is the Weyl group,  $\overline{\mathfrak{a}^+}$  is the closed Weyl chamber and  $\Gamma$  is the kernel of the exponential mapping for the maximal torus etc. If we can show that any derivative falling on  $k(t, H)$  in effect brings down a factor of  $|H|$ , then the  $m$ -th derivative can be estimated to give

$$K_t^{-m}(g, g^*) \leq C(1 + |H|^2)^{2(m+d)} \Phi(H) e^{\frac{1}{2t}|H|^2}.$$

This will then complete the proof of the necessity.

We now give some details of the above sketch of the proof. In [13], the author has proved that there is a polynomial  $P$  such that the estimate

$$|p_{\gamma_0}(t, H)| \leq P(t^{-1/2}|\gamma_0|)$$

holds. This has been stated and proved as Proposition 3 in [13]. For our proof, we need to get estimates for sums of the form

$$p_{\gamma_0, j}(t, H) = \pi(H)^{-1} \sum_{\gamma \in W \cdot \gamma_0} \pi(H - \frac{1}{2i}\gamma) (H, \gamma)^j e^{\frac{i}{t}(H, \gamma)}.$$

We claim that

$$|p_{\gamma_0, j}(t, H)| \leq C_{j, t} |H|^j P_j(t, |\gamma_0|)$$

for some polynomials  $P_j(t, \cdot)$ . This will give us the required estimate. As in [13], we can assume that  $t = 1$ . We indicate the proof when  $j = 1$ , the general case being very similar.

Consider the operators  $I_\alpha$  defined by ( see [13])

$$I_\alpha f(x) = \int_0^\infty f(x - t\alpha) dt,$$

which invert the directional derivative operators  $D_\alpha$ . For any distribution supported on a cone over  $\Delta^+$ , we can define  $I_\alpha T$  by duality (cf. Definition 8 in [13]). In [13], the author has proved that the convex hull of the support of the distribution  $S = I_{\alpha_1} I_{\alpha_2} \dots I_{\alpha_k} T$  is contained in the convex hull of the support of  $T$ , whenever  $T$  is a compactly supported distribution, which is alternating with respect to the action of

the Weyl group. (This is proved in Lemma 9 of [13].) Let  $\mathcal{F}$  be the Euclidean Fourier transform. Let  $T$  denote the Fourier transform of the distribution  $\pi(H)p_{\gamma_0,1}(1, H)$ , which can be written as

$$T = c \sum_{\gamma \in W.\gamma_0} D_\gamma T_\gamma,$$

where  $T_\gamma$  is the Fourier transform of  $\pi(H - \frac{1}{2i}\gamma)e^{i(H,\gamma)}$ . It is clear that  $T$  is alternating and hence Lemma 9 of [13] applies.

As in [13], we set  $S_\gamma = I_{\alpha_1}I_{\alpha_2}\dots I_{\alpha_k}T_\gamma$  and note that  $S_\gamma$  is a finite linear combination of distributions of the form  $(\alpha_{i_1}, \gamma)\dots(\alpha_{i_l}, \gamma)I_{\alpha_{i_1}}\dots I_{\alpha_{i_l}}\delta_\gamma$ . Defining  $S = I_{\alpha_1}I_{\alpha_2}\dots I_{\alpha_k}T$ , we get

$$S = c \sum_{\gamma \in W.\gamma_0} D_\gamma S_\gamma$$

and therefore,

$$\mathcal{F}^{-1}S(H) = c(\pi(H))^{-1}\mathcal{F}^{-1}T(H) = c'p_{\gamma_0,1}(1, H).$$

Thus, we need to estimate  $\mathcal{F}^{-1}S(H)$ . If  $E$  is the convex hull of the support of  $S$ , then by Lemma 9 (of [13]) it is contained in the convex hull of  $W.\gamma_0$ . This follows from the fact that  $T_\gamma$  are linear combinations of

$$(\alpha_{i_1}, \gamma)\dots(\alpha_{i_l}, \gamma)D_\gamma D_{\alpha_{i_{l+1}}}\dots D_{\alpha_{i_k}}\delta_\gamma.$$

Finally, if  $\varphi$  is any nonnegative  $C_0^\infty$  function supported in a small neighbourhood  $E_\epsilon$  of  $E$  and identically one on another (smaller) neighbourhood of  $E$ , then  $(S, f) = \sum_{\gamma \in W.\gamma_0} (D_\gamma(\varphi S_\gamma), f)$  for any test function  $f$ , as can be easily checked. This gives us

$$\mathcal{F}^{-1}S(H) = c \sum_{\gamma \in W.\gamma_0} (H, \gamma)\mathcal{F}^{-1}(\varphi S_\gamma)(H),$$

which leads to the estimate

$$|\mathcal{F}^{-1}S(H)| \leq C|H||\gamma_0| \sum_{\gamma \in W.\gamma_0} |\mathcal{F}^{-1}(\varphi S_\gamma)(H)|.$$

The above sum is bounded by

$$\sum_{\gamma \in W.\gamma_0} \int \varphi(H)d|S_\gamma| \leq \sum_{\gamma \in W.\gamma_0} |S_\gamma|(E_\epsilon).$$

Since  $S_\gamma$  is a linear combination of the positive measures  $(\alpha_{i_1}, \gamma) \dots (\alpha_{i_l}, \gamma) I_{\alpha_{i_1}} \dots I_{\alpha_{i_l}} \delta_\gamma$  the measure  $|S_\gamma|(E_\epsilon)$  can be estimated as in [13] to give the required estimate

$$|\mathcal{F}^{-1}S(H)| \leq C_\epsilon P_1(1, \gamma_0) |H|.$$

This completes the proof of the theorem. □

### Acknowledgments

The author wishes to thank the referee for his thorough reading of the previous version of this paper and making several useful remarks. He pointed out several inaccuracies, demanded clarifications of several points and suggested a reorganisation of the paper all of which have considerably improved the readability of the paper. The author wishes to thank E. K. Narayanan for answering several naive questions on the structure theory of semisimple Lie groups. He is also thankful to Bernhard Krötz for pointing out an error in a previous version of this paper. This work is supported by a grant from UGC under SAP.

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